Reduced density matrices and decoherence in quantum cosmology

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We investigate a quantum cosmological model consisting of inhomogeneous massless minimally coupled scalar field perturbations on a closed Friedmann-Robertson-Walker minisuperspace model with a spatially homogeneous massless minimally coupled scalar field. We discuss how to define a reduced density matrix by summing over the perturbations in the full density matrix, using the approximate Hilbert-space structure that exists for the perturbation wave function when the minisuperspace part of the wave function is of the WKB form. We then concentrate on two particular candidates for a reduced density matrix and discuss their relation to particle creation effects in quantum field theory on curved spacetime. Our results do not suggest that decoherence in the reduced density matrices could be directly identified as a lack of interference between the classical trajectories that correspond to a WKB minisuperspace part of the total wave function.

I. INTRODUCTION

The Universe on the large scale behaves classically to a high degree of accuracy. What is observed in cosmology is mainly the electromagnetic field coming from stars, galaxies, and clusters of galaxies. From it we deduce the redshifts and the recessing velocities of the galaxies, their masses, and their positions with respect to us. All this assumes a classical Friedmann-Robertson-Walker (FRW) model where galaxies are thought of as essentially point particles. On the other hand, we believe that all matter in the Universe is fundamentally described by quantum fields, and we expect that also the gravitational field will ultimately be described in a similar way. The bold hypothesis made in quantum cosmology is that the whole Universe is described quantum mechanically by a single wave function. To make predictions for the large-scale structure of the Universe from quantum cosmology, it is of crucial importance to understand how and when this wave function can describe phenomena that are perceived as classical. A main problem here is to understand exactly what is meant by "classical." This has not yet been solved satisfactorily, although important progress has recently been made.

It has often been argued in quantum mechanics and quantum cosmology that an essential ingredient of recovering classical behavior from quantum theory is to have a wave function of the WKB form.[1–3] As the phase of a WKB wave function approximately obeys the classical Hamilton-Jacobi equation, it is expected that this wave function should give rise to essentially the same correlations between the configuration-space variables and their momenta as what would be obtained by interpreting the phase as Hamilton's principal function in the classical theory. In this way one recovers from the wave function a certain family of solutions to the classical theory, and it is assumed that the quantum system at the classical limit corresponds to a statistical ensemble of these classical trajectories.

There have been some attempts[4–6] to use the Wigner function as a tool for justifying the above interpretation of a WKB wave function in terms of the Hamilton-Jacobi theory. This uses the suggestion by Geroch and Hartle that predictions in quantum cosmology could be obtained from peaks in the wave function of the Universe.[2,7] A more careful analysis,[8–10] however, demonstrates that there exist in general many peaks in the Wigner function distribution which do not exhibit the expected classical correlations. It was also shown[9,10] that, adopting the peaking interpretation, an essential element to predict classical correlations in the WKB wave function was the introduction of some form of coarse graining.

Another much more widely recognized characteristic of classical behavior is decoherence or lack of quantum interference between the classical configurations.[11–22] A possible quantum-mechanical mechanism giving rise to decoherence is the interaction of the system of interest with an environment. This possibility has recently been intensely investigated in the context of measurement theory and might answer long-standing questions.[12,14] Starting with a closed quantum-mechanical system, such as in quantum cosmology of the whole Universe, one considers a reduced density matrix or a coarse-grained decoherence functional for a few "large" degrees of freedom, obtained by summing over all the unobserved or irrelevant degrees of freedom. The latter then effectively act as a bath for the former ones. Even if the division between the system and the environment remains to some extent arbitrary, this mechanism seems reasonable, since we never observe but a small fraction of all the degrees of freedom of the Universe.

It is also possible that decoherence could provide a justification of the interpretation of a WKB wave function in terms of the classical correlations given by the Hamilton-Jacobi theory. This is supported by the results in Ref. 10, where it was shown that, in the simple cosmological minisuperspace model studied in Refs. 15–20, summing over unobserved inhomogeneous degrees of freedom for a scalar field is equivalent to doing a coarse graining of the momenta of the Universe.

In nonrelativistic quantum mechanics described by a wave function $\psi(\mathbf{x},t)$ obeying the Schrödinger equation, the usual tool for investigating decoherence is the density matrix

$$\rho(\mathbf{x},t;\mathbf{x}',t')=\psi^*(\mathbf{x},t)\psi(\mathbf{x}',t') \; . \tag{1.1}$$

Reduced density matrices may be constructed from ρ by dividing the configuration-space variables $\mathbf{x}$ into "large" ones and "small" ones and tracing ρ over the small variables. Although this procedure is applicable for arbitrary t and t', the resulting reduced density matrix would usually be interpreted in terms of physical decoherence only at the equal time limit $t=t'$. As the time parameter t in the Schrödinger equation is in principle unobservable,[23] it may be possible to recover the same results also without explicitly referring to the equal-time limit,[24] but it remains nevertheless true that the presence of the explicit time variable in the reduced density matrix makes it easy to make a connection between "configurations" and "trajectories:" The latter may simply be thought of as the evolutions of the former in the time parameter t. Decoherence in the configuration-space variables $\mathbf{x}$ can thus be interpreted in terms of decoherence between classical time evolutions.

In quantum cosmology the situation is different. The wave function $\Psi(h_{ij},\phi)$ is a functional of the metric h_{ij} and matter fields ϕ on a three-dimensional surface. Neither the wave function nor the density matrix

$$\rho(h_{ij},\phi;h'_{ij},\phi')=\Psi^*(h_{ij},\phi)\Psi(h'_{ij},\phi') \tag{1.2}$$

has an explicit time argument. It is again at least formally possible to divide the variables into large and small ones and to construct reduced density matrices by tracing over the small ones, but the relation between decoherence in the configuration space and decoherence between trajectories is now less clear. For example, if there is just a single large variable and the wave function is rapidly oscillating in this variable, then this variable can be identified with time in a straightforward way and the resulting reduced density matrix is analogous to a quantum-mechanical reduced density matrix with $t\neq t'$ but with all the configuration-space variables $\mathbf{x}$ traced out.[17] On the other hand, if there is more than one large variable and the wave function is rapidly oscillating in these variables, one can by the WKB approximation introduce a vector field which can be thought of as $\partial/\partial t$,[25–29] but this vector field does not single out a unique time coordinate in the configuration space unless one imposes further conditions.[30]

Another important issue is what is meant by tracing over the small variables. Even if one does not assume a Hilbert-space structure for the full theory, tracing over the small variables appears to require assuming at least an approximate Hilbert-space structure for these variables.[15–18] The resulting reduced density matrix may then depend on how this approximate Hilbert space is chosen. Another way to put this is that, whereas the full density matrix is a biscalar on the full configuration space, no apparent geometrical interpretation exists for the reduced density matrix except at the limit where the two arguments coincide.

Yet another important point is that when there are an infinite number of small degrees of freedom, the expressions for the reduced density matrices may have to be regularized.[31,32] In quantum cosmology it is usual to take the large degrees of freedom to consist of a spatially homogeneous minisuperspace model and treat all the remaining degrees of freedom as perturbations by a multipole expansion on this background.[15–18] There may therefore arise the need to regularize sums and products over the infinite number of multipoles. As methods based on $3+1$ split mode sums can give misleading results for the renormalized energy-momentum tensor in quantum field theory on curved spacetime,[33] one would ideally like to strive for a covariant regularization of the multipole sums also with the reduced density matrix. Although such a four-dimensional regularization has been considered in the case where the two arguments of the reduced density matrix coincide,[34] it appears uncertain whether something similar can be done away from the coincidence limit where the geometrical interpretation of the reduced density matrix is less clear.

There have been proposals for treating the sums over the multipoles in the reduced density matrix by introducing a cutoff in the number of the modes. One suggestion was to let the cutoff go to infinity;[16,18] this leads to "perfect decoherence" in the model studied by Kiefer[16] and, also, more generally at least for the total three-volume.[18] One may, however, question the consistency of this suggestion on the grounds that the semiclassical expansion of the Wheeler-DeWitt equation already assumes either a finite cutoff in the number of the multipoles or a regularization of the zero-point energy of the multipoles. Roughly speaking, the reason is that the multipole modes contribute to the potential of the minisuperspace Wheeler-DeWitt equation by their zero-point energy. If this energy is not regularized, it becomes infinite when the cutoff is pushed to infinity, and thus the assumption that the multipoles are perturbations in the minisuperspace Wheeler-DeWitt equation is not satisfied.[33]

A second possibility for a cutoff would be to use physically motivated arguments to introduce a natural limiting scale for the multipole modes. One could, for example, sum only over spatial wavelengths greater than the Planck scale.[16] A suggestion by Halliwell in the de Sitter minisuperspace model was to sum over modes whose wavelength is larger than the de Sitter horizon.[17] This suggestion could perhaps be generalized to more complicated minisuperspace backgrounds by taking the cutoff to be at the instantaneous Hubble radius, which is well defined provided the minisuperspace part of the wave function consists of a single rapidly oscillating $\exp(iS)$ component. The appeal in this suggestion is that modes larger than the instantaneous Hubble radius (which in the de Sitter model coincides with the horizon) are certainly unobservable, but it is perhaps hard to justify why one should then be interested in the radius of the Universe whose wavelength is the largest possible of all.

In this paper we shall investigate the above issues in a quantum cosmological model whose minisuperspace part consists of the closed Friedmann model with a spatially homogeneous, massless, minimally coupled scalar field. This minisuperspace model has thus two configuration-space variables, which is the minimum number that allows nontrivial minisuperspace dynamics: The general solution to the minisuperspace classical equations of motion has two genuine constants of integration. The role of the environment will be taken by another massless and minimally coupled but inhomogeneous scalar field, treated as a perturbation on the minisuperspace background. The inhomogeneous modes of the gravitational field and those of the minisuperspace scalar field will be omitted: This is self-consistent in the sense of the dynamics, since the omitted modes would not couple to our environment scalar field to quadratic order in the action. Also, the total reduced density matrix would factorize into a contribution from our environment field and a contribution from the omitted fields. Our motivation for introducing the environment scalar field is that we expect its contribution to the reduced density matrix to qualitatively reflect the properties of the contribution from the modes we have omitted, yet without the technical difficulties of having to couple scalar and gravitational perturbations to each other. To examine the validity of this expectation will be left a subject to future work.

The appeal of our model is that it combines nontrivial minisuperspace dynamics to a nontrivial reduced density matrix, but remains sufficiently simple to be solvable. For any minisuperspace wave function of the Lorentzian semiclassical form $\exp(iS)$ in our model it is possible to find in an essentially closed form the solutions to the next-to-leading-order functional Schrödinger equation in the semiclassical expansion of the total Wheeler-DeWitt equation. The reduced density matrix can then be expressed as a sum involving Legendre functions. One can thus experiment with different choices for the perturbation quantum state and different definitions of the reduced density matrix, and one can compare the resulting reduced density matrices to the classical minisuperspace trajectories obtained from the minisuperspace part of the wave function by Hamilton-Jacobi theory. We find that the suppression of the off-diagonal elements in our reduced density matrices is at its weakest when the perturbations are chosen to be in their adiabatic vacuum state. This supports the expectation that the reduced density matrices in some sense do describe the interaction between the background and perturbations, since this interaction is expected to be at its smallest for the adiabatic perturbation vacuum. However, our models do not suggest a direct identification of decoherence in the reduced density matrix as lack of interference between the classical minisuperspace trajectories.

In Sec. II we present our minisuperspace model and describe the quantum field theory of the environment scalar field on a fixed classical solution of the minisuperspace model. In Sec. III we quantize the full model consisting of both the minisuperspace and environment. We perform the expansion of the Wheeler-DeWitt equation to the next-to-leading order around the minisuperspace model using the by now well-known technique,[26–29] but paying special attention to the factor ordering in the total Wheeler-DeWitt equation and to the way the total wave function is factorized into the minisuperspace and perturbation parts. None of this would be important if the eventual aim were a covariant regularization of the divergent quantities, as with the energy-momentum tensor in Refs. 2 and 33; however, we shall find that a discussion of the factor ordering is essential when we wish to define a reduced density matrix and discuss its interpretation in terms of the geometry of superspace. We are able to give a geometrical interpretation to the reduced density matrix only when the arguments coincide, and we argue that no purely geometrical definition exists away from this coincidence limit. We first construct a reduced density matrix using the definition which has appeared in most of the previous literature.[15–18] We then propose an alternative definition which we argue to be geometrically equally consistent, but potentially more directly related to particle-creation effects.

In Sec. IV we compute the two reduced density matrices for three concrete examples with minisuperspace wave functions of the Lorentzian semiclassical form $\exp(iS)$. The peaking in the reduced density matrices in these examples bears no apparent relation to the classical minisuperspace trajectories obtained by interpreting S as Hamilton's principal function. Although the Lagrangian of the total system contains no explicit coupling term between the minisuperspace scalar field and environment, the reduced density matrices are seen to depend in general on both of the two minisuperspace variables in a nontrivial way. The reason is that the reduced density matrix is affected not only by the explicit couplings in the Lagrangian, but also by the form assumed by the minisuperspace wave function.

The results are summarized and discussed in Sec. V. In Appendix A we show how a solution to the functional Schrödinger equation on classical solutions to the minisuperspace model can be lifted into a solution to the more general functional Schrödinger equation that appears in the semiclassical expansion of the Wheeler-DeWitt equation. Finally, in Appendix B we discuss the definition of the reduced density matrix away from the coincidence limit both in quantum mechanics and quantum cosmology.

II. MASSLESS SCALAR FIELD ON A CLASSICAL MINISUPERSPACE BACKGROUND

We consider a minisuperspace model which consists of spatially homogeneous closed ($k=+1$) Friedmann model with a massless, minimally coupled scalar field. The metric ansatz is

$$ds^2=\rho^2[-N^2(t)dt^2+a^2(t)d\Omega_3^2]\ , \tag{2.1}$$

where $d\Omega_3^2$ is the metric on the unit three-sphere and $\rho^2=2G/3\pi$. The scalar field Φ is taken to be constant on the spatial surfaces, $\Phi=\Phi(t)$. The action is a sum of the Einstein-Hilbert-York-Gibbons-Hawking gravitational action[35,36]

$$S_g=\frac{1}{16\pi G}\int_{\mathcal{M}} d^4x(-g)^{1/2}R+\frac{1}{8\pi G}\int_{\partial\mathcal{M}} d^3x(h)^{1/2}K \tag{2.2}$$

and the massless scalar field action

$$S_\Phi=-\frac{1}{2}\int_{\mathcal{M}} d^4x(-g)^{1/2}g^{\mu\nu}(\partial_\mu\Phi)(\partial_\nu\Phi)\ . \tag{2.3}$$

The cosmological constant has been taken to vanish. Inserting the ansatz into (2.2) and integrating over the spatial surfaces gives the minisuperspace action

$$S_M=\int L\,dt\ , \tag{2.4}$$

where the minisuperspace Lagrangian is given by

$$L_M=\frac{N}{2}\left[-a\left[\frac{\dot a}{N}\right]^2+a^3\left[\frac{\dot\phi}{N}\right]^2+a\right]\ . \tag{2.5}$$

The overdot denotes d/dt, and we have defined $\phi=(2^{1/2}\pi\rho)\Phi$. This minisuperspace action is well known to reproduce correctly the full equations of motion for our ansatz.

The classical solutions of the model are well known.[37–40] In the gauge $N=a^{-1}$, the general solution can be written as

$$a^2(t)=(D^2-4t^2)^{1/2}\ , \tag{2.6a}$$

$$\phi(t)=\phi_0+\frac{1}{4}\ln\left[\frac{D+2t}{D-2t}\right]\ , \tag{2.6b}$$

$$N(t)=\frac{1}{a(t)}\ . \tag{2.6c}$$

Here D and ϕ_0 are constants of integration, satisfying $|D|>0$ and $-\infty<\phi_0<\infty$. The coordinate time t takes the range $-\frac{1}{2}|D|<t<\frac{1}{2}|D|$. All the solutions start and end at curvature singularities at $t=\mp\frac{1}{2}|D|$, at which $|\phi|$ diverges. The maximum value of $a(t)$ on a given solution is $\sqrt{|D|}$, and ϕ_0 is the value of $\phi(t)$ at the moment of maximum expansion. Note that D is the value of the momentum conjugate to ϕ. The proper time $\int N\,dt$ can be expressed as a function of t in terms of elliptic integrals.[40] Eliminating t, a solution with given D and ϕ_0 traces in the configuration space the curve

$$a^2=\frac{|D|}{\cosh[2(\phi-\phi_0)]}\ . \tag{2.7}$$

Let us now regard one of these minisuperspace classical solutions as a given background spacetime, with given values of the constants (D,ϕ_0), and let us introduce on this spacetime an inhomogeneous massless minimally coupled scalar field σ with the action

$$S_\sigma=-\tfrac{1}{2}\int_{\mathcal{M}} d^4x(-g)^{1/2}g^{\mu\nu}(\partial_\mu\sigma)(\partial_\nu\sigma)\ . \tag{2.8}$$

We decompose σ as

$$\sigma(t,\mathbf{x})=\rho^{-1}\sum_{nlm} f_{nlm}(t)Q_{nlm}(\mathbf{x})\ , \tag{2.9}$$

where $\mathbf{x}$ denotes the spatial coordinates and the scalar hyperspherical harmonics $Q_{nlm}(\mathbf{x})$ are the normalized eigenfunctions of the scalar Laplacian on the unit three-sphere. The index n takes the values $1,2,\ldots,$ the corresponding eigenvalues of the Laplacian being $-(n^2-1)$. Further properties of Q_{nlm} can be found in Ref. 41.

With the background ansatz (2.1), S_σ takes the form

$$S_\sigma=\sum_n S_n\ , \tag{2.10}$$

$$S_n=\int dt\frac{N}{2}\left[a^3\frac{\dot f_n^2}{N^2}-a(n^2-1)f_n^2\right]\ . \tag{2.11}$$

Here, and from now on, we suppress the degeneracy indices l and m. In the subsequent sums and products over n, each n will therefore stand for n^2 separate terms. Inserting the background solution (2.6) and writing $t=\frac{1}{2}|D|x$, we obtain

$$S_n=|D|\int dx\left[(1-x^2)\left[\frac{df_n}{dx}\right]^2-\left[\frac{n^2-1}{4}\right]f_n^2\right]\ . \tag{2.12}$$

The classical equation of motion for f_n is the Legendre equation of order $\frac{1}{2}(n-1)$:[42]

$$\left[\frac{d}{dx}(1-x^2)\frac{d}{dx}\right]f_n+\left[\frac{n^2-1}{4}\right]f_n=0\ . \tag{2.13}$$

The two independent solutions for f_n are the Legendre functions $P_\nu(x)$ and $Q_\nu(x)$, where $\nu=\frac{1}{2}(n-1)$.

We quantize the field σ first in the functional Schrödinger representation. The Hamiltonian form of S_n is

$$S_n=\int d\tau\left[p_n\frac{df_n}{d\tau}-H_n\right]\ , \tag{2.14}$$

where τ is the proper time defined by $d\tau=N\,dt$ and the Hamiltonian is given by

$$H_n=\frac{1}{2}\left[\frac{dx}{d\tau}\right]\left[\frac{p_n^2}{2|D|(1-x^2)}+2|D|\left[\frac{n^2-1}{4}\right]f_n^2\right]\ . \tag{2.15}$$

The components of the functional Schrödinger equation thus read

$$i\frac{\partial\chi_n}{\partial\tau}=\hat H_n\chi_n\ , \tag{2.16}$$

where

$$\hat{H}_n=\frac{1}{2}\left[\frac{dx}{d\tau}\right]\left[-\frac{1}{2|D|(1-x^2)}\frac{\partial^2}{\partial f_n^2}+2|D|\left[\frac{n^2-1}{4}\right]f_n^2\right]. \quad (2.17)$$

The total wave functional χ is given by the product

$$\chi=\prod_n \chi_n . \quad (2.18)$$

We shall limit ourselves to investigating Gaussian states of the form

$$\chi_n=A_n(x)\exp\left[i|D|(1-x^2)\frac{F_n'(x)}{F_n(x)}f_n^2\right], \quad (2.19)$$

where the prime denotes a derivative with respect to x. It would be straightforward to investigate also more general states. Inserting this ansatz into the Schrödinger equation (2.16) gives two independent equations for $F_n(x)$ and $A_n(x)$. One of these is the Legendre equation (2.13) for $F_n(x)$, which implies that $F_n(x)$ must be a linear combination of the Legendre functions $P_\nu(x)$ and $Q_\nu(x)$ with $\nu=\frac{1}{2}(n-1)$. The remaining equation involving both $F_n(x)$ and $A_n(x)$ can then be integrated, with the result that $A_n(x)$ is proportional to $(F_n)^{-1/2}$.

In order for the solutions for χ_n to be normalizable, it is seen from (2.19) that F_n must contain both P_ν and Q_ν with nonvanishing coefficients. Without loss of generality we can set the coefficient of P_ν to unity and take

$$F_n(x)=P_\nu(x)+wQ_\nu(x) , \quad (2.20)$$

where w is a complex number with a nonvanishing imaginary part. In general, w could be chosen to depend on n, as well as on the suppressed degeneracy indices. For simplicity we shall take w to be the same number for all the multiple modes.

Using the Wronskian property of the Legendre functions,[42] χ_n can be written as

$$\chi_n=A_n\exp[-(\alpha_n+i\beta_n)f_n^2] , \quad (2.21)$$

where

$$\alpha_n=\frac{|D|\mathrm{Im}(w)}{|F_n(x)|^2} , \quad (2.22a)$$

$$\beta_n=-\frac{|D|}{2}(1-x^2)\frac{d}{dx}\ln[|F_n(x)|^2] . \quad (2.22b)$$

For normalizability α_n must be positive, and we must therefore have $\mathrm{Im}(w)>0$. Taking χ_n to be normalized to unity according to

$$1=\int df_n\chi_n^*\chi_n , \quad (2.23)$$

we can choose the phase so that A_n is given by

$$A_n=\left[\frac{2|D|\mathrm{Im}(w)}{\pi}\right]^{1/4}F_n^{-1/2} . \quad (2.24)$$

We have thus found in the functional Schrödinger picture a one-parameter family of Gaussian states, parametrized by the complex number w with $\mathrm{Im}(w)>0$. One can verify by standard techniques[43] that these states are the no-particle states associated with the Heisenberg picture "positive"-frequency mode functions

$$\varphi_n(x)=\frac{1}{\sqrt{|D|\mathrm{Im}(w)}}\times[P_\nu(x)+w^*Q_\nu(x)] , \quad \nu=\frac{n-1}{2} . \quad (2.25)$$

We shall call these states w vacua.

An especially interesting vacuum is obtained when $w=2i/\pi$. In this case $\varphi_n(x)$ has the large-n expansion

$$\varphi_n(x)=\frac{1}{[|D|\mathrm{Im}(w)n(1-x^2)]^{1/2}}\times\exp\left[i\left[\frac{n}{2}\arccos x-\frac{\pi}{4}\right]\right]+O\left[\frac{1}{n^{3/2}}\right] . \quad (2.26)$$

The expression on the right-hand side of (2.26) is the zeroth-order adiabatic positive-frequency solution of the mode equation (2.13) at $n\to\infty$. This holds for all values of x, even though not uniformly, as the expansion given in (2.26) is valid only for $n^2(1-x^2)\gg 1$. The vacuum with $w=2i/\pi$ is therefore an adiabatic vacuum for all times.[43]

If $w\neq 2i/\pi$, α_n and β_n in (2.22) will for large n be rapidly oscillating functions of x. For the adiabatic vacuum, $w=2i/\pi$, the oscillations cancel and we have the asymptotic expansions

$$\alpha_n=\frac{n|D|}{2}(1-x^2)^{1/2}+O\left[\frac{1}{n}\right] , \quad (2.27a)$$

$$\beta_n=-\frac{|D|x}{2}+O\left[\frac{1}{n^2}\right] , \quad (2.27b)$$

where the omitted terms are slowly varying functions of x. Note that the adiabatic vacuum is symmetric with respect to the moment of maximum expansion, in the sense that $\alpha_n(x)=\alpha_n(-x)$, $\beta_n(x)=-\beta_n(-x)$.

III. QUANTIZATION OF THE FULL MODEL AND THE REDUCED DENSITY MATRIX

We shall now embark on quantizing the combined system of the background minisuperspace model coupled to the scalar field perturbations. The total action is

$$S=S_M+S_\sigma , \quad (3.1)$$

where S_M and S_σ are given, respectively, by (2.4) and (2.11). To be consistent, the quantization must respect the assumption under which the action (3.1) has been derived, namely, that the inhomogeneous scalar field σ be a small perturbation on the minisuperspace background. Although it is well known how to implement this assumption (see, for example, Refs. 26–29), there are two issues here that merit a brief discussion.

First, although the background model contains the spatially homogeneous mode of the massless scalar field Φ, this field should not be thought of as a "large" homogeneous mode of the massless perturbation scalar field σ. The reason is that inhomogeneous perturbations in Φ and in the metric would couple to each other already to quadratic order in the action,[27] and it would be inconsistent to retain just the inhomogeneous perturbations in Φ while setting the gravitational perturbations to zero. A more appropriate interpretation of the action (3.1) is that the inhomogeneous perturbations in both Φ and the metric have been set to zero, and σ is a new, separate massless scalar field on this background. This is consistent for describing the dynamics of σ to the extent that its back reaction on the minisuperspace is neglected, since the inhomogeneous perturbations in Φ and the metric would not couple to σ to quadratic order in the action. This is, however, not consistent for investigating decoherence, since the inhomogeneous perturbations in Φ and the metric would contribute to the reduced density matrix by a factor which would be expected to be of the same order of magnitude as the factor coming from σ. Our motivation for working with the action (3.1) is that S_σ could be hoped to model qualitatively the effects that would arise from inhomogeneous perturbations in Φ and the metric, while avoiding the technical complications of having to couple gravitational perturbations to matter perturbations.

Second, since our background model contains both gravity and matter, the assumption that σ be small compared with the background variables cannot be implemented by an expansion with Planck mass as the large expansion parameter. Rather, the appropriate expansion can be obtained, for example, by making in (3.1) the replacement $S_M \rightarrow MS_M$, regarding M as a "large" parameter, and at the end setting $M=1$. The parameter M is thus not to be identified with any particular constant of nature, but it merely keeps track of the physical assumption that the background action be large compared with the perturbation action.

The Hamiltonian form of the total action is

$$S=\int dt\left[\dot{\alpha}p_\alpha+\dot{\phi}p_\phi+\sum_n \dot{f}_n p_n - N\mathcal{H}\right] \tag{3.2}$$

where the total super-Hamiltonian $\mathcal{H}$ is given by

$$\mathcal{H}=\mathcal{H}_M+\sum_n \mathcal{H}_n\ , \tag{3.3}$$

$$\mathcal{H}_M=\tfrac{1}{2}[e^{-3\alpha}(-p_\alpha^2+p_\phi^2)-e^{\alpha}]\ , \tag{3.4}$$

$$\mathcal{H}_n=\tfrac{1}{2}[e^{-3\alpha}p_n^2+e^{\alpha}(n^2-1)f_n^2]\ . \tag{3.5}$$

We have defined $a=e^\alpha$. To promote $\mathcal{H}$ into a quantum super-Hamiltonian operator $\hat{\mathcal{H}}$, one must make a choice of factor ordering. We adopt here the usual geometrical viewpoint that the ordering should be covariant in the configuration space, so that the first- and second-derivative terms should combine into the scalar Laplacian with respect to the metric appearing in the total action.[44] (The effective potential term which could arise from the zeroth-derivative terms[45,46] will not be important to the order in which we shall expand the wave function.) The configuration space with all the perturbation modes $\{f_n\}$ is infinite dimensional and nonflat, and a problem of regularization therefore emerges already at the formal level of writing down an expression for a covariantly ordered $\hat{\mathcal{H}}$. We shall treat this problem by introducing an explicit cutoff m in the number of the perturbation modes. A possible physical interpretation of this cutoff will be discussed later in this section.

The covariantly ordered super-Hamiltonian operator is given by

$$\hat{\mathcal{H}}=\hat{\mathcal{H}}_M+\hat{\mathcal{H}}_{\rm ord}+\sum_n \hat{\mathcal{H}}_n\ , \tag{3.6}$$

where

$$\hat{\mathcal{H}}_M=\frac{1}{2}\left[e^{-3\alpha}\left[\frac{\partial^2}{\partial\alpha^2}-\frac{\partial^2}{\partial\phi^2}\right]-e^{\alpha}\right]\ , \tag{3.7}$$

$$\hat{\mathcal{H}}_{\rm ord}=\frac{3m}{4}e^{-3\alpha}\frac{\partial}{\partial\alpha}\ , \tag{3.8}$$

$$\hat{\mathcal{H}}_n=\frac{1}{2}\left[-e^{-3\alpha}\frac{\partial^2}{\partial f_n^2}+e^{\alpha}(n^2-1)f_n^2\right]\ . \tag{3.9}$$

The geometrical meaning of the various terms is as follows. The kinetic term in $\hat{\mathcal{H}}_M$ is $-\frac{1}{2}\nabla^2$, where ∇^2 is the Laplacian in the minisuperspace metric $f_{\alpha\beta}$ which appears in the minisuperspace Lagrangian (2.5):

$$f_{\alpha\beta}dq^\alpha dq^\beta=e^{3\alpha}(-d\alpha^2+d\phi^2)\ . \tag{3.10}$$

Here, and in what follows, we denote by $\{q^\alpha\}$ the minisuperspace coordinates $\{\alpha,\phi\}$. The total kinetic term in $\hat{\mathcal{H}}$ is $-\frac{1}{2}\tilde{\nabla}^2$, where $\tilde{\nabla}^2$ is the Laplacian in the metric $\tilde{f}_{\alpha\beta}$ which appears in the total Lagrangian:

$$\tilde{f}_{\alpha\beta}dz^\alpha dz^\beta=e^{3\alpha}\left[-d\alpha^2+d\phi^2+\sum_n df_n^2\right]\ . \tag{3.11}$$

Here, and in what follows, we denote by $\{z^\alpha\}$ the coordinates on the total configuration space, including both the minisuperspace coordinates $\{q^\alpha\}$ and the perturbation coordinates $\{f_n\}$. The difference between $-\frac{1}{2}\tilde{\nabla}^2$ and $-\frac{1}{2}\nabla^2$ when operating on functions of the minisuperspace coordinates is made up for by the cutoff-dependent term $\hat{\mathcal{H}}_{\rm ord}$ [Eq. (3.8)].

In the previous analyses of the reduced density matrix in Refs. 15–18, the ordering adopted for the super-Hamiltonian corresponds to omitting the term $\hat{\mathcal{H}}_{\rm ord}$. Although appealing by virtue of the absence of explicitly cutoff-dependent terms which would operate only on the background variables, such an ordering cannot be motivated solely by the geometry of the configuration space. We shall return to this below.

The total Wheeler-DeWitt equation reads

$$\hat{\mathcal{H}}\Psi(\alpha,\phi,\{f_n\})=0\ . \tag{3.12}$$

We seek for solutions of the form

$$\Psi=C(\alpha,\phi)\exp[iS(\alpha,\phi)]\prod_n \tilde{\chi}_n(\alpha,\phi,f_n)\ , \tag{3.13}$$

respecting the assumption that the perturbations be small compared to the background. Expanding the Wheeler-

DeWitt equation to next-to-leading order in the perturbations (for example, by introducing the "large" parameter M as described above) yields the two equations

$$0=-\left[\frac{\partial S}{\partial\alpha}\right]^2+\left[\frac{\partial S}{\partial\phi}\right]^2-e^{4\alpha}, \tag{3.14}$$

$$\begin{aligned}0=-\frac{i}{2}\left[\prod_n\chi_n\right]\left[C\nabla^2 S+2f^{\alpha\beta}\frac{\partial S}{\partial q^\alpha}\frac{\partial C}{\partial q^\beta}\right]\\+C\left[-if^{\alpha\beta}\frac{\partial S}{\partial q^\alpha}\frac{\partial}{\partial q^\beta}\left[\prod_n\chi_n\right]\right.\\\left.+\left[\sum_n\hat{\mathcal{H}}_n\right]\left[\prod_n\chi_n\right]\right],\end{aligned} \tag{3.15}$$

where ∇^2 is as above and we have defined

$$\chi_n=\exp\left[\frac{3\alpha}{4}\right]\tilde{\chi}_n\ . \tag{3.16}$$

The factor $\exp(3\alpha/4)$ between χ_n and $\tilde{\chi}_n$ arises from the term $\hat{\mathcal{H}}_{\rm ord}$ in $\hat{\mathcal{H}}$. If $\hat{\mathcal{H}}_{\rm ord}$ were dropped, χ_n in (3.15) would be replaced by $\tilde{\chi}_n$.

The leading-order equation (3.14) is the Hamilton-Jacobi equation for the background minisuperspace model. In the next-to-leading-order equation (3.15), there is a freedom in dividing the total phase of the wave function into $C(\alpha,\phi)$ and into the perturbation wave functions χ_n. We fix part of this freedom by setting the two terms in (3.15) to be individually zero and the rest by demanding the second term in (3.15) to factorize in the χ_n in a standard way. This gives the well-known equations

$$0=C\nabla^2 S+2f^{\alpha\beta}\frac{\partial S}{\partial q^\alpha}\frac{\partial C}{\partial q^\beta}, \tag{3.17}$$

$$if^{\alpha\beta}\frac{\partial S}{\partial q^\alpha}\frac{\partial}{\partial q^\beta}\chi_n=\hat{\mathcal{H}}_n\chi_n\ . \tag{3.18}$$

The reason for this choice for the phases is that Eq. (3.17) is the usual semiclassical prefactor equation in minisuperspace, and the wave function $\Psi_M=C\exp(iS)$ is therefore a semiclassical solution to the covariantly ordered background Wheeler-DeWitt equation $\hat{\mathcal{H}}_M\Psi_M=0$. Equation (3.18), on the other hand, is the usual form of the functional Schrödinger equation for the perturbation wave functions χ_n. It should be emphasized that if one wishes to discuss the back reaction of the perturbations on the background using a covariant regularization method to eliminate the cutoff, the total phase would need to be divided between C and the perturbation wave functions in a different way.[3,33]

Given an arbitrary solution of the background Hamilton-Jacobi equation (3.14), solutions to Eq. (3.18) can be generated in a way described in Appendix A, by an uplift of the solutions found in Sec. II to the conventional functional Schrödinger equations (2.16). These solutions to (3.18) can be normalized to unity according to

$$1=\int df_n\chi_n^*(\alpha,\phi;f_n)\chi_n(\alpha,\phi;f_n)\ . \tag{3.19}$$

Explicit examples will be given in the next section.

We now turn to a discussion of density matrices and their reductions. To begin, let $\Psi(z)$ be a solution to the full Wheeler-DeWitt equation (3.12). We define the full density matrix $\rho(z,z')$ as

$$\rho(z,z')=\Psi^*(z)\Psi(z')\ . \tag{3.20}$$

We have for brevity dropped the indices from the coordinates z. Clearly, $\rho(z,z')$ is a biscalar on the full configuration space. Since

$$\rho(z,z)=|\Psi(z)|^2\ , \tag{3.21}$$

one can construct from ρ a coordinate-invariant Hawking-Page "probability" $P_{\tilde{V}}$ for a domain $\tilde{V}$ in the full configuration space by[44]

$$P_{\tilde{V}}=\int_{\tilde{V}}(-\tilde{f})^{1/2}dz^\alpha\rho(z,z)\ . \tag{3.22}$$

Suppose now that $\Psi(z)$ takes the semiclassical form (3.13)–(3.19). From the minisuperspace part of $\Psi(z)$, one can construct a minisuperspace density matrix $\rho_0(q,q')$ as

$$\rho_0(q,q')=\Psi_0^*(q)\Psi_0(q')\ , \tag{3.23}$$

where we have again dropped the indices from the minisuperspace coordinates. $\rho_0(q,q')$ is clearly a biscalar on the minisuperspace. We are interested in the relation between $\rho(z,z')$ and $\rho_0(q,q')$.

At the coincidence limit $q=q'$, a relation between the two density matrices can be easily written in terms of the Hawking-Page probability. Let the domain $\tilde{V}$ in (3.22) be a product of a domain V in the minisuperspace and a full infinite range in all the f_n's. $P_{\tilde{V}}$ now reduces to

$$P_{\tilde{V}}\approx\int_V(-f)^{1/2}dq^\alpha\rho_0(q,q)\ , \tag{3.24}$$

which is the minisuperspace Hawking-Page probability for $\Psi_0(q)$. To arrive at (3.24), we have used the normalization (3.19) and the fact that the nontrivial factor between $(-f)^{1/2}$ and $(-\tilde{f})^{1/2}$ is cancelled by the factor appearing in (3.16).

When $q\neq q'$, however, the relation between ρ and ρ_0 is less clear. The idea to be carried over from quantum mechanics would be to form a reduced density matrix $\rho_{\rm red}(q,q')$ by taking a trace of $\rho(z,z')$ over the perturbations. As it is not clear whether the full quantum theory should have a Hilbert-space structure, it is usually assumed that the trace over the perturbations can be taken using the approximate Hilbert-space structure of the perturbation mode wave functions $\chi_n(q;f_n)$. The reduced density matrix would thus take the form

$$\rho_{\rm red}(q,q')\approx\rho_0(q,q')\mathcal{F}(q,q')\ , \tag{3.25}$$

where the quantity $\mathcal{F}(q,q')$ is constructed from the perturbation wave functions χ_n by tracing over the perturbation degrees of freedom. We shall refer to $\mathcal{F}(q,q')$ as the influence functional.

For consistency, $\mathcal{F}(q,q')$ should be equal to unity at the coincidence limit $q=q'$, but it can in general be nontrivial for $q\neq q'$, where it would be expected to describe the decohering effects of the perturbations on the background. The issue now is how $\mathcal{F}(q,q')$ should be defined.

To our knowledge no geometrically motivated construction of $\mathcal{F}(q,q')$ has been given in models of the kind we are considering. One approach advocated in the literature[15–18] is to consider the functional

$$\mathcal{F}_1(q,q')=\prod_n \int df_n \chi_n^*(q;f_n)\chi_n(q';f_n) \ . \tag{3.26}$$

This functional would be obtained if one tacitly chose to omit the factor-ordering term $\mathcal{H}_{\rm ord}$ from the total super-Hamiltonian and to integrate $\rho(z,z')$ with $f_n=f_n'$ without any measure factors. In the presence of $\mathcal{H}_{\rm ord}$, $\mathcal{F}_1$ can be arrived at by integrating $\rho(z,z')$ over the f_n's with an added measure factor $\exp[3m(\alpha+\alpha')/4]$, which could be argued to arise as a point-split version of the factor $\exp(3m\alpha/2)$ between $(-\tilde{f})^{1/2}$ and $(-f)^{1/2}$. However, neither of these ways of arriving at $\mathcal{F}_1$ relies solely on the geometry of the configuration space. We are therefore prompted to ask whether $\mathcal{F}_1$ is the only conceivable candidate for an influence functional that would correspond to the intuitive idea of tracing over the perturbations.

In defining $\mathcal{F}_1$, one sums over the perturbations by setting $f_n=f_n'$. Let us perform a coordinate transformation on the full configuration space from the coordinates $\{q^\alpha,\{f_n\}\}$ to a new set of coordinates $\{q^\alpha,\{y_n\}\}$, where $y_n=af_n$. Suppose now we wish to sum over the environment by setting $y=y'$. As the full density matrix is a biscalar, the only freedom in this summing is in the measure of the dy_n integrals, and this is fixed by the requirement that the influence functional be equal to unity at $q=q'$. We thus arrive at an influence functional given by

$$\mathcal{F}_2(q,q')=\prod_n \int dy_n (aa')^{-1/2}\chi_n^*\left(q;\frac{y_n}{a}\right)\chi_n\left(q';\frac{y_n}{a'}\right). \tag{3.27}$$

We propose that $\mathcal{F}_2$ can be understood as a result of tracing over the perturbations by setting $y_n=y_n'$. Note that although both $\mathcal{F}_1$ and $\mathcal{F}_2$ are equal to unity at the coincidence limit, they are not equal at generic values of the arguments.

Further candidates for an influence functional could be constructed by choosing yet different ways of summing over the perturbations. We are not aware of consistency criteria which would allow one to choose one of these candidates over the others, and we shall argue in Appendix B that a similar ambiguity may arise even in simple quantum-mechanical models. The choice of an influence functional becomes therefore an issue of recognizing the relevant physical criteria. In the rest of this paper, we shall confine our attention to $\mathcal{F}_1$ and $\mathcal{F}_2$. The motivations behind introducing $\mathcal{F}_2$ will be explained shortly.

When q and q' are close, it is a straightforward exercise in the asymptotic expansions of Legendre functions[47] to find the leading form of $\mathcal{F}_1$ and $\mathcal{F}_2$ at the limit of large cutoff. From (2.21) we have

$$\begin{aligned}|\mathcal{F}_1(q,q')|&=\prod_n\left[1-\left[\frac{\Delta\alpha_n}{2\bar{\alpha}_n}\right]^2\right]^{1/4}\\&\quad\times\left[1+\left[\frac{\Delta\beta_n}{2\bar{\alpha}_n}\right]^2\right]^{-1/4}\\&\approx\exp\left\{-\frac{1}{4}\sum_n\left[\left[\frac{\Delta\alpha_n}{2\bar{\alpha}_n}\right]^2+\left[\frac{\Delta\beta_n}{2\bar{\alpha}_n}\right]^2\right]\right\},\end{aligned} \tag{3.28}$$

where $\bar{\alpha}_n=(\alpha_n+\alpha_n')/2$, $\Delta\alpha_n=(\alpha_n-\alpha_n')$, $\Delta\beta_n=(\beta_n-\beta_n')$, and both α_n and β_n are functions of the minisuperspace coordinates in the way described above. For $|\mathcal{F}_2|$ we obtain a similar expression with α_n and β_n replaced, respectively, by α_n/a^2 and β_n/a^2. Let us introduce a new cutoff $n_{\rm max}$ related to the cutoff m by

$$m=\sum_{n=1}^{n_{\rm max}} n^2=\frac{n_{\rm max}(n_{\rm max}+1)(2n_{\rm max}+1)}{6} , \tag{3.29}$$

so that the new cutoff corresponds to including all the perturbation modes with $n\le n_{\rm max}$. For a generic perturbation vacuum $w\neq 2i/\pi$ we obtain

$$-\ln(|\mathcal{F}_1|)\approx\frac{n_{\rm max}^3}{12}\left[\left[\frac{\Delta a}{\bar{a}}\right]^2+\frac{K^2(\Delta x)^2}{(1-\bar{x}^2)^2}\right]+O(n_{\rm max}^2) , \tag{3.30a}$$

$$-\ln(|\mathcal{F}_2|)\approx\frac{n_{\rm max}^3K^2(\Delta x)^2}{12(1-\bar{x}^2)^2}+O(n_{\rm max}^2) , \tag{3.30b}$$

where $\bar{a}_n=(a+a')/2$, $\Delta a=(a-a')$, $\bar{x}_n=(x+x')/2$, and $\Delta x=(x-x')$. The constant K^2 is related to w by

$$K^2=\frac{1}{128R^2V^2}(R^4+V^2)[(R^2+1)^2-4V^2] , \tag{3.31}$$

where

$$\begin{aligned}V&=\frac{\pi}{2}\mathrm{Im}(w) ,\\R&=\frac{\pi}{2}|w| .\end{aligned} \tag{3.32}$$

The $O(n_{\rm max}^2)$ terms in (3.30) are rapidly oscillating functions of $\bar{x}$. For the adiabatic vacuum, $w=2i/\pi$, $K=0$, and we have

$$-\ln(|\mathcal{F}_1|)\approx\frac{n_{\rm max}(n_{\rm max}+1)(2n_{\rm max}+1)}{24}\left[\frac{\Delta a}{\bar{a}}\right]^2+O(n_{\rm max}) , \tag{3.33a}$$

$$-\ln(|\mathcal{F}_2|)\approx\frac{n_{\rm max}(\Delta x)^2}{(1-\bar{x}^2)^4}+\text{const} . \tag{3.33b}$$

These expansions are valid when the cutoff satisfies

$$n_{\rm max}^2(1-\bar{x}^2)\gg 1 . \tag{3.34}$$

A first observation is that for both $\mathcal{F}_1$ and $\mathcal{F}_2$ the suppression of the off-diagonal elements is at its weakest when the perturbations are in the adiabatic vacuum.

This supports the expectation that the reduced density matrices do in some sense describe the interaction between the background and perturbations, since this interaction is expected to be at its smallest for the adiabatic perturbation vacuum. Another way to say this is that the adiabatic vacuum state is in our models the closest one can get to a state which would contain no "particle creation" on a curved background.[43]

A second observation concerns the difference between $\mathcal{F}_1$ and $\mathcal{F}_2$. We see that $\mathcal{F}_2$ depends on the background coordinates only through the functions $x(\alpha,\phi)$ and $x'(\alpha',\phi')$, which are determined by the background wave function, whereas $\mathcal{F}_1$ contains an additional explicit dependence on a and a'. This result is not restricted to the approximative formulas (3.30), but holds, in fact, exactly. The reason is that $\mathcal{F}_2$ only depends on α_n/a^2 and β_n/a^2 and the corresponding primed quantities, which by virtue of (2.22) and (2.6a) are functions of, respectively, x and x' only. We therefore see that $|\mathcal{F}_2|$ is peaked at $x=x'$ independently of the cutoff, with the cutoff affecting only the sharpness of this peaking. With $|\mathcal{F}_1|$, on the other hand, the cutoff affects the shape of the peak in a more complicated way, since the divergent next-to-leading terms in (3.30) depend on both Δa and Δx. One can verify that the exact $|\mathcal{F}_2|$ is obtained from the exact $|\mathcal{F}_1|$ by setting $\Delta a=0$ in the explicit dependence on a and a', but retaining the implicit dependence on a and a' through x and x'.

The explicit dependence of $|\mathcal{F}_1|$ on a and a' is of purely kinematic origin. It arises from the change in the three-volume, but is unrelated to the choice of the background wave function or perturbation vacuum. If one expects the reduced density matrix to describe the interaction between the background and perturbations via particle-creation effects, one might therefore be inclined to favor $\mathcal{F}_2$ over $\mathcal{F}_1$ as a candidate for a physically relevant influence functional. This is our motivation for introducing $\mathcal{F}_2$. We shall discuss this issue further in Appendix B in the context of conformally coupled scalar field perturbations on a de Sitter minisuperspace model.

The suggestion that decoherence be related to particle creation might also give rise to a natural cutoff. Roughly speaking, a massless minimally coupled scalar field becomes unstable at wavelengths larger than the instantaneous Hubble radius. Taking the cutoff $n_{\max}$ to correspond to a wavelength of the order of the instantaneous Hubble radius would thus correspond to tracing over only those modes of the scalar field where one expects large interaction between the scalar field and gravity. In the case of de Sitter space, the Hubble radius coincides with the horizon, and this cutoff is the same as the one advocated by Halliwell,[17] although for other fields or other spacetimes it is different. For a more general scalar field, on a FRW background it is possible to find a similar criterion for the cutoff by examining the equation of motion for a mode of wave number k in conformal time,

$$\frac{d^2\chi_k}{d\eta^2}+W_k^2(\eta)\chi_k=0 \;, \tag{3.35}$$

where $\phi_k=\chi_k/a$ and W_k is the effective frequency given by

$$W_k^2(\eta)=k^2+m^2a^2+(\xi-\tfrac{1}{6})Ra^2 \;, \tag{3.36}$$

m being the mass and ξ the coupling constant to the Ricci scalar R. Whenever W_k^2 is negative, this will lead to an instability of the scalar field, and we can say that the gravitational and scalar fields interact effectively. The cutoff would thus be at the value of k separating positive and negative values of W_k^2. We note that a cutoff of this kind would not give the result obtained in Ref. 16 where a scalar or fermion field with a Planck mass in its ground state decoheres the wave function of the Universe efficiently.

In our model the instantaneous Hubble radius cutoff translates into

$$n_{\max}\approx\frac{|\bar{x}|}{(1-\bar{x}^2)^{1/2}} \;. \tag{3.37}$$

Although this cutoff is too low for the asymptotic formulas (3.30) and (3.33) to hold, it is still possible to infer some qualitative features of our influence functionals with this cutoff from the above general discussion. In the special case that $x(a,\phi)$ is independent of ϕ, we see that $|\mathcal{F}_1|$ is peaked at $a=a'$, whereas for a generic $x(a,\phi)$, $|\mathcal{F}_1|$ is peaked at the full coincidence limit $a=a'$, $\phi=\phi'$. Similarly, we see that $|\mathcal{F}_2|$ is always peaked at $x=x'$. It is, however, difficult to estimate the sharpness of the peak in $|\mathcal{F}_2|$ and, for generic $x(a,\phi)$, even the overall shape of the peak in $|\mathcal{F}_1|$.

IV. EXAMPLES

We have seen that the decoherence displayed by our reduced density matrices depends on the form of the minisuperspace wave function in a nontrivial way. In this section we shall illustrate this with concrete examples. We shall in particular be interested in how the decoherence in the reduced density matrices is related to the minisuperspace classical trajectories.

Let us first reiterate what happens for a generic minisuperspace wave function of the Lorentzian semiclassical form $\exp(iS)$, where S is a solution to the minisuperspace Hamilton-Jacobi equation (3.14). The corresponding one-parameter set of classical solutions is found by solving the Hamilton-Jacobi equations of motion

$$p_\alpha=\frac{\partial S}{\partial q^\alpha} \;, \qquad \dot{q}^{\,\alpha}=N\frac{\partial \mathcal{H}_0}{\partial p_\alpha} \;. \tag{4.1}$$

This set can at least locally be expressed in terms of the general solution (2.6) by giving one relation between the two integration constants D and ϕ_0. With this relation Eqs. (2.6) and the definition $t=\frac{1}{2}|D|x$ can then be used to solve for x as a function of a and ϕ.

A picture of the situation is most easily drawn in the minisuperspace null coordinates (u,v) defined by[40]

$$u=\frac{a^2}{2}e^{-2\phi}\ ,$$
$$v=\frac{a^2}{2}e^{2\phi}\ , \tag{4.2}$$

in which all the classical trajectories are straight lines. In Fig. 1 we show a one-parameter set of classical solutions corresponding to a "generic" choice for S. Since u and v are linear functions of x on each of these classical solutions, with $x=\pm 1$ at the two ends on the boundary of the configuration space, it follows that the curves of constant $x(a,\phi)$ in Fig. 1 consist of points which divide the classical trajectories into two parts with a constant length ratio. For example, the curve $x=0$ consists of the middle points of the trajectories.

We shall now look at three specific examples for the minisuperspace wave function.

A. Separation of variables in (a,ϕ)

One set of exact solutions to the minisuperspace Wheeler-DeWitt equation can be found by separating this equation in the coordinates (a,ϕ).[37–40] We take the separation constant negative and consider the solutions

$$\Psi_\kappa^{\pm}=e^{i\kappa\phi}I_{\mp i|\kappa|/2}\left[\frac{a^2}{2}\right]\ , \tag{4.3}$$

where $-\infty<\kappa<\infty$ and $I_{\mp i|\kappa|/2}$ is a modified Bessel function of purely imaginary order.[42] For $|\kappa|\rightarrow\infty$ with fixed a, we have the asymptotic expansion[48]

$$\Psi_\kappa^{\pm}\approx\frac{e^{\pm i\pi/4}e^{|\kappa|\pi/4}}{(\pi)^{1/2}(\kappa^2-a^4)^{1/4}}\exp(iS_\kappa^{\pm})[1+O((\kappa^2-a^4)^{-1/2})]\ , \tag{4.4}$$

where

$$S_\kappa^{\pm}=\kappa\phi\pm\frac{|\kappa|}{2}\operatorname{arccosh}\left[\frac{|\kappa|}{a^2}\right]\mp\tfrac{1}{2}(\kappa^2-a^4)^{1/2}\ . \tag{4.5}$$

Here $S_\kappa^{\pm}$ is an exact solution to the Hamilton-Jacobi equation (3.14). Matching (4.4) with $\kappa\neq 0$ to a semiclassical solution of the modified Bessel equation, one sees that $\Psi_\kappa^{\pm}$ takes the Lorentzian semiclassical form for $0<a^2<|\kappa|$. This semiclassical form remains valid as $a\rightarrow 0$, although at this limit we would no longer expect to be able to treat the inhomogeneous modes as perturbations. Even without taking the inhomogeneous modes into account, the semiclassical form breaks down near $a^2\approx|\kappa|$, which is the transition region between oscillatory and exponential behavior of $I_{\mp i|\kappa|/2}(a^2/2)$.

The one-parameter families of classical Lorentzian solutions corresponding to the wave functions $\Psi_\kappa^{\pm}$, $\kappa\neq 0$, can be expressed in terms of the general solution (2.6) by setting $D=\kappa$ and letting ϕ_0 range over all real values. However, from Ψ_κ^{+} one recovers only the "expanding" halves of these solutions, $-\frac{1}{2}|D|<t<0$. Similarly, from Ψ_κ^{-} one recovers only the "collapsing" halves of these solutions, $0<t<\frac{1}{2}|D|$.

The trajectories corresponding to Ψ_κ^{+} are shown in Fig. 2 in the case $\kappa>0$. The trajectories start from a singularity at $a=0$ and "end" near the maximum expansion envelope $a^2=\kappa$, where the Lorentzian semiclassical approximation to $\Psi_\kappa^{\pm}$ breaks down. The trajectories corresponding to Ψ_κ^{-} would "start" near the maximum expansion envelope and run into a singularity at $a=0$, thus giving the missing halves of the trajectories shown in Fig. 2.

The function $x(a,\phi)$ is given by

$$x=\mp\left[1-\frac{a^4}{\kappa^2}\right]^{1/2}\ , \tag{4.6}$$

where the upper and lower signs correspond, respectively, to the upper and lower signs in $\Psi_\kappa^{\pm}$. The surfaces of constant x are therefore surfaces of constant a. Thus the decoherence in both $|\mathcal{F}_1|$ and $|\mathcal{F}_2|$ is completely independent of ϕ.

B. Separation of variables in (u,v)

As a second example, we consider background wave functions obtained by separating the Wheeler-DeWitt equation in the coordinates (u,v). The separation constant is taken to be purely imaginary. These solutions can be written as[40]

$$\Psi_{\phi_0}^{\pm}=\exp(iS_{\phi_0}^{\pm})\ , \tag{4.7}$$

where

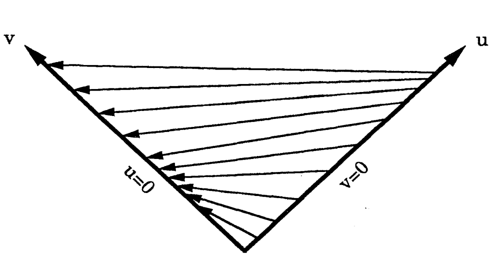

FIG. 1. "Generic" one-parameter family of classical trajectories (2.6) in the coordinates (u,v). The configuration space is the region of positive u and v. The surfaces of constant a are hyperbolas uv=const, and surfaces of constant ϕ are straight lines through the origin.

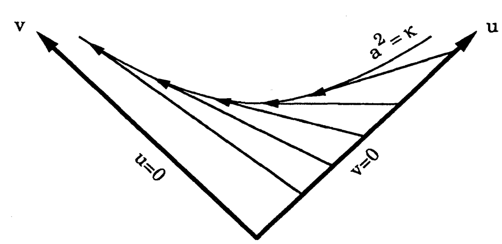

FIG. 2. Classical trajectories corresponding to the wave function Ψ_κ^{+} (4.3) in the case $\kappa>0$. The arrow points to the direction of increasing coordinate time t in (2.6). All these trajectories start from the boundary of the configuration space, where they have a curvature singularity at $a=0$, and they end near the maximum expansion envelope $a^2=\kappa$, where the Lorentzian semiclassical approximation to $\Psi_\kappa^{\pm}$ breaks down.

$$S^{\pm}_{\phi_0} = \pm\frac{a^2}{2}\sinh[2(\phi-\phi_0)]$$
$$= \pm\tfrac{1}{2}(ve^{-2\phi_0} - ue^{2\phi_0}) \; . \quad (4.8)$$

Here $S^{\pm}_{\phi_0}$ are exact solutions to the Hamilton-Jacobi equation (3.14). As the notation suggests, the classical trajectories corresponding to $\Psi^{\pm}_{\phi_0}$ can be expressed in terms of the general solution (2.6) by setting the constant ϕ_0 equal to the value appearing in $S^{\pm}_{\phi_0}$. For $S^{+}_{\phi_0}$ the constant D ranges only over all positive values, so that $\phi(t)$ in these solutions increases in the same direction as the coordinate time t (with the convention that the lapse is kept positive). Similarly, for $S^{-}_{\phi_0}$ the constant D ranges only over all negative values, and the solutions are time inverses of those obtained from $S^{+}_{\phi_0}$. In the coordinates (u,v) these trajectories are given by parallel straight lines, the common direction being related to ϕ_0 (Fig. 3).

The function $x(a,\phi)$ is given by

$$x = \pm\tanh[2(\phi-\phi_0)]$$
$$= \pm\frac{v-ue^{4\phi_0}}{v+ue^{4\phi_0}} \; , \quad (4.9)$$

where the upper and lower signs correspond, respectively, to the upper and lower signs in $\Psi^{\pm}_{\phi_0}$. The surfaces of constant x are surfaces of constant ϕ. Thus the decoherence in $|\mathcal{F}_2|$ is completely independent of a, whereas the decoherence in $|\mathcal{F}_1|$ depends on both a and ϕ in a nontrivial way.

C. Point-source outside the configuraton space

As a last example, we consider a family of background wave functions defined by

$$\Psi^{\pm}_{(u_0,v_0)} = \frac{1}{\sqrt{v-v_0}}\exp[\pm i\sqrt{(v-v_0)(u_0-u)}] \; , \quad (4.10)$$

where $u_0>0$ and $v_0 \le 0$ are parameters. The square root in the exponent is defined to be positive for $(v-v_0)(u_0-u)>0$ and positive imaginary for $(v-v_0)(u_0-u)<0$; this can be rephrased by writing the square root as $\sqrt{(v-v_0)(u_0-u)+i\epsilon}$ with $\epsilon\to 0_+$. Although $\Psi^{\pm}_{(u_0,v_0)}$ has an apparent singular source at $(u,v)=(u_0,v_0)$, this source lies outside the configuration space, and the wave function is a solution to the minisuperspace Wheeler-DeWitt equation.

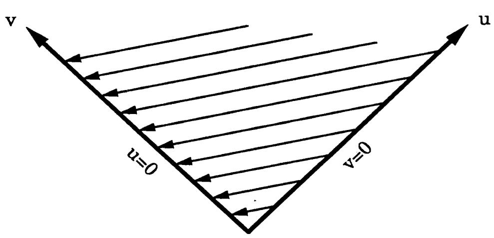

FIG. 3. Classical trajectories corresponding to the wave function $\Psi^{+}_{\phi_0}$ [Eq. (4.7)]. The arrow points to the direction of increasing coordinate time. All these trajectories have the same slope, given by $dv/du = -e^{4\phi_0}$.

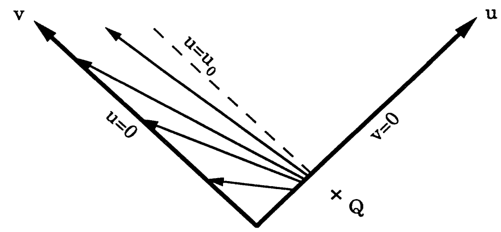

FIG. 4. Classical trajectories corresponding to the wave function $\Psi^{+}_{(u_0,v_0)}$ [Eq. (4.10)] in the case $u_0>0$, $v_0<0$. The arrow points to the direction of increasing coordinate time. All these trajectories would intersect at the fictitious point Q $(u=u_0,\; v=v_0)$, which lies outside the actual configuration space.

For $0<u<u_0$, $\Psi^{\pm}_{(u_0,v_0)}$ takes a Lorentzian semiclassical form. The corresponding Lorentzian solutions can be expressed in terms of the general solution (2.6) as the one-parameter set satisfying

$$D = \mp(u_0e^{2\phi_0} + v_0e^{-2\phi_0}) \; , \quad (4.11)$$

with the restriction that

$$e^{4\phi_0} > -\frac{v_0}{u_0} \; . \quad (4.12)$$

The upper and lower signs match with the upper and lower signs in $\Psi^{\pm}_{(u_0,v_0)}$. In the (u,v) plane these trajectories are straight lines whose extensions beyond the actual configuration space would cross at the singular source at $(u,v)=(u_0,v_0)$ (Fig. 4).

The function $x(a,\phi)$ is given by

$$x = \pm\frac{uv_0+vu_0-2uv}{vu_0-uv_0} \; , \quad (4.13)$$

with signs again matching with those in $\Psi^{\pm}_{(u_0,v_0)}$. For $v_0<0$ the surfaces of constant x are hyperbolas in the (u,v) plane, and in the limiting case $v_0=0$ they degenerate into surfaces of constant u. Thus, in either case, both $|\mathcal{F}_1|$ and $|\mathcal{F}_2|$ depend on both a and ϕ in a nontrivial way.

V. SUMMARY AND DISCUSSION

We have investigated the reduced density matrix in a quantum cosmological model consisting of a closed Friedmann-Robertson-Walker model with inhomogeneous minimally coupled scalar field perturbations. The background was taken to include a spatially homogeneous massless scalar field, distinct from the perturbation scalar field: This gives the minisuperspace model nontrivial dynamics, but allows the perturbation functional Schrödinger equations to be solved in terms of known functions. After a general discussion of how to construct reduced density matrices by summing the full density matrix over the perturbations, we concentrated on two particular choices. The first of these gives the reduced density matrix which has been discussed in the previous literature,[15–18] whereas the second one is constructed so as to be independent of the nondynamical volume factor in the wave function. In both of these reduced density matrices, we found the suppression of the off-diagonal elements to be at its weakest when the perturbations are chosen to be in their adiabatic vacuum state. This supports the expectation that the reduced density matrices in some sense do describe the interaction between the background and perturbations, since this interaction is expected to be at its smallest for the adiabatic perturbation vacuum.

The second one of our reduced density matrices was found to be sensitive both to the choice of the minisuperspace wave function and perturbation vacuum. In the first reduced density matrix, the suppression of the off-diagonal elements contains, roughly speaking, all the suppression that is present in the second one, plus an additional suppression factor. This additional suppression arises solely from the nondynamical volume factor in the wave function, and it is independent of both the choice of the perturbation vacuum and the choice of the minisuperspace wave function. One might therefore favor the second of our reduced density matrices as the one being more directly related to the interaction between the background and perturbations.

We have here only looked at wave functions whose minisuperspace part consists of a single $\exp(iS)$ component. It would be straightforward to generalize the analysis to the case where the wave function is a linear combination of such terms. Assuming the perturbation vacuum is the same for all the components, the individual terms in the total reduced density matrix are obtained from the results in Sec. III by using for $x(a,\phi)$ in each term the function computed from the appropriate S. In the cross terms, in particular, $x(a,\phi)$ and $x'(a',\phi')$ are therefore different functions of their respective arguments. In our second reduced density matrix, the cross terms are again peaked at $x=x'$, but what this implies in the configuration space depends on the detailed form of the respective S's. For our first reduced density matrix, the peaking in the cross terms is, in addition, affected by difference in the scale factors.

An issue that remained largely open was the interpretation of our reduced density matrices in terms of decoherence between spacetimes. Although we saw in Sec. IV how to relate graphically in the configuration space the "minimum decoherence" surfaces $x=\text{const}$ to the one-parameter family of trajectories that correspond to the minisuperspace part of the wave function, it is not clear what the suppression or lack of suppression in the reduced density matrix should be understood as implying about the interferences between the trajectories. The problem is, as explained in the Introduction, that the absence of an explicit time variable prohibits interpreting decoherence in the configuration space directly as decoherence between different histories.

This suggests that a reduced density matrix of the kind we have considered may not as such be the appropriate object for identifying interference between spacetimes. An alternative possibility might be to try and construct a density matrix as a point-split version of the Klein-Gordon probability density and then to reduce this density matrix by summing over the unimportant degrees of freedom.[17] The appeal in this suggestion is that the Klein-Gordon probability density is more directly related to the ordinary low-energy Schrödinger probabilities than the squared absolute value of the total wave function.[3] It appears likely, however, that the reducing procedure of such a density matrix would raise ambiguities similar to those we discussed in Sec. III. A more drastically different possibility would be not to start from a wave function, but to adopt a spacetime, or history, as the fundamental element in the theory and construct a decoherence functional which is defined directly in terms of histories.[21,49,50] It would be interesting to understand whether any of the physical questions that can be posed in the decoherence functional approach would be answerable in terms of reduced density matrices of the kind considered in this paper.

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APPENDIX A: SOLUTION TO THE PERTURBATION SCHRÖDINGER EQUATIONS

In this appendix we show how to write solutions to the Schrödinger equations which arise in the expansion of the Wheeler-DeWitt equation around a minisuperspace model, assuming that solutions to the perturbation Schrödinger equations on a sufficiently large set of classical solutions to the minisuperspace background model are known. The construction is straightforward, but it has to our knowledge not been previously given.

Consider a system (minisuperspace) of $n>1$ degrees of freedom, $\{q^\alpha\}$, described by the action

$$S=\int(\dot{q}^\alpha p_\alpha - N\mathcal{H})dt \ , \tag{A1}$$

$$\mathcal{H}=\tfrac{1}{2}f^{\alpha\beta}(q)p_\alpha p_\beta + U(q) \ . \tag{A2}$$

Here, and in what follows, q in the arguments of the functions will denote all the configuration-space variables $\{q^\alpha\}$. Let x be a new degree of freedom, and let H be an operator of the form $H(q;x,\partial/\partial x)$ acting on functions of $\{q^\alpha\}$ and x. We wish to find solutions to the differential equation

$$if^{\alpha\beta}\frac{\partial S}{\partial q^\alpha}\frac{\partial}{\partial q^\beta}\chi(q;x)=H\chi(q;x) \ , \tag{A3}$$

where $S(q)$ is a given nonconstant function on the configuration space.

Let us consider the integral curves of the vector field $V=f^{\alpha\beta}(\partial S/\partial q^\alpha)(\partial/\partial q^\beta)$. We label these integral curves by $n-1$ constants $\{K_i\}$, and we introduce along each of the curves a parameter t satisfying

$$f^{\alpha\beta}\frac{\partial S}{\partial q^\alpha}\frac{\partial t}{\partial q^\beta}=\frac{1}{N}\ , \tag{A4}$$

where $N(q)$ is a given function. For simplicity we assume that the integral curves of V are not closed (this will be the case in the main text). We now make a smooth global choice for the zeros of the parameters t along the different curves; this can be accomplished, for example, by introducing in the configuration space a surface of codimension 1 which intersects each of the integral curves once and setting $t=0$ on this surface. We can then write the integral curves as

$$q^\alpha(t)=G^\alpha(\{K_i\},t)\ , \tag{A5}$$

where $\{G^\alpha\}$ are n functions of the $n-1$ constants $\{K_i\}$ and the parameter t.

Equations (A5) can be inverted to solve for $\{K_i\}$ and t as functions of $\{q^\alpha\}$. Let the solution be given by

$$\begin{aligned} K_i&=\tilde{K}_i(q)\ , \\ t&=\tilde{t}(q)\ . \end{aligned} \tag{A6}$$

For a point q^α in the configuration space, $\tilde{K}_i(q)$ gives the values of the constants $\{K_i\}$ on the integral curve that passes through q^α, and $\tilde{t}(q)$ gives the value of the parameter t on this curve at q^α.

Assume now that there exists a function $F(\{K_i\},t;x)$ which satisfies the equation

$$i\frac{\partial F}{\partial t}=[N(q)H(q;x,\partial/\partial x)]_G F\ , \tag{A7}$$

where the subscript G indicates that $\{q^\alpha\}$ in H and N are taken to be functions of $\{K_i\}$ and t by (A5). Consider the function $\chi(q;x)$ defined by

$$\chi(q;x)=[F(\{K_i\},t;x)]_{(\tilde{K}_i,\tilde{t})}\ , \tag{A8}$$

where the subscript $(\tilde{K}_i,\tilde{t})$ indicates that $\{K_i\}$ and t in F are taken to be functions of $\{q^\alpha\}$ by (A6). Since

$$f^{\alpha\beta}\frac{\partial S}{\partial q^\alpha}\frac{\partial}{\partial q^\beta}\tilde{K}_i(q)=0 \tag{A9}$$

and

$$f^{\alpha\beta}\frac{\partial S}{\partial q^\alpha}\frac{\partial}{\partial q^\beta}\tilde{t}(q)=\frac{1}{N}\ , \tag{A10}$$

by construction, we have, using (A7),

$$\begin{aligned} if^{\alpha\beta}\frac{\partial S}{\partial q^\alpha}\frac{\partial}{\partial q^\beta}\chi(q;x)&=\frac{i}{N(q)}\left[\frac{\partial}{\partial t}F(\{K_i\},t;x)\right]_{(\tilde{K}_i,\tilde{t})} \\ &=\frac{i}{N(q)}\{[N(q)H(q;x,\partial/\partial x)]_G F(\{K_i\},t;x)\}_{(\tilde{K}_i,\tilde{t})} \\ &=H(q;x,\partial/\partial x)\chi(q;x)\ . \end{aligned} \tag{A11}$$

Therefore, $\chi(q;x)$ satisfies Eq. (A3), which is what we wanted.

The situation of interest for the main text is when $S(q)$ satisfies the Hamilton-Jacobi equation

$$\frac{1}{2}f^{\alpha\beta}\frac{\partial S}{\partial q^\alpha}\frac{\partial S}{\partial q^\beta}+U(q)=0\ , \tag{A12}$$

and H is of the form of a quantum-mechanical Hamilton operator for x:

$$H=-\frac{1}{2}g(q)\frac{\partial^2}{\partial x^2}+W(q,x)\ . \tag{A13}$$

(Here x corresponds to the perturbation modes f_n of the main text.) In this case the integral curves of V are solutions to the classical equations of motion for $\{q^\alpha\}$, with proper time given by $\int N\,dt$, and the differential equation (A3) can be interpreted as a collection of Schrödinger equations for x, one on each of these classical solutions. In (A7) the dependence of these Schrödinger equations on the classical solution has been isolated into the constants $\{K_i\}$, and the individual Schrödinger equations (A7) are consequently easier to solve than the collective equation (A3). The prescription given by (A8) tells how to lift the solutions to the individual Schrödinger equations (A7) into a solution to the collective Schrödinger equation (A3).

A situation of special interest is when the Schrödinger equation

$$i\frac{\partial}{\partial t}\psi(q(t);x)=N(q(t))H(q(t);x,\partial/\partial x) \tag{A14}$$

can be solved for the general solution $q^\alpha(t)$ of the classical equations of motion, involving $2(n-1)$ constants of integration. In this case one has at hand the solutions to (A7) for every possible choice for the Hamilton-Jacobi function $S(q)$. This is the situation occurring in the main text.

APPENDIX B: REDUCED DENSITY MATRIX

In Sec. III we defined the reduced density matrix by summing the full density matrix over the unobserved degrees of freedom. In this summing there remained a freedom which we were not able to fix by geometric arguments alone, except at the limit where the two arguments of the reduced density matrix coincide. In this appendix we shall illustrate this phenomenon in a simple quantum-mechanical example and discuss the relation of this example to a closely analogous situation in quantum cosmology.

Consider a quantum-mechanical system consisting of two uncoupled harmonic oscillators with unit mass and unit frequency, with position coordinates x and y. The metric from which the kinetic term of the action can be derived is just the flat metric

$$ds^2=dx^2+dy^2 . \tag{B1}$$

Let the system be in its quantum-mechanical ground state described by the wave function

$$\Psi(x,y)=\frac{1}{\sqrt{\pi}}e^{-(x^2+y^2)/2} . \tag{B2}$$

(The pure phase coming from the time dependence of the wave function is here inessential and can be omitted.) Suppose now we wish to regard the variable x as a "system" and the variable y as an "environment" and form the reduced density matrix in x. A natural way to do this would be to form the full density matrix from (B2), set $y=y'$, and integrate over y, with the result

$$\rho_{\rm red}(x,x')=\frac{1}{\sqrt{\pi}}e^{-(x^2+x'^2)/2} . \tag{B3}$$

This obviously is a density matrix corresponding to a pure state in x, signaling the absence of interactions between x and y. The probability measure for x becomes just

$$dP_x=\frac{e^{-x^2}}{\sqrt{\pi}}dx . \tag{B4}$$

Suppose now we make a coordinate transformation in the configuration space by

$$\begin{aligned} u&=x ,\\ v&=x^n y , \end{aligned} \tag{B5}$$

where n is some positive integer. The metric (B1) reads now

$$\begin{aligned} ds^2=&(1+n^2v^2u^{-2n-2})du^2-2nvu^{-2n-1}du\,dv\\ &+u^{-2n}dv^2 , \end{aligned} \tag{B6}$$

with determinant $f=u^{-2n}$. We now write the full density matrix in the new coordinates, set $v=v'$, and integrate over v, and recover a new reduced density matrix given by

$$\tilde{\rho}_{\rm red}(u,u')=\left[\frac{2}{\pi(u^{-2n}+u'^{-2n})}\right]^{1/2}e^{-(u^2+u'^2)/2} . \tag{B7}$$

As the new coordinate u is simply the old coordinate x, $\rho_{\rm red}$ and $\tilde{\rho}_{\rm red}$ both depend only on the system, but not on the environment. Further, it is easily seen that the probability measure for u obtained from the diagonal part of $\tilde{\rho}_{\rm red}$ with the determinant factor $\sqrt{f}=u^{-n}$ agrees with the probability measure (B4); however, the ratio of the off-diagonal terms to the diagonal terms is different in $\rho_{\rm red}$ and $\tilde{\rho}_{\rm red}$. Thus having changed the way of summing over the environment has changed the reduced density matrix. Geometrically, this goes back to the observation that summing over $y=y'$ with fixed (x,x') does not correspond to the same configurations as summing over $v=v'$ with fixed (u,u'). Another way to see this is to realize that, although $x=u$, their momenta are not equal.

In this simple example it is obvious that the relevant reduced density matrix is the one given by (B3), indicating no correlations between the two oscillators and no decohering from summing over the environment. We did not attempt above to rigorously justify (B7) as arising from a quantum-mechanical tracing, and we paid no attention, for example, to the singularity of the transformation (B5) at $x=0$. In quantum cosmological situations it is, however, less clear how to choose a reduced density matrix which could be understood as a result of a quantum-mechanical tracing, one reason for this being that the total wave function may not live in any Hilbert space. One may therefore ask whether the reduced density matrices obtained in the quantum cosmological models investigated in Refs. 15–18 should be interpreted in terms of physically observable decoherence.

Consider, for example, the de Sitter minisuperspace model of Ref. 17 with scale factor a, and let the environment consist of a massless conformally coupled scalar field ϕ. If we now do the summing over the environment in the multipole coefficients of the field ϕ itself, we get a result analogous to (B7), having an appearance indicating "decoherence." However, if we instead do the summing in the multipole coefficients of the rescaled field $a\phi$, we get a result analogous to (B3), having an appearance indicating "no decoherence." To interpret either of these reduced density matrices in terms of physically relevant decoherence, it would thus appear necessary to relate them to more directly observable quantities. If, in particular, we expect the reduced density matrix to be related to particle creation in curved space, the conformal invariance of the environment in this model suggests that the physically correct answer should be "no decoherence."

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