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Stability of flat space, semiclassical gravity, and higher derivatives

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Flat space is shown to be perturbatively stable, to first order in $\hbar$, against quantum fluctuations produced in semiclassical (or $1/N$ expansion) approximations to quantum gravity, despite past indications to the contrary. It is pointed out that most of the new "solutions" allowed by the semiclassical corrections do not fall within the perturbative framework, unlike the effective action and field equations which generate them. It is shown that excluding these nonperturbative "pseudosolutions" is the only self-consistent approach. The remaining physical solutions do fall within the perturbative formalism, do not require the introduction of new degrees of freedom, and suffer none of the pathologies of unconstrained higher-derivative systems. As a demonstration, a simple model is solved, for which the correct answer is not obtained unless the nonperturbative pseudosolutions are excluded. The presence of the higher-derivative terms in the semiclassical corrections may be related to nonlocality.

I. INTRODUCTION

Our everyday experience tell us that flat space is stable (or at least very metastable) against small perturbations in matter or curvature. Theoretically this has been shown to be the case for classical general relativity (and matter obeying the dominant energy condition) by the proof of the positive-energy theorem.[1] It has been suggested, however, that quantum corrections to classical general relativity might change this result. Issues of stability in quantum mechanics can be trickier and more subtle than in classical mechanics, but nonetheless there have been several strong indications of the instability of gravity when coupled to quantum fields.

Attempts to examine quantum effects on gravity have been made using semiclassical and $1/N$ expansion approximations. In semiclassical approximations, it appeared that the gravitational curvature could either grow very large on a time scale of order of the Planck time or generate large-scale radiation production with this frequency.[2,3] In $1/N$ approximations, where gravity is quantized as well as the matter fields, it appeared that the expectation value of the energy could be lowered from that of flat space, and that the gravitational propagator contained tachyonic modes, both of which imply instability.[4] These calculations are particularly disturbing because they hint that flat space is unstable against quantum perturbations, in contradiction with our everyday experience. Because the field equations for the semiclassical and $1/N$ systems contain terms that are higher derivative than in the classical Einstein system, the solution space is potentially larger than in the lowest-order case (here, lowest order means lowest order in $\hbar$, even if there is also an expansion in powers of $1/N$). New solutions arising only from the presence of higher derivatives describe the instabilities found above.

Recent work, however, sheds new light on the relationship between the higher-derivative terms and the full, nonperturbative system from which they arise. The semiclassical analyses[2,3] begin by assuming that it is appropriate to perturbatively expand the effective action describing geometry in the presence of matter fields (and so also the field equations) in powers of $\hbar$. In the case of gravity, to lowest (zeroth) order in $\hbar$, the effective action is just the classical Einstein-Hilbert action. The first-order correction contains terms that are second order in time derivatives [see Eq. (1)]. These give rise in their field equations to terms that are fourth order in time derivatives, and therefore to entirely new families of solutions not present in the lowest order, second-order differential equation. Most of these new solutions are not perturbatively expandable in $\hbar$ (where "perturbatively expandable in $\hbar$" is defined as analytic in $\hbar$ as $\hbar \to 0$), and so, if used, violate the initial perturbative ansatz. In fact, neither the expanded action nor the expanded field equations, if evaluated at a new, nonperturbative solution, remain perturbative expansions in $\hbar$. To be internally consistent, the solution space must be restricted to only solutions perturbatively expandable in $\hbar$. It had been hoped, or perhaps tacitly assumed, that, despite this inconsistency, the apparently new solutions would give insight to the behavior of the solutions of the full, nonperturbative effective action. While this cannot be explicitly ruled out, a more likely explanation is that the higher-derivative terms are not related to nonperturbative behavior of solutions of the full action, but instead arise from perturbatively expanding a nonlocal expression. This is a common feature of perturbatively expanded nonlocal actions, as the examples below will show. In these cases the higher-derivative terms that arise do not correspond in any way to nonperturbative behavior of the full action, but they would give rise to false, nonperturbative "pseudosolutions" if the perturbative ansatz were abandoned halfway through the calculation. These pseudosolutions are never perturbatively expandable in $\hbar$, even in the case where the action and field equations are perturbative expansions. A self-consistent method for restricting solutions to remain within the perturbative framework is presented below.

Even if the nondynamical higher derivatives appear for reasons other than nonlocality, the nonperturbative pseudosolutions must still be excluded for self-consistency, if the action itself is a perturbative approximation. Whatever the full quantum theory of gravity may be, it is expected to possess a low-energy effective action, of which the first few terms of the truncated perturbative expansion in $\hbar$ would be semiclassical gravity. By remaining within the perturbative framework, although nonperturbative information is lost or hidden, at least self-consistency is maintained. If one were to abandon the perturbative ansatz once new solutions were found outside the domain of formal perturbative expansions, false conclusions could easily be drawn, and because self-consistency would be lost, the relationship between the effective theory and the full theory would be lost as well.

The case of $1/N$ expansion[4] is perhaps more subtle, since the higher derivatives occur to first order in $\hbar$ but lowest order in $1/N$. Requiring that solutions be perturbatively expandable in $1/N$, but making no restrictions based on analyticity in $\hbar$, does not modify the original predictions of the model. There may be nothing inconsistent in this, but the prediction of instability of flat space still remains. By additionally requiring that all solutions be perturbatively expandable in $\hbar$ (as well as $1/N$), we change the predictions of the model (as will be demonstrated below). Furthermore, if we adopt this additional requirement, the behavior of solutions of both the semiclassical and $1/N$ models will be similar (in this linearized approximation). We will impose the ansatz of perturbative expandability in $\hbar$ on the $1/N$ model, and so we need not further distinguish between the semiclassical and the modified $1/N$ case.

It is shown below that, to first order in $\hbar$, semiclassical corrections do not engender instabilities. This result is twofold. First, we show that the calculation itself can be done within the self-consistent perturbative framework. Second, we show that the particular result obtained is that flat space remains stable (to first order in $\hbar$). This is in contrast with physical systems that do develop true instabilities when perturbative quantum corrections are considered (e.g., tachyonic mass corrections to a classically massless particle). The final conclusion is that semiclassical gravity does not contradict experiment in nearly flat regions of spacetime.

II. QUANTUM CORRECTIONS TO GRAVITY

Some quantum corrections to gravity can be calculated without the full quantum theory. One approach is the semiclassical method, in which purely classical gravity is driven by the expectation value of quantum matter. This approximation should be valid in many interesting cases, where the gravitational part of the wave function of spacetime behaves strongly semiclassically, but quantum effects are important for the matter fields. Important examples are the backreaction of Hawking radiation on the metric of a large evaporating black hole, and the backreaction of particles created in the transition from an inflationary era to a radiation-dominated era. The semiclassical approximation would be expected to break down in situations where the effect of the quantum matter on gravitation is to drive it into a regime of high (Planck scale) curvature, such as the final stages of an evaporating black hole, or at very early times in the Universe. Solutions produced by the semiclassical approach that make predictions in such a regime should not be considered physical results.

One quite general approach to semiclassical approximations of quantum gravity was implemented by Horowitz,[3] using Wald's stress-energy axioms[5] to constrain the form of the semiclassical field equations. Another method, even more general in some respects, is the $1/N$ approximation of Hartle and Horowitz,[4] which quantizes gravity coupled to N matter fields, and then examines the large-N limit. The first term in the $1/N$ expansion gives a semiclassical-like field equation (where the terms lowest and first order in $\hbar$ are both lowest order in $1/N$), but higher-order corrections in $1/N$ are (in principle) calculable as well, a feature lacking when gravitation is kept strictly classical.

All of these approaches to quantum corrections to gravity share common features. The effective field equations are higher order in time derivatives than the classical equation, and these higher-order terms have small coefficients (proportional to $\hbar$). If taken seriously, higher derivatives mean that twice as much initial data must be specified to evolve the system forward in time, or, in the variational formulation, twice as much data must be specified on the boundaries. In the initial data formulation, not only must the metric and its first derivative be specified, but also the second and third derivatives. In the variational formulation, not only must the metric be specified on the boundary (or boundaries), but also its first derivative. It would make semiclassical gravity very different from almost all other physical dynamical theories, which are almost always second order in time. Furthermore, as higher-order corrections are considered when the gravitational field is also quantized, terms proportional to higher powers and higher derivatives of curvature are expected. This would have the bizarre effect of requiring more and more initial data to be specified as terms of (supposedly) less and less importance are considered.

For the moment, let us consider only corrections first order in $\hbar$, which make the field equations fourth order in time. The effective action takes the form

$$S_{\text{eff}}=\frac{1}{16\pi G}\int d^4x\sqrt{g}\,[-2\Lambda+R+\alpha R^2+\beta R_{ab}R^{ab}+\gamma(\text{terms nonlocal in curvature})]+(\text{surface}+\text{terms})\,, \tag{1}$$

where α, β, and γ are all proportional to $\hbar$ (the terms nonlocal in curvature in the action lead to purely local terms in the field equations; they should not be confused with nonlocality in the equations of motion discussed at length below). We use the conventions $c=1$, $\eta_{ab}=(-+++)$, $R^l_{\;man}=\partial_a\Gamma^l_{\;mn}+\cdots$, and $R^l_{\;mln}=R_{mn}$. In the semiclassical case were found tachyonic and exponentially growing fluctuations, both of which strongly indicate an instability of flat space.[3] In the $1/N$ expansion were found fluctuations of negative energy and also tachyonic poles in the gravitational propagator.[4] In all such cases, choosing certain values of some parameters could lessen some of the unstable behavior, but for no combinations could the instabilities be made to vanish (this is true even if the probably unphysical Landau ghost discussed by Hartle and Horowitz is discounted as an instability[4]), with the exception of the case $\alpha>0$, $\beta=0$, $\gamma=0$,[6] which is less interesting due to its nonrenormalizability and the absence of the trace anomaly.

An insufficiently stressed property of the solutions contributing to the above instabilities is that, despite the fact that both the effective action and the field equations governing the quantum corrections are perturbative expansions in $\hbar$, most of the solutions are not perturbatively expandable in $\hbar$ (i.e., not analytic functions of $\hbar$ as $\hbar\to0$). This can be seen from the field equations, which for a metric g are roughly

$$\ddot g=f^{\rm cl}(g,\dot g)+\hbar f^1(g,\dot g,\ddot g,\dddot g,g^{(4)})\;, \tag{2}$$

where overdots represent time derivatives. In order to invert this equation for the highest derivative of the metric, it is necessary to divide by $\hbar$. On dimensional grounds, the natural time scale is the Planck time $t_{\rm Pl}=(G\hbar)^{1/2}$ and solutions generally behave as functions of $t/t_{\rm Pl}$. We are faced with the prospect of nonperturbative solutions to a perturbative expansion. The presence of still higher derivatives in still-higher-order corrections makes the situation look even stranger.

Short of giving up completely, on the grounds that semiclassical gravity might be irredeemably inconsistent, there are two directions to proceed. The first is to accept the new nonperturbative solutions as valid. This has been the more popular path historically. There is some hope that the nonperturbative solutions are actually giving some (unexpected) insight into the nonperturbative behavior of the full quantum gravitational theory. There is little motivation for this, since semiclassical gravity is only expected to approximate a perturbative expansion of the full theory. In any truncated perturbative expansion, nonperturbative behavior has necessarily already been lost.

The second path is to take the perturbative expansion seriously and exclude all solutions not perturbatively expandable in $\hbar$ as fictitious. This is the approach we put forward in this paper. The primary advantage of this approach is self-consistency: the effective action is a formal perturbative expansion, the field equations are formal perturbative expansions, and so should be the solutions. Furthermore, the action and the field equations lose their interpretation as a perturbative expansion if evaluated at nonperturbative extrema. That is, the "higher-order" terms are not higher order when evaluated on a nonperturbative "pseudosolution." Unless the perturbative expansion holds at the extrema, there is no reason the effective action should be expected to approximate the full action in any sense, evaluated near the extrema. The applicability of perturbation theory to the stability of action-based physical systems is discussed in the Appendix. The second benefit to taking the perturbative expansion seriously is that the solution space does not grow as the perturbative order is increased. A result of Jaén, Llosa, and Molina[7] shows that, to any order, the same amount of initial data will suffice for all solutions analytic in the perturbative expansion parameter of any system of the form

$$L=\frac{1}{2}\sum_{a=1}^{N}m_a\dot q_a^{\,2}+\sum_{l=0}^{n}\varepsilon^l V_l\left[q,\frac{dq}{dt},\ldots,\frac{d^lq}{dt^l}\right]+O(\varepsilon^{n+1})\;, \tag{3}$$

where ε is the perturbative expansion parameter and m_a is the mass of particles $a=1,\ldots,N$, and the matrices $\partial^2V_l/\partial q_a^{(l)}\partial q_b^{(l)}$ are regular. Their proof demonstrates that all but N of the momenta of this system cannot be inverted within the formalism of perturbative expansions, corresponding to the presence of constraints, which are shown to be second-class constraints. The constrained system has the same number of degrees of freedom for any n, including $n=0$. This result can be generalized to more complicated systems, as will be done below for linearized gravity, which potentially has additional fields present in the first-order correction not present in the classical action.

To reiterate, the advantage of taking the perturbative expansion seriously is self-consistency: (1) the initial action and field equations are formal perturbative expansions and now the solutions are also formal perturbative expansions; (2) the number of degrees of freedom of the system is fixed and does not depend on the order to which the expansion is taken; (3) the system plus the constraints necessary to exclude the nonperturbative pseudosolutions is strongly equivalent (in the sense of Dirac constrained systems) to a second-order system, and thus has none of the pathologies of unconstrained higher-derivative theories. The consequences of losing self-consistency are the appearance of spurious solutions to the truncated series, not related to any solutions of the full action. These spurious solutions occur even in simple examples (as shown below), and must be excluded if solutions to the truncated expansion are to approximate solutions to the full action.

Even if the more consistent, perturbative direction is taken, one might still reasonably ask why the extra solutions that must be excluded arise at all. What is the purpose of the higher derivatives in the effective action and field equations? There may be several answers to this question, but an answer common to many theories based on effective actions is that the higher derivatives come from nonlocality. This is discussed next.

III. NONLOCALITY, PERTURBATIVE EXPANSIONS WITH HIGHER DERIVATIVES

Nonlocality is a feature often displayed in theories based on effective actions, i.e., a theory made simpler by integrating out some subset of its degrees of freedom. Effective actions describe theories with "action at a distance" since some fields have been deprived of their dynamical status. One example of a theory described by an effective action is semiclassical electrodynamics, where the electromagnetic fields are classical but the quantum nature of the matter fields are retained.[8] Another is the Wheeler-Feynman theory of classical electrodynamics, in which electrons interact nonlocally via half-retarded/half-advanced potentials, without dynamical electromagnetic fields.[9] Since Einstein gravity is nonrenormalizable, it is likely that it is not a fundamental theory but, rather, the low-energy limit of an effective theory based on some larger, fundamental "theory of everything" (perhaps string theory). The effective low-energy theory predicted by superstrings will be discussed below.

Nonlocal theories for which the nonlocality is regulated by a small, dimensionful parameter can produce higher derivatives when perturbatively expanded in that parameter. For instance, a function that is nonlocal in time, such as $x(t+\varepsilon t')$, can be expanded in powers of ε. For example,

$$x(t+\varepsilon t') = \sum_{n=0}^{\infty} \frac{(\varepsilon t')^n}{n!} \frac{d^n x(t)}{dt^n} . \tag{4}$$

In this way an infinite sum of individually local, higher-derivative terms can represent a nonlocal expression. The full nonlocal theory may or may not contain behavior usually associated with purely higher-derivative theories (e.g., additional degrees of freedom, lack of a lowest-energy state; see Eliezer and Woodard[10] for a lucid presentation of higher derivatives and nonlocality). If such an expansion is used for a nonlocal action, any finite truncation of the sum may behave very differently from the full theory. In particular, the number of degrees of freedom of the truncated sum appears to depend on the degree of truncation, whereas the number of degrees of freedom of the full theory is fixed. The only solution to this problem is to agree that for any finite truncation one will only examine consistent perturbative solutions. Such an agreement does not deny the existence of possible nonperturbative behavior of the full theory, but it does acknowledge that such behavior is inaccessible in the perturbative expansion already performed. At the very least, nonlocal theories demonstrate how higher derivatives may appear in an approximate theory and not represent dynamical degrees of freedom.

A simple example of a nonlocal theory can help develop some intuition for the subject. The model is of a nonlocal harmonic oscillator (for a fuller treatment, including quantization, see Simon[11]). The potential of this harmonic oscillator is nonlocal in the sense that it depends not only on the position of the spring at a specific instant, but also on the position in the past and future, with heavier weighting of times near the present. This model simply displays the effects of nonlocality and the appearance of higher derivatives in a perturbative expansion, and it has the important advantage of being exactly soluble. The model's equation of motion is

$$\ddot{x}(t) = -\omega_0^2 \int_0^\infty ds\, e^{-s} \tfrac{1}{2}[x(t+\varepsilon s)+x(t-\varepsilon s)] , \tag{5}$$

where $\varepsilon\omega_0 < 1$. In the limit $\varepsilon \to 0$, we regain the simple harmonic-oscillator equation $\ddot{x} = -\omega_0^2 x$. The two-parameter family of exact solutions is given by

$$x = A \cos(\omega t + \phi) , \tag{6}$$

where A and ϕ depend on the initial conditions and

$$\begin{aligned}\omega^2 &= \omega_0^2 \left(\tfrac{1}{2} + \tfrac{1}{2}\sqrt{1+4\varepsilon^2\omega_0^2}\right)^{-1} \\ &= \omega_0^2(1-\varepsilon^2\omega_0^2 + 2\varepsilon^4\omega_0^4 + \cdots)\end{aligned} \tag{7}$$

is the new effective frequency due to nonlocal effects.

One may also solve the system perturbatively and compare the result with the exact solution. Since both the equation of motion and the general solution are perturbatively expandable in ε, there should be no obstacles. The equation of motion becomes

$$\ddot{x} = -\omega_0^2(x + \varepsilon^2 \ddot{x} + \varepsilon^4 x^{(4)} + \varepsilon^6 x^{(6)} + \cdots) . \tag{8}$$

There appears to be an arbitrarily high number of degrees of freedom due to the infinite sum of higher derivatives. In fact, we know that the exact solution has only two arbitrary parameters, so all other degrees of freedom must be excluded implicitly in demanding that the sum converge. If we truncate at any finite order, though, we lose the implicit constraints, and we must then explicitly exclude nonperturbative solutions. Truncating (8) at ε^0 or ε^2 and solving gives no trouble because the equation of motion remains second order and gives the correct answers,

$$\begin{aligned}&\varepsilon^0\!: \; x = A\cos(\omega_0 t + \phi) , \\ &\varepsilon^2\!: \; x = A\cos(\omega_2 t + \phi) ,\end{aligned} \tag{9}$$

to the appropriate order in ε, where $\omega_2^2 = \omega_0^2(1-\varepsilon^2\omega_0^2 + \cdots) = \omega^2 + O(\varepsilon^4)$ is an easily calculable function of ε and ω_0. Truncating (8) at higher orders, however, gives extra "pseudosolutions" occur that are not perturbatively expandable in ε:

$$\begin{aligned}&\varepsilon^4\!: \; x = A\cos(\omega_4 t+\phi) + B\cos(\gamma t + \psi), \quad \gamma \sim \frac{1}{\varepsilon}\frac{1}{\varepsilon\omega_0} , \\ &\varepsilon^6\!: \; x = A\cos(\omega_6 t+\phi) + B_+\cos(\gamma_+ t + \psi_+) + B_-\cos(\gamma_- t+\psi_-), \quad \gamma_\pm \sim \frac{1}{\varepsilon}\frac{1}{\sqrt{\pm i\varepsilon\omega_0}} ,\end{aligned} \tag{10}$$

and so on, where $\omega_{2n}^2 = \omega^2 + O(\varepsilon^{2n+2})$ is a calculable function of ε and ω_0 in each case.

Thus, this simple model is an explicit example of how abandoning the perturbative formalism for the solution simply gives the wrong answer. Retaining the perturbative formalism (that is, excluding, by the appropriate constraints, all nonperturbative results) gives the correct answer, to any order. We see that when the order of derivatives grows with the order of expansion, it is an obvious symptom of nonlocality. It alerts us that the higher derivatives do not represent dynamical degrees of freedom but are an artifact of the expansion. Keeping only perturbative solutions is the only self-consistent path available.[12]

Solving for all exact solutions of the truncated expansion and then discarding those not perturbatively expandable, while a valid procedure, is computationally wasteful and may not always be possible. A more feasible prescription is to solve the equations of motion while remaining, at every step, strictly within the perturbative formalism.

"Strictly within the perturbative formalism" means that, in solving the field equations, all expressions must be polynomials (formal expansions) in the perturbative constant, up to the specified order of the truncation. Only operations which preserve the formal expansion are permitted. One may consider the perturbative expansion parameter to be not an ordinary number, but an abstract object with no multiplicative inverse (once the perturbative order is set). Division by terms containing the perturbative constant is *forbidden* (though multiplying by a reciprocal, *if it exists*, is allowed), once the perturbative order is set. Note that the strictly perturbative formalism implies that if $f(x)+\varepsilon g(x)=0+O(\varepsilon^2)$, and f and g are both zeroth order in ε, then both f and g must vanish independently. Note also that the vanishing of the product of two terms does not guarantee that either must vanish [e.g., $\varepsilon\times\varepsilon=0+O(\varepsilon^2)$]. Algebraically speaking, the system is a commutative ring with zero divisors, where the role of zero element is played by $O(\varepsilon^{N+1})$.[7]

To make these ideas more concrete we will solve the example system above by this method, truncated to powers of ε^4. The equation of motion is

$$\ddot{x} + \omega_0^2 x + \varepsilon^2\omega_0^2\ddot{x} + \varepsilon^4\omega_0^2 x^{(4)} = O(\varepsilon^6) . \tag{11}$$

Dividing by ε^4 is forbidden if the equation is to remain a perturbative expansion to $O(\varepsilon^4)$. Instead we multiply by ε^4,

$$\varepsilon^4\ddot{x} + \varepsilon^4\omega_0^2 x = O(\varepsilon^6) , \tag{12}$$

take two time derivatives,

$$\varepsilon^4 x^{(4)} + \varepsilon^4\omega_0^2\ddot{x} = O(\varepsilon^6) , \tag{13}$$

and substitute back into (11) to get

$$\ddot{x}(1+\varepsilon^2\omega_0^2 - \varepsilon^4\omega_0^4) + \omega_0^2 x = O(\varepsilon^6) . \tag{14}$$

We are still forbidden to divide by any expression containing ε, but we may still multiply by the reciprocal if it exists. Since

$$(1+\varepsilon^2\omega_0^2 - \varepsilon^4\omega_0^4)(1-\varepsilon^2\omega_0^2 + 2\varepsilon^4\omega_0^4) = 1 + O(\varepsilon^6) , \tag{15}$$

the final form of the equation of motion is

$$\ddot{x} + \omega_0^2(1-\varepsilon^2\omega_0^2 + 2\varepsilon^4\omega_0^4)x = O(\varepsilon^6) . \tag{16}$$

Compare this with (7) to see that this gives the correct answer to the full equation of motion (to order ε^4), and compare with the first line of (10) to see that this also agrees with the method of solving for all solutions and afterwards excising all nonperturbative pseudosolutions. That we are not missing any perturbative solutions is guaranteed by (3).

IV. QUANTUM CORRECTIONS TO GRAVITY REVISITED

We may now consider these ideas in the specific context of quantum corrections to gravity. Whatever properties the full quantum theory of gravity may have, it is expected to posess a low-energy effective action that can be expanded in powers of the Planck time $t_{\rm Pl} = (\hbar G)^{1/2}$, and there is no reason to suspect that the expansion ends at any finite order. For example, superstrings predict an effective low-energy theory with an infinite expansion given by[13]

$$S = \frac{1}{2\alpha'}\int d^d x\sqrt{g}\left[R - \frac{\alpha'}{4}R_{abcd}R^{abcd} + \frac{\alpha'}{4}\nabla^2 R + (\text{matter}) + O(\alpha'^2)\right] \tag{17}$$

at the tree level, where α' is the slope parameter, with dimensions of $l_{\rm Pl}^2$. On dimensional grounds, higher-order corrections will be accompanied by higher powers of curvature and its derivatives, giving higher and higher time derivatives. Einstein gravity itself is nonrenormalizable, and so makes no predictions concerning the form of higher-order terms in the expansion. Nevertheless, to the extent that any approximation giving an action with a first-order correction in $\hbar$ is to agree with predictions of the full theory, it must be treated as giving the first few terms of a larger expansion. Since nonlocality is a common feature of effective actions, it is quite plausible that all higher-derivative terms arise from the perturbative expansion of nonlocality, and, therefore, that the nonperturbative pseudosolutions should be excluded. Still, even if the nondynamical higher derivatives appear for reasons other than nonlocality, the nonperturbative pseudosolutions must still be excluded for self-consistency, if the action itself is a perturbative approximation. Information of nonperturbative solutions has already been lost in making the perturbative approximation of the action and field equations. It is impossible to tell whether the nonperturbative pseudosolutions are at all related to any lost nonperturbative solutions, but excluding them is at least self-consistent.

The effects of excluding the pseudosolutions are several. First, we show that there are no new degrees of freedom or fields. The most general higher derivative, semiclassical corrections[3,4] can be written most concisely in terms of the Fourier transform

$$S_{\rm eff}=\int \frac{d^4k}{(2\pi)^4}\sqrt{-g}\left\{\kappa^{-2}R+a\ln\left[\frac{k^2}{\mu^2}\right]C^*_{abcd}C^{abcd}+\left[b\ln\left[\frac{k^2}{\mu^2}\right]+\alpha\right]R^*R\right\}+O(\hbar^2)\,, \tag{18}$$

where an asterisk denotes complex conjugation, and a, b, and α are all proportional to $\hbar$, and their exact values depend on which matter fields couple to gravity and which regularization scheme is chosen in the process of renormalization. C_{abcd} is the Weyl tensor. Following Stelle,[14] we decompose the linearized metric into transverse traceless, transverse, and longitudinal components:

$$\begin{aligned} h_{ij}&=h_{ij}^{TT}+h_{ij}^{T}+k_i\xi_j+k_j\xi_i\,,\\ h_{i0}&=h_{i0}^{T}+k_i\xi_0+k_0\xi_i\,,\\ \tilde h_{00}&=h_{00}-2k_0\xi_0\,, \end{aligned} \tag{19}$$

where

$$\begin{aligned} &h^T=h_{ii}-\mathbf{k}^{-2}k_ik_jh_{ij},\quad h_{ij}^T=\tfrac{1}{2}(\delta_{ij}h^T-\mathbf{k}^{-2}k_ik_jh^T)\,,\\ &h_{ij}^{TT}=(\delta_{ik}-\mathbf{k}^{-2}k_ik_k)(\delta_{lj}-\mathbf{k}^{-2}k_lk_j)h_{kl}\\ &\qquad-\tfrac{1}{2}(\delta_{ij}-\mathbf{k}^{-2}k_ik_j)(\delta_{kl}-\mathbf{k}^{-2}k_kk_l)h_{kl}\,,\\ &h_{i0}^T=h_{i0}-\mathbf{k}^{-2}(k_lk_ih_{0l}+k_lk_0h_{il}-\mathbf{k}^{-2}k_ik_kk_jk_0h_{jk})\,,\\ &\xi_i=\mathbf{k}^{-2}(k_jh_{ij}-\tfrac{1}{2}\mathbf{k}^{-2}k_kk_jk_ih_{kj})\,,\\ &\xi_0=\mathbf{k}^{-2}(k_ih_{0i}-\tfrac{1}{2}\mathbf{k}^{-2}k_ik_jk_0h_{ij})\,, \end{aligned} \tag{20}$$

and h_{ij}^{TT}, h^T, h_{i0}^T, and $\tilde h_{00}$ are invariant under the transformation $h_{mn}\to h_{mn}+k_{(m}\eta_{n)}$ for arbitrary η_n. Inserting this decomposition into the linearized action gives

$$S^{\rm lin}_{\rm eff}=\int\frac{d^4k}{(2\pi)^4}\{\tfrac{1}{2}h_{ij}^{TT*}(\kappa^{-2}-fk^2)k^2h_{ij}^{TT}-\tfrac{1}{4}h^{T*}[\kappa^{-2}+(\tfrac{1}{3}f+4g)k^2]k^2h^T+\mathrm{Re}\,h^{T*}[\kappa^{-2}+(-\tfrac{1}{3}f+2g)k^2]\tilde h_{00}-\tilde h_{00}^*(\tfrac{1}{3}f+g)\mathbf{k}^4\tilde h_{00}-h_{0i}^{T*}(\kappa^{-2}-fk^2)\mathbf{k}^2h_{0i}^T\}+O(\hbar^2)\,, \tag{21}$$

where $f=a\ln(k^2/\mu^2)$, $g=b\ln(k^2/\mu^2)+\alpha$, and the field equations are given by

$$\delta S^{\rm lin}_{\rm eff}=0+O(\hbar^2)\,. \tag{22}$$

Since this is independent of ξ_m, the ξ_m are the natural gauge variables of the linearized system. In the classical limit, $f=g=0$, and the reader may verify that only the h_{ij}^{TT} are dynamical in this limit. Following the same steps as for the simple model above, multiply (22) by $\hbar$ to get

$$\hbar\delta S^{\rm lin}_{\rm eff}=0+O(\hbar^2) \tag{23}$$

which is equivalent to

$$\begin{aligned} &\hbar\Box h_{ij}^{TT}=0+O(\hbar^2)\,,\\ &\hbar h^T=\hbar\tilde h_{00}=\hbar h_{i0}^T=0+O(\hbar^2)\,. \end{aligned} \tag{24}$$

Recall that division by $\hbar$ is not allowed if we are to remain at the same order. Since all corrections to the field equations are of the form of (24), they also vanish (to this order). The only solutions to the linearized field equations that are perturbatively expandable in $\hbar$ are the same as the solutions to the classical equations, but now to one higher order in $\hbar$: $\Box h_{ij}^{TT}=0+O(\hbar^2)$. There cannot be any other solutions perturbatively expandable in $\hbar$ because of the second-class constraints associated with the momenta and time derivatives of h^T, $\tilde h_{00}$, and h_{i0}^T and remaining within the perturbative formalism. The momenta cannot be inverted within the confines of strict perturbation theory, signaling the presence of primary constraints. These constraints, along with their associated secondary constraints, do not commute; i.e., they are second class. The result is that h^T, $\tilde h_{00}$, and h_{i0}^T are not dynamical fields. The only field degrees of freedom are those of the graviton (h_{ij}^{TT}). This should not be too surprising in the context of the Stelle's analysis. The apparently new degrees of freedom found there corresponded to particles with masses inversely proportional to the Planck length. Any excitations of those false degrees of freedom would result in frequencies also of order the Planck scale, corresponding to solutions that diverge as $\hbar\to 0$.

It is the fictitious degrees of freedom excised above that are responsible for indications of the instability of flat space. Previous analyses of the stability of flat space found "solutions" to the semiclassical equations with behavior $\sim t/t_{\rm Pl}$. For instance, Horowitz and Wald find modes of real or imaginary frequency $(48\pi\alpha)^{-1/2}$ [where $\alpha\propto\hbar$ is defined in Eq. (1)] which lead to instabilities either from runaway solutions or enormous radiation production.[2] Below we will reanalyze in detail the energy analysis of Hartle and Horowitz[4] in the perturbative formalism. Generalizing these techniques to other analyses of the stability of flat space-time is straightforward.

The energy analysis of Hartle and Horowitz[4] computes the minimum energy among all states for which the expectation value of the metric is a given stationary geometry satisfying the constraints of the system. The answer may be expressed in terms of the effective action by[15]

$$E[g]=-\frac{S_{\rm eff}[g]}{T}\,, \tag{25}$$

where T is the time integrated over in evaluating S_{eff} and g is the stationary geometry. The original analysis found that this quantity can be made negative for some deformations of the metric (the energy vanishes for flat space), indicating an instability of flat space. When the analysis is reperformed within the formalism of perturbation theory, as will be seen, no such indications are found.

The linearized effective action can be written

$$S_{\text{eff}}[h]=-\frac{1}{2}\int\frac{d^4k}{(2\pi)^4}(h^{ab})^*(\dot{G}_{ab}-f\dot{A}_{ab}-g\dot{B}_{ab})+O(\hbar^2)\ , \tag{26}$$

where a single overdot denotes the linearized approximation (not a time derivative), and

$$\begin{aligned}\dot{A}_{ab}&=2k^2\dot{G}_{ab}+\tfrac{2}{3}(k^2\eta_{ab}-k_ak_b)\dot{R}\ ,\\ \dot{B}_{ab}&=2(k^2\eta_{ab}-k_ak_b)\dot{R}\ ,\\ \dot{G}_{ab}&=\tfrac{1}{2}[k^2h_{ab}-2k^ck_{(a}h_{b)c}+k_ak_bh^c{}_c\\ &\qquad+\eta_{ab}(k^ck^dh_{cd}-k^2h^c{}_c)]\ ,\\ \dot{R}&=-\dot{G}^a{}_a=k^2h^a{}_a-k^ak^bh_{ab}\ .\end{aligned} \tag{27}$$

We must also, however, include the new second-class constraints documented in the previous section, i.e., that there are no new degrees of freedom. The constraints (23) can be summarized covariantly as

$$\hbar\dot{G}_{ab}=0+O(\hbar^2) \tag{28}$$

and can also be derived by putting the system in canonical form, but retaining the perturbative expansion formalism. There the momenta cannot be inverted within the perturbative formalism, which leads to new primary constraints[7] (in addition to the usual first-class constraints of general relativity), which in this case is (28). Its accompanying secondary constraint is the time derivative of (28), and the two constraints are second class (i.e., they do not commute), reflecting the fact that the number of field degrees of freedom is smaller than is expected in a higher-derivative action (in contrast with the still present first-class constraints of general relativity, which signify gauge freedom). Both $f\dot{A}_{ab}$ and $g\dot{B}_{ab}$ are proportional to $\hbar\dot{G}_{ab}$, and so vanish (to this order), leaving the effective action equal to the classical action. The action, field equations, and usual first-class constraints are all the same as the classical case (but now to higher order), and so the energy functional is also the same:

$$E[h_{ij}]=\frac{1}{4}\int\frac{d^3k}{(2\pi)^3}(\mathbf{k}^2|h_{ij}^{TT}|^2)\geq 0+O(\hbar^2)\ . \tag{29}$$

Thus, remaining in the perturbative framework guarantees that the energy of flat space cannot be lowered perturbatively, to first order in $\hbar$.

The same constraint, (28), applies to all semiclassical expansions about flat space and vacuum matter [it does not apply, for instance, to semiclassical expansions in the presence of a cosmological constant, where, for quantum corrections to de Sitter space, we would have $\hbar G_{ab}=\hbar\Lambda g_{ab}+O(\hbar^2)$]. Any examination of corrections to other gravitational behavior must also take this constraint into account. For instance, the structure of the graviton propagator, which without (28) would have tachyonic poles at Planck-like frequencies,[4] behaves exactly as the classical propagator.

Excluding the nonperturbative pseudosolutions by no means proves that flat space is stable against quantum effects of gravity, but as judged by the consistent method followed here, there is no indication of any instability. Furthermore, the inconsistent solutions which did signal instability are likely to be misleading. It is not ruled out that higher order or nonperturbative behavior (inaccessible, by definition, in this formalism, but also by construction, in approximating the action as a truncated perturbative expansion) could make flat space unstable or metastable. But at least for the moment, the issue of stability of flat space is no reason to question general relativity as an approximation to nature, nor to question the present methods of obtaining first-order quantum corrections to the field equations of gravity.

V. CONCLUSION

Semiclassical and other more systematic approaches (such as $1/N$ expansions) to quantum corrections of gravity give an effective action and field equations which are truncated perturbative expansions in powers of $\hbar$. In the case of gravity, the perturbative corrections have the form and dimension of curvature squared terms (though the effective actions may not be entirely expressible in terms consisting of only the metric and curvature), which leads to a higher-derivative theory, i.e., fourth order in time. Corrections of still higher order, expanded in powers of the Planck length squared (i.e., powers of $\hbar$), are expected to be of even higher order in time derivatives. If taken seriously, new solutions to the higher-order equations make the system both qualitatively and quantitatively different from the classical case, leading to, among other symptoms, the instability of flat space. Two important features of these apparently new solutions also point the way to the cure. First, most of the new "solutions" are not perturbatively expandable in powers of $\hbar$, in contrast to the effective action and field equations. Second, the order of the derivatives increase with increasing perturbative order. These make it plausible that the higher derivatives arise from a perturbative expansion of a nonlocal system and not from any dynamical considerations. Nonlocality is to be expected in the low-energy effective action describing gravity in the low-curvature limit (as in all effective actions). Still, even if the higher-derivative terms arise for reasons other than nonlocality, the pseudosolutions must still be excluded for self-consistency if the effective action examined is a truncated perturbative expansion in powers of $\hbar$. This process does not deny the existence of possible nonperturbative behavior of the full theory, but it does acknowledge that such behavior is inaccessible in the perturbative expansion already performed.

The cure is merely to take the perturbative expansion seriously and to exclude all "pseudosolutions" not perturbatively expandable in $\hbar$. This is necessary for self-consistency: (1) the initial action and field equations are formal perturbative expansions in $\hbar$ and now the solutions are also formal perturbative expansion in $\hbar$; (2) the number of degrees of freedom of the system is fixed and does not depend on the order to which the expansion is taken; (3) the system plus the constraints necessary to exclude the nonperturbative pseudosolutions is strongly equivalent (in the sense of Dirac constrained systems) to a second-order system, and thus has none of the pathologies of unconstrained higher-derivative theories. Otherwise, the penalty is spurious solutions to the field equations, unlikely to be related to solutions of the full nonperturbative field equations. A simple model has been provided for which retaining nonperturbative degrees of freedom (as is usually done for semiclassical gravity) gives the wrong answer, and excluding them gives the correct answer. It also demonstrates that the presence of higher-derivative terms in the action and field equations does not automatically require that they will have dynamical consequences as such.

The effect of excluding nonperturbative pseudosolutions from semiclassical gravity is to restore stability to flat space from quantum corrections, at least perturbatively to first order in $\hbar$. This result is not guaranteed merely by the process of excluding nonperturbative pseudosolutions, since even perturbative quantum corrections can result in instability. Stability is not proven or guaranteed to all orders or against nonperturbative behavior, but, to this order, semiclassical gravity does not contradict experiment in nearly flat regions of spacetime.

There are other contexts, e.g., cosmology, in which semiclassical gravity has been used without excluding nonperturbative pseudosolutions. Any proposal that depends crucially on the nonperturbative behavior is flawed for the same reasons.

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APPENDIX

Here we investigate the effects of the perturbative expansion in examining the stability of a system through its action (which can be effective or exact). Let the full (nonperturbative) action $\Gamma^\alpha[\varphi]$ be a functional of a field φ (not necessarily a scalar) and a function of the parameter α. φ is limited to a particular class of functions, S, i.e., $\varphi \in S$. For example, S could be the class of functions held fixed at the boundaries, or of functions and their first derivatives held fixed at the boundaries. We require that $\Gamma^\alpha[\varphi]$ have two properties. First, it must be perturbatively expandable in α:

$$\Gamma^\alpha \cong \Gamma^\alpha_{\rm pert} = \Gamma_0 + \alpha\Gamma_1 + \cdots \; , \qquad \text{where } \Gamma_n \equiv \frac{1}{n!}\frac{\partial^n \Gamma}{\partial \alpha^n}\bigg|_{\alpha=0} \; , \tag{A1}$$

where "$\cong$" has the specific meaning of "possesses an asymptotic expansion equal to." [Note that, for instance, any term of the nonperturbative form $\exp(-1/\alpha)$ could be added to the left-hand side of (A1) and the right-hand side would remain unchanged.] Second, there must exist an extremum to the action that is also perturbatively expandable in α.

$$\exists \bar\varphi^\alpha \text{ such that (a) } \bar\varphi^\alpha \in S \; , \qquad \text{(b) } \frac{\delta\Gamma}{\delta\varphi}\bigg|_{\bar\varphi^\alpha} = 0 \; , \qquad \text{(c) } \bar\varphi^\alpha \cong \bar\varphi^\alpha_{\rm pert} = \bar\varphi_0 + \alpha\bar\varphi_1 + \cdots \; . \tag{A2}$$

For any action with these properties, the following statements can be proven just from the theory of asymptotic expansions (or strict perturbation theory). First

$$0 = \frac{\delta\Gamma^\alpha}{\delta\varphi}\bigg|_{\bar\varphi^\alpha} \cong \frac{\delta\Gamma^\alpha_{\rm pert}}{\delta\varphi}\bigg|_{\bar\varphi^\alpha_{\rm pert}} \; ; \tag{A3}$$

i.e., the perturbative expansion of the extremal field configuration is also an extremal field configuration of the perturbatively expanded action. This follows trivially from the definition of an asymptotic expansion. The same holds true (with identical proofs) for any number of functional derivatives, and, in particular, for the second functional derivative, which must be positive definite if the system is to be stable:

$$\frac{\delta^2\Gamma^\alpha_{\rm pert}}{\delta^2\varphi}\bigg|_{\bar\varphi^\alpha_{\rm pert}} \cong \frac{\delta^2\Gamma^\alpha}{\delta^2\varphi}\bigg|_{\bar\varphi^\alpha} \; . \tag{A4}$$

A stronger statement than (A3) is that

$$\text{if } \frac{\delta\Gamma^\alpha_{\rm pert}}{\delta\varphi}\bigg|_{\varphi^\alpha_{\rm pert}} = 0, \quad \text{where } \varphi^\alpha_{\rm pert} = \varphi_0 + \alpha\varphi_1 + \cdots \; , \tag{A5}$$

then

$$\varphi^\alpha_{\rm pert} = \bar\varphi^\alpha_{\rm pert} \; ,$$

or, if an extremum of the perturbatively expanded action is itself perturbatively expandable, then it equals the perturbatively expanded extremum of the full action. This is most important because it shows that the perturbatively expandable extrema of the perturbatively expanded action is related in a well-defined way to the exact extrema of the full action. No such statement can be made if an extremum of the perturbatively expanded action is not itself a perturbative expansion, and in fact if

$$\frac{\delta^2\Gamma^\alpha_{\rm pert}}{\delta^2\varphi}\bigg|_{\varphi_{\rm test}} = 0, \quad \text{where } \varphi_{\rm test} \neq \varphi_0 + \alpha\varphi_1 + \cdots \tag{A6}$$

(a nonperturbative extremum is chosen), then, in general,

$\Gamma^{\alpha}[\varphi_{\rm test}] \not\approx \Gamma^{\alpha}_{\rm pert}[\varphi_{\rm test}]$; i.e., the perturbative action, when evaluated at a nonperturbative extremum of itself, it is not even approximately equal to the full action evaluated at the same test function. The proof of (A5) is straightforward. The left-hand side of the first equation of (A5) is an infinite polynomial in α, every term of which must vanish independently (since α is arbitrary). The vanishing of each term determines ϕ_n uniquely (so long as ϕ_0 is unique). Thus the infinite polynomial $\varphi^{\alpha}_{\rm pert}$ is unique, and must equal $\overline{\varphi}^{\,\alpha}_{\rm pert}$ by (A2).

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