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General-relativistic celestial mechanics. I. Method and definition of reference systems

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We present a new formalism for treating the general-relativistic celestial mechanics of systems of N arbitrarily composed and shaped, weakly self-gravitating, rotating, deformable bodies. This formalism is aimed at yielding a complete description, at the first post-Newtonian approximation level, of (i) the global dynamics of such N-body systems ("external problem"), (ii) the local gravitational structure of each body ("internal problem"), and, (iii) the way the external and the internal problems fit together ("theory of reference systems"). This formalism uses in a complementary manner $N+1$ coordinate charts (or "reference systems"): one "global" chart for describing the overall dynamics of the N bodies, and N "local" charts adapted to the separate description of the structure and environment of each body. The main tool which allows us to develop, in an elegant manner, a constructive theory of these $N+1$ reference systems is a systematic use of a particular "exponential" parametrization of the metric tensor which has the effect of linearizing both the field equations, and the transformation laws under a change of reference system. This linearity allows a treatment of the first post-Newtonian relativistic celestial mechanics which is, from a structural point of view, nearly as simple and transparent as its Newtonian analogue. Our scheme differs from previous attempts in several other respects: the structure of the stress-energy tensor is left completely open; the spatial coordinate grid (in each system) is fixed by algebraic conditions while a convenient "gauge" flexibility is left open in the time coordinate [at the order $\delta t = O(c^{-4})$]; the gravitational field locally generated by each body is skeletonized by particular relativistic multipole moments recently introduced by Blanchet and Damour, while the external gravitational field experienced by each body is expanded in terms of a particular new set of relativistic tidal moments. In this first paper we lay the foundations of our formalism, with special emphasis on the definition and properties of the N local reference systems, and on the general structure and transformation properties of the gravitational field. As an illustration of our approach we treat in detail the simple case where each body can, in some approximation, be considered as generating a spherically symmetric gravitational field. This "monopole truncation" leads us to a new (and, in our opinion, improved) derivation of the Lorentz-Droste-Einstein-Infeld-Hoffmann equations of motion. The detailed treatment of the relativistic motion of bodies endowed with arbitrary multipole structure will be the subject of subsequent publications.

I. INTRODUCTION AND OVERVIEW

A. Motivation and brief historical overview

The problem of describing the dynamics of N gravitationally interacting extended bodies is the cardinal problem of any theory of gravity. Within the framework of Newton's theory this problem, called "celestial mechanics," has been thoroughly investigated (see, e.g., Tisserand[1]). Very shortly after the discovery of Einstein's theory of gravity, Einstein,[2] Droste,[3] de Sitter,[4] and Lorentz and Droste[5] devised an approximation method (called "post-Newtonian") which allowed them to compare general relativity with Newton's theory of gravity, and to predict several "relativistic effects" in celestial mechanics, such as the relativistic advance of the perihelion of planets, and the relativistic precession of the Moon's orbit. This post-Newtonian approach to general-relativistic celestial mechanics was subsequently developed (and completed) by many authors, notably by Fock[6] (for a review of the development of the problem of motion in general relativity see, e.g., Damour[7]). However, the great increase in precision of current, and foreseeable, observational techniques in the solar system makes it now necessary to reconsider this traditional (post-Newtonian) way of tackling the gravitational dynamics of N-body systems.

Indeed, modern technology is giving us access to high-precision data on both the global celestial mechanics of the solar system, and the local relativistic gravitational environment of the Earth, and on the way they fit together. We have in mind high-precision techniques such as the following. Concerning the global mechanics of the solar system: radar ranging to the planets (with, e.g., a few meters accuracy for the Viking landers on Mars), laser ranging to the Moon (few centimeters level), and the timing of millisecond pulsars (sub-microsecond level). Concerning the local environment of the Earth: the comparison, at the 100 nanosecond level, of stable atomic clocks (via, for instance, the Global Positioning System) and laser ranging to artificial satellites (such as LAGEOS, at the 1-cm level). Concerning the fitting of the local Earth environment to the global structure of the solar system, and/or of the external universe at large, we have in mind, in particular, the very long baseline interferometry technique, which determines baselines on the surface of the Earth, and the position of the rotation pole, with centimeter accuracy, the length of the day at the few millisecond level, and relative angles between distant objects, as seen on the Earth, with a precision better than a milliarcsecond. For an introduction to these techniques, and a review of their impact on general relativity see Soffel.[8]

In order to match the high precision of this wealth of (present and forseeable) data, one needs a correspondingly accurate relativistic theory of celestial mechanics able to describe both the global gravitational dynamics of a system of N extended bodies, the local gravitational structure of each, arbitrarily composed and shaped, rotating deformable body, and the way each of these N local structures meshes into the global one. The traditional post-Newtonian approach to relativistic celestial mechanics fails, for both conceptual and technical reasons, to bring a satisfactory answer to this problem. This traditional post-Newtonian approach uses only one global coordinate system $x^\mu=(ct,x,y,z)$, to describe an N-body system, and models itself on the Newtonian approach to celestial mechanics consisting of decomposing the problem into two subproblems [Tisserand[1] (Vol. I, pp. 51–52); Fock[6]]: (i) *the external problem,* to determine the motion of the centers of mass of the N bodies; (ii) *the internal problem,* to determine the motion of each body around its center of mass. However, the treatments of both subproblems in the traditional post-Newtonian approach are unsatisfactory.

The *external problem* is attacked by introducing some collective variable, say $z^i(t)$, $i=1,2,3$, generalizing the Newtonian center of mass, i.e., describing the overall motion of each body as seen in the global coordinate system x^μ. Then, one attempts to derive some (translational) "equations of motion" for $z^i(t)$ by integrating over each considered body the local law of balance of energy and momentum, i.e., the covariant conservation of the stress-energy tensor,

$$\nabla_\nu T^{\mu\nu}=0 \; . \tag{1.1}$$

However, the various definitions of the position in the global coordinate system of the center of mass z^i used in post-Newtonian investigations have never been quite satisfactory, especially when considering rotating bodies. Moreover, the final equations of motion for $z^i(t)$ contain various other collective variables ("spin" and higher "multipole moments") describing the gravitational structure of each body as seen in the global system x^μ, which are not related in a simple way to any physical "local" multipole moments, defined, say, in an operational way by the motion of artificial satellites around each body.

Concerning the treatment of the *internal problem* in the usual post-Newtonian approach, it is even more unsatisfactory for the following reasons. In Newtonian celestial mechanics the introduction of nonrotating accelerated "mass-centered frames" associated with each body, i.e., of local coordinates

$$X^i=x^i-z^i(t) \; , \tag{1.2}$$

where $i=1,2,3$ and where z^i denotes the global coordinates of the center of mass, serves both a kinematical and a dynamical purpose. The kinematical usefulness of the local coordinates X^i stems from the fact that they are "comoving" with the considered body, while their dynamical usefulness comes from the fact that they succeed in decoupling, to a large degree, the "internal" from the "external" problem. Indeed, with respect to these local frames X^i the external gravitational field is greatly "effaced"[7] in the sense that the effective external gravitational potential acting locally on the body and its environment,

$$U^{\text{eff}}(X^i)=U^{\text{ext}}(z^i+X^i)-U^{\text{ext}}(z^i)-\frac{d^2z^i}{dt^2}X^i \; , \tag{1.3}$$

is essentially reduced to tidal forces.

For a long time, the relativistic internal problem has been given only little attention, and many authors working in the global post-Newtonian framework have, more or less implicitly, assumed that the usual Newtonian formula (1.2) was sufficient for defining a useful "mass-centered frame" in Einsteinian gravity. In principle, this view is admissible because the coordinate systems are arbitrary in general relativity, and the definition (1.2) is as kinematically useful as in Newtonian gravity. However, the formula (1.2) does not define a dynamically useful mass-centered frame in general relativity, in the sense that it does not efface the external gravitational field down to tidal effects but, instead, introduces in the description of the internal dynamics of the body many external "relativistic" effects proportional to the square of the orbital velocity or the external gravitational potential. As discussed in Ref. 7 the latter effects come from the fact that the external description (x^μ-coordinate representation) of each body contains many "apparent deformations" which are not intrinsic to the body (notably the "Lorentz contraction," linked with the orbital velocity, and the "Einstein contraction," linked with the external gravitational potential).

As emphasized by Damour,[7] those technical defects of the usual global post-Newtonian approach are partly rooted in, and certainly further enhanced by, the conceptual defect of surreptitiously introducing a kind of "neo-Newtonian"[9] interpretation of general relativity, by which the global coordinates $t\equiv x^0/c,(x,y,z)\equiv(x^i)$, $i=1,2,3$ are implicitly identified with the absolute time and absolute space of Newtonian theory. This implicit conceptual reduction of Einstein's theory to the Procustean bed of Newton's framework is liable to cause technical mistakes when one forgets the existence of the "apparent deformations" alluded to above.

In recent years, several authors have tried to remedy some of the defects of the traditional post-Newtonian approach to the N-body problem. For instance, Martin *et al.*[10] and Hellings[11] have tried, in an essentially heuristic manner, to explicitly include the main apparent deformations due to the use of an external coordinate representation. A more ambitious approach consists of defining a local comoving frame by using, not a kinematical criterion [like in Eq. (1.2)], but a dynamical one: i.e., to find a useful relativistic definition of an accelerated frame of reference with respect to which the external gravitational effects are strongly effaced. In the simple case of negligibly self-gravitating test bodies moving in a background gravitational field (e.g., an artificial satellite around the Earth) such "external-gravitational-field-effacing" frames are the well-known "locally inertial frames" which can be explicitly constructed by means of Fermi coordinates based on the center-of-mass world line (see, e.g., Synge,[12] and Misner, Thorne and Wheeler[13]). In the more subtle case of (possibly strongly) self-gravitating *test* bodies (i.e., of mass much smaller than the masses of the other bodies) it has been argued as early as 1921 by Weyl[14] that such frames should exist, and be the locally inertial frames (or Fermi frames) of the "external space-time" generated by the masses of the other bodies only. Weyl[14] used this argument to conclude that test bodies (even self-gravitating ones) follow geodesics of the "external space-time." This heuristic reasoning has been later taken up,[15–17] although it never became clear what could be rigorously proven with its help (because of the lack of mathematical control on the limiting process which defines what one means by "test-body" and "external universe").

Concerning nontest bodies (of mass comparable to the masses of the other bodies), some authors (in particular Bertotti[18]) remarked that, at the first post-Newtonian approximation, the orbital motion, according to the Lorentz-Droste-Einstein-Infeld-Hoffmann equations of motion,[19] of one self-gravitating body member of an N-body system, could be interpreted as the motion of a test body in some effective external gravitational field. This remark, together with the previous results for test bodies, suggested that it should be always possible to define good "external-gravitational-field-effacing frames" around any body A, abstracted from an N-body system, by constructing some "locally inertial coordinate system" in some "effective external gravitational field." At the heuristic level, such a construction has been more or less explicitly assumed by many authors.[13,20–24] More explicit results on such local external-field-effacing frames have been obtained in the study of the motion of strongly self-gravitating bodies (neutron stars or black holes), because this was a problem where the standard only global-frame approach was definitely inadequate to derive results needed for astrophysical applications. In particular, D'Eath[25] and Damour,[26] in their studies of binary systems of gravitationally condensed bodies, have made an explicit use of local external-field-effacing coordinate systems, $X^\alpha=(cT,X^a)$ (one for each body), linked with the global coordinate system x^μ, covering the binary system, by transformation formulas of the type ($a=1,2,3$)

$$x^\mu(T,X^a)=z^\mu(T)+e^\mu_a(T)X^a+O((X^a)^2)+\cdots\,, \tag{1.4}$$

and have derived the constraints on the functions $z^\mu(T)$, $e^\mu_a(T)$, imposed by the requirement of a suitable effacement in the X^α system. Other explicit results about such good local frames were also obtained in the study of weakly self-gravitating bodies, treated at the first post-Newtonian approximation, notably through the introduction of "generalized Fermi coordinates" by Ashby and Bertotti[27,28] (see also Soffel *et al.*[29] and the contributions of Bertotti, of Boucher, of Fukushima, Fujimoto, Kinoshita, and Aoki, and of others in Kovalevsky and Brumberg[30]).

More recently a notable progress in the theory of such local relativistic frames (at the post-Newtonian approximation, relevant to systems of N weakly self-gravitating bodies) has been achieved by Brumberg and Kopejkin in a series of publications[31–34] (see also Voinov[35]). Their approach combines the usual post-Newtonian-type expansions with the multipole expansion formalisms for internally generated,[36–38] and externally generated,[24,39,40] gravitational fields, and with asymptotic matching techniques.[25,26] We believe, however, that the approach by Brumberg and Kopejkin has several drawbacks: *ad hoc* assumptions about the structure of various expansions (as, e.g., in the coordinate transformation between global and local coordinates) are made, which are only partially justified by some later consistency checks; the scheme is confined to a particular model for the matter (isentropic perfect fluid) and rigidly restricts itself to considering only some special (harmonic) coordinate system; moreover, their approach is basically incomplete in that it neither describes the full multipole moment structure of the bodies with post-Newtonian accuracy, nor gets (translational or rotational) equations of motion with full post-Newtonian accuracy.

B. Outline of the method used in this paper

In this paper, we introduce a new formalism for treating the general-relativistic celestial mechanics of systems of N arbitrarily composed and shaped, weakly self-gravitating, rotating, deformable bodies. This formalism yields a complete description, at the first post-Newtonian level, of the global dynamics of such N-body systems ("external problem"), the local gravitational structure of each body ("internal problem"), and the way they fit together ("relativistic theory of reference systems"). This new scheme successfully overcomes, in our opinion, the problems encountered by previous approaches (notably the one of Brumberg and Kopejkin[31–35]): only very general assumptions are made for the structure of the formalism which is developed in a constructive way by proving a number of theorems; the structure of the stress-energy tensor of the matter is left completely open; the scheme is formulated in a certain "gauge-invariant" way which leaves a convenient flexibility in the choice of the time gauge [at the order $\delta t=O(c^{-4})$]; the scheme describes with full post-Newtonian accuracy the gravitational structure of each body by means of a set of multipole moments which are linked in an operational way to what can be observed in the local gravitational environment of each body; finally, the scheme succeeds in getting translational and rotational equations of motion with full post-Newtonian accuracy, and inclusion of all multipole moments, for the N-body system. Our approach does not use any asymptotic matching technique but takes advantage of two different recent progresses in the first post-Newtonian approximation method: (i) linearization of Einstein's field equations by means of certain "exponential parametrization" of the metric tensor (introduced by Blanchet and Damour,[41] and Blanchet, Damour, and Schäfer[42]), and (ii) the definition, by Blanchet and Damour[41] (BD), of a set of post-Newtonian multipole moments of an isolated body given as compact support integrals of the stress-energy tensor of the matter. A third basic element of the present approach is our way of restricting (without fixing completely) the coordinate freedom inherent to the theory of general relativity. We do that not by imposing one of the two *differential* coordinate conditions generally used in the post-Newtonian literature (namely "harmonic gauge" versus "standard post-Newtonian gauge") but by imposing, in all coordinate systems, some *algebraic* conditions on the metric coefficients, which can be written as $(i,j=1,2,3)$

$$g_{00}g_{ij}=-\delta_{ij}+O(1/c^4)\ . \tag{1.5}$$

This condition can be described by saying that the spatial coordinates are "conformally Cartesian" or "isotropic." This condition encompasses both usual choices and is, at once, more flexible (for the time gauge) and more rigid (for the space gauge) than either one of them. It plays an important technical role in freezing down the spatial coordinate freedom to a level which is nearly the usual freedom in Newtonian celestial mechanics (arbitrary choice of a time-dependent spatial origin and of a time-dependent rotation matrix).

In this first paper we shall lay the foundations of our formalism, with special emphasis on the definition and properties of the N local reference systems adapted to the description of each individual body, on the way these local coordinate systems X_A^α $(\alpha=0,1,2,3;A=1,\dots,N)$ mesh into the global ("common-view") coordinate system x^μ $(\mu=0,1,2,3)$, and on the general structure and transformation properties of the gravitational field, as seen in the various reference frames. We shall also illustrate our approach by treating in detail the simple case where each body can, in some approximation, be considered as generating a locally spherically symmetric gravitational field (monopole truncation). This will give us a new (and, in our opinion, improved) derivation of the well-known Lorentz-Droste[5]-Einstein-Infeld-Hoffmann[19] equations of motion.

In subsequent publications, we will tackle, by means of our formalism, the external problem of the post-Newtonian motion of N extended arbitrarily shaped bodies, and give various applications of this formalism (completing existing results).

Our presentation will go through the following main steps. In Sec. II, we study the post-Newtonian theory of reference systems. We first consider the constraints on a general coordinate transformation $x^\mu=f^\mu(X^\alpha)$, decomposed as

$$x^\mu(X^\alpha)=z^\mu(X^0)+e^\mu_a(X^0)X^a+\xi^\mu(X^0,X^a)\ , \tag{1.6a}$$

with

$$\xi^\mu=O((X^a)^2)\ , \tag{1.6b}$$

coming from the requirement that the metric admits a post-Newtonian expansion of the usual type,

$$h_{00}=O(c^{-2}),\quad h_{0i}=O(c^{-3}),\quad h_{ij}=O(c^{-2})\ , \tag{1.7a}$$

in both coordinate systems. We then show that if we further demand that the spatial coordinates be "conformally Cartesian" (or "isotropic"), i.e., that

$$h_{ij}=h_{00}\delta_{ij}\ , \tag{1.7b}$$

holds in both coordinate systems, this restricts very much the elements z^μ, e^μ_a, and ξ^μ of the coordinate transformation, Eq. (1.6), and leaves essentially only the usual Newtonian freedom of choosing an arbitrarily moving origin $[z^i(t)]$ and a slowly changing SO(3) rotation matrix (in e^i_a).

In Sec. III, we show (after Refs. 41 and 42) how the use of an exponential parametrization of the metric coefficients,

$$\begin{aligned} &g_{00}=-\exp(-2w/c^2),\quad g_{0i}=-4w_i/c^3\ ,\\ &g_{ij}=\delta_{ij}\exp(+2w/c^2)\ , \end{aligned} \tag{1.8}$$

linearizes Einstein's field equations. We work within a class of "spatially conformally Cartesian" coordinates (which encompasses the two usual choices: harmonic coordinates and "standard post-Newtonian gauge") defined by the condition (1.7b) and show that this leaves open a certain gauge freedom for the "four-potential" $a_\mu=(cw,-4w_i)$, formally identical to the one of Maxwell's theory:

$$w'=w-\frac{1}{c^2}\partial_t\lambda\ , \tag{1.9a}$$

$$w_i'=w_i+\frac{1}{4}\partial_i\lambda\ . \tag{1.9b}$$

By analogy to Maxwell's theory we then introduce some gauge-invariant gravitoelectric and gravitomagnetic fields. In Sec. IV, we study the transformation properties of the gravitational potentials (w,w_i) under a coordinate transformation (1.6). If a certain body A is considered we uniquely split the global potentials w_μ into a self-part and an external part, where the self-part describes the gravitational action of body A itself while the external part originates from all other bodies different from A. A similar split is introduced for the potentials W^A_α seen in the local frame of body A. The transformation laws of the various self- and external potentials are given in explicit form. In Sec. V we present the general formulation of our method. In Sec. VI the self-potentials of some body in the corresponding local frame are expanded in terms of post-Newtonian mass (M_L) and spin (S_L) multipole moments (BD moments). The external (tidal) potentials are expanded in terms of electriclike (G_L) and magneticlike (H_L) post-Newtonian tidal moments. Finally, in Sec. VII the general structure of the post-Newtonian global equations of motion for a system of N gravitationally interacting bodies with arbitrary mass- and spin-multipole moments is given. As a simple application a new improved derivation of the well known Lorentz-Droste-Einstein-Infeld-Hoffmann equations of motion for "spherical bodies without spin" is presented.

II. POST-NEWTONIAN THEORY OF REFERENCE SYSTEMS

A. Notation, conventions, and terminology

We use the signature $-+++$; spacetime indices go from 0 to 3 and are denoted by greek indices, while space indices (1 to 3) are denoted by latin indices. We use Einstein's summation convention for both types of indices, whatever be the position of repeated indices. The flat metric is denoted by f, with components diag$(-1,+1,+1,+1)$ in Lorentzian coordinates. The *absolute* value of the determinant of a covariant metric $g_{\mu\nu}$ is denoted by $g\equiv -\det g_{\mu\nu}$. Our curvature conventions are those of Misner, Thorne, and Wheeler.[13] Parentheses denote symmetrization, e.g.,

$$T_{(ij)}\equiv \tfrac{1}{2}(T_{ij}+T_{ji}) .$$

As usual G denotes Newton's gravitational constant and c the velocity of light (we do not use units where $G=c=1$). In post-Newtonian expansions we shall sometimes abbreviate the order symbol $O(c^{-n})$ simply by $O(n)$. As we shall consider $N+1$ different coordinate systems we shall consistently use the following conventions: The N bodies will be labeled by upper case latin indices $A,B,C=1,\ldots,N$. This body-labeling index will indifferently appear as subscript or superscript according to the positions of other possible tensor indices. The "global" (or "common view") coordinates used for describing the overall dynamics of the N-body system will be denoted by $(x^\mu)\equiv(ct,x^i)$ with spacetime indices taken from the second part of the greek alphabet $(\mu,\nu,\lambda,\ldots)$ and space indices from the second part of the latin alphabet $(i,j,k,\ldots)$. The corresponding global metric is $g_{\mu\nu}(x^\lambda)$, and we shall try to use systematically lower case latin letters to denote quantities belonging to this global frame. By contrast, each of the N "local" coordinate systems, used for describing the internal dynamics of each body, will be denoted by $(X_A^\alpha)\equiv(cT_A,X_A^a)$ with spacetime indices taken from the first part of the greek alphabet $(\alpha,\beta,\gamma,\ldots)$ and space indices from the first part of the latin alphabet $(a,b,c,\ldots)$. The corresponding local metric is $G_{\alpha\beta}(X^\gamma)$ (where we shall often omit the body label A if it is clear from the context), and we shall try to systematically use upper case latin letters for quantities belonging to a local frame.

When dealing with sequences of spatial indices we shall use the condensed notation introduced by Blanchet and Damour:[37] a spatial multi-index containing l indices is simply denoted L (and K for k indices etc.), i.e., $L\equiv i_1i_2\cdots i_l$ if it belongs to the global system, and $L\equiv a_1a_2\cdots a_l$, if it belongs to a local one (we shall use this notation only for space indices, not for spacetime ones). When several multi-indices appear simultaneously, different carrier letters (or primes) must be used. When needed we also use $L-1\equiv i_1i_2\cdots i_{l-1}$. A multisummation is always understood for repeated multi-indices

$$S_LT_L=\sum_{i_1\cdots i_l} S_{i_1\cdots i_l}T_{i_1\cdots i_l} .$$

Given a spatial vector v^i, its lth tensorial power is denoted by $v^L\equiv v^{i_1}v^{i_2}\cdots v^{i_l}$. Also, $\partial_L\equiv\partial_{i_1}\cdots\partial_{i_l}$. The symmetric and trace-free (STF) part of a spatial tensor will be denoted by angular brackets (or by a caret when no ambiguity arises)

$$\mathrm{STF}_{i_1\cdots i_l}(T_{i_1\cdots i_l})\equiv T_{\langle i_1\cdots i_l\rangle}\equiv \hat{T}_L \equiv \sum_{k=0}^{[l/2]} a_k^l\delta_{(i_1i_2}\cdots\delta_{i_{2k-1}i_{2k}} \times S_{i_{2k+1}\cdots i_l)j_1j_1\cdots j_kj_k} ,$$

where

$$S_L=T_{(L)}$$

is the symmetric part of $T_{i_1\cdots i_l}$,

$$a_k^l=(-)^k\frac{l!}{(l-2k)!}\frac{(2l-2k-1)!!}{(2l-1)!!(2k)!!} ,$$

and where $[l/2]$ denotes the integer part of $l/2$, δ_{ij} the Kronecker symbol, and the double factorial sign means $p!!\equiv p(p-2)\cdots(2 \text{ or } 1)$ (we shall denote the Levi-Civita alternating symbol by ϵ_{ijk}, with $\epsilon_{123}=+1$). For instance,

$$\hat{T}_{ij}\equiv T_{(ij)}-\tfrac{1}{3}\delta_{ij}T_{ss} ,$$

$$\hat{T}_{ijk}\equiv T_{(ijk)}-\tfrac{1}{5}[\delta_{ij}T_{(kss)}+\delta_{jk}T_{(iss)}+\delta_{ki}T_{(jss)}] .$$

Note that we shall freely lower or raise spatial indices by means of the Kronecker symbol ("flat Euclidean metric"). For more details about the algebra of STF Cartesian tensors see Thorne[36] and Appendix A of Blanchet and Damour.[37]

Finally, we wish to clarify some points of terminology which have caused a lot of confusion in the relativistic literature on "reference frames." All over this paper, we will use interchangeably the expressions "coordinate systems," "reference systems," or "reference frames" and we shall always mean by these expressions what mathematicians call a "(coordinate) chart," i.e., a map from an open subset of a four-dimensional differentiable manifold onto an open subset of $\mathbb{R}^4$, or, in simpler terms, a continuous labeling of spacetime points by quadruplets of real numbers. Even when we speak of "local frames" we mean some (local) coordinate chart, and never only of a vectorial basis of the tangent space at some point. Also, we shall never introduce "moving frames" ("repères mobiles") in the sense of Cartan, i.e., four-dimensional fields of orthonormalized vectorial bases (also called "tetrad fields," or "vierbeins"). We think that what is needed for applications of general relativity in N-body systems is always, and only, the clear definition of various spacetime charts (one global chart covering the whole system, plus N more local charts covering in better details what happens in some spacetime "tube" containing one of the bodies) together with the explicit expression of the metric coefficients in each chart.

B. z-e-ξ decomposition of a general coordinate transformation

Let us be given a four-dimensional smooth differentiable (abstract) manifold V_4, endowed with N (abstract) world lines $\mathcal{L}_A$ ($A=1,\ldots,N$). Let $\mathcal{T}_A \subset V_4$ (a topological "tube") be some open neighborhood of the line $\mathcal{L}_A$. Let us say that a coordinate chart X_A^α (from the atlas of V_4) is *adapted* to the world line $\mathcal{L}_A$ if it is a map from (the abstract tube) $\mathcal{T}_A$ into $\mathbb{R}^4$ which sends the world line $\mathcal{L}_A$ onto the "time axis" of $\mathbb{R}^4$, i.e., the set of quadruplets (S,0,0,0) with $S \in \mathbb{R}$. When such an adapted chart is given it endows the line $\mathcal{L}_A$ with the following (abstract) structures.

(i) A parametrization of $\mathcal{L}_A$ by means of the real parameter S such that

$$X_A^\alpha(P(S))=(S,0,0,0)\ , \tag{2.1}$$

for some point $P \in \mathcal{L}_A$.

(ii) A one-parameter family of vectorial bases $\mathbf{e}_\alpha^A(S)$ along $\mathcal{L}_A$, defined as

$$\mathbf{e}_\alpha^A(S)=\left.\frac{\partial}{\partial X_A^\alpha}\right|_{P(S)\in\mathcal{L}_A} \tag{2.2}$$

(by "vectorial basis" we mean here only a basis of the tangent space to V_4 at P, i.e., a set of four linearly independent vectors). This vectorial basis is adapted to $\mathcal{L}_A$ in the sense that its first vector

$$\mathbf{e}_0^A(S)=\left.\frac{\partial}{\partial S}\right|_{P(S)} \tag{2.3}$$

is the tangent vector to the parametrized line $\mathcal{L}_A$. Note that all the structures that we are discussing here are defined on a differentiable manifold V_4 independently of any (pseudo-) Riemannian metric. This is why we do not call the vectorial basis $\mathbf{e}_\alpha$ a "tetrad," as this terminology usually conveys the meaning of a vectorial basis orthonormalized with respect to some given metric. It is only at a much later stage in the development of our formalism that it will turn out to be convenient (though not strictly necessary) to orthonormalize the vectorial basis $\mathbf{e}_\alpha$ with respect to a particular metric (which will *not* be the real spacetime metric). Note, also, that we prefer not to use the words "vectorial frame" as the word "frame" has been overused in general relativity, and we wish to reserve the word "frame" to allude to a coordinate chart.

Let us now assume that, in addition to the local coordinate chart X_A^α, one is also given a global coordinate chart x^μ with a domain all over the manifold V_4. Actually, we need only the domain of x^μ to consist of a "big spacetime tube" which contains all the N local tubes $\mathcal{T}_A$, $A=1,\ldots,N$, but, to simplify the language, we shall consider x^μ as covering all V_4. Let us now consider each of the N coordinate transformations from local to global coordinates:

$$x^\mu=f_A^\mu(X_A^0,X_A^1,X_A^2,X_A^3)\ . \tag{2.4}$$

Let us define

$$z_A^\mu(S)\equiv f_A^\mu(S,0,0,0)\ , \tag{2.5a}$$

$$e_{Aa}^\mu(S)\equiv\frac{\partial f_A^\mu}{\partial X_A^a}(S,0,0,0)\ , \tag{2.5b}$$

$$\xi_A^\mu(S,X_A^a)\equiv f_A^\mu(S,X_A^a)-f_A^\mu(S,\mathbf{0})-X_A^a\frac{\partial f_A^\mu}{\partial X_A^a}(S,\mathbf{0})\ . \tag{2.5c}$$

It is easily seen that $z_A^\mu(S)$ is nothing but the x^μ-coordinate representation of the abstract S-parametrized world line $\mathcal{L}_A$ [as defined by Eq. (2.1)], while $e_{Aa}^\mu(S)$ are just the x^μ components of the three abstract vectors $\mathbf{e}_a^A(S)$ defined by Eq. (2.2). Indeed,

$$\mathbf{e}_a^A(S)=\left.\frac{\partial}{\partial X_A^a}\right|_{\mathcal{L}_A}=e_{Aa}^\mu(S)\left.\frac{\partial}{\partial x^\mu}\right|_{\mathcal{L}_A}\ , \tag{2.6a}$$

while

$$\mathbf{e}_0^A(S)=\left.\frac{\partial}{\partial S}\right|_{\mathcal{L}_A}=e_{A0}^\mu(S)\left.\frac{\partial}{\partial x^\mu}\right|_{\mathcal{L}_A} \tag{2.6b}$$

tells us that

$$e_{A0}^\mu(S)=\frac{dz_A^\mu(S)}{dS}\ . \tag{2.7}$$

With this notation, the coordinate transformation (2.4) reads

$$x^\mu(X^\alpha)=z^\mu(X^0)+e_a^\mu(X^0)X^a+\xi^\mu(X^0,X^a)\ , \tag{2.8a}$$

where we have omitted the labeling index A, and where we know [from the definition (2.5c), remembering that coordinate transformations in the atlas of a smooth differentiable manifold are by definition smooth in their variables] that

$$\xi^\mu(X^0,X^a)=O((X^a)^2)\ \text{ as } X^a\to 0 \text{ with fixed } X^0\ . \tag{2.8b}$$

By a slight abuse of notation we shall often in the following write $z^\mu(X^0)=z^\mu(T)$, etc. when using $T\equiv X^0/c$ as time coordinate. The Jacobian matrix of the coordinate transformation (2.4), i.e.,

$$A_\alpha^\mu(X^\beta)\equiv\frac{\partial x^\mu(X^\beta)}{\partial X^\alpha} \tag{2.9}$$

reads ($X^0\equiv cT$)

$$A_0^\mu(X^\beta)=e_0^\mu(T)+\frac{de_a^\mu(T)}{c\,dT}X^a+\frac{1}{c}\frac{\partial\xi^\mu}{\partial T}\ , \tag{2.10a}$$

$$A_a^\mu(X^\beta)=e_a^\mu(T)+\frac{\partial\xi^\mu}{\partial X^a}\ . \tag{2.10b}$$

Note that, because of our normalization for the parameter S ($=cT$ taken along $\mathcal{L}_A$), and because of Eq. (2.7) the first term on the right-hand side of Eq. (2.10a) reads explicitly

$$e_0^\mu(T)=\frac{dz^\mu(T)}{cdT} . \tag{2.10c}$$

C. Constraints on z, e, and ξ from post-Newtonian assumptions

Up to now we have not introduced any metric structure in our four-dimensional manifold V_4. Let us now assume that we are given a "post-Newtonian (PN-) type metric," i.e., a one-parameter sequence of metrices (with parameter $\epsilon=1/c$, the inverse velocity of light), which, when $\epsilon=1/c\rightarrow 0$, deviates, in some coordinate system, from a flat (Minkowskian) metric only by terms of order $1/c^2$ in the time-time and space-space components and of order $1/c^3$ in the mixed time-space components. If we define a metric deviation h in each one of our $N+1$ coordinate systems by

$$g_{\mu\nu}(x^\lambda)=f_{\mu\nu}+h_{\mu\nu}(t,x^i) , \tag{2.11a}$$

$$G^A_{\alpha\beta}(X^\gamma)=f_{\alpha\beta}+H^A_{\alpha\beta}(T,X^a) \tag{2.11b}$$

[where f denotes the same numerical matrix, diag$(-1,+1,+1,+1)$, in each chart], we are requesting the following assumptions to hold.

PN assumptions for the metric:

$$\begin{aligned} &h_{00}(t,\mathbf{x})=O(c^{-2}),\quad h_{0i}(t,\mathbf{x})=O(c^{-3}) ,\\ &h_{ij}(t,\mathbf{x})=O(c^{-2}) ;\\ &H^A_{00}(T,\mathbf{X})=O(c^{-2}),\quad H^A_{0a}(T,\mathbf{X})=O(c^{-3}) ,\\ &H^A_{ab}(T,\mathbf{X})=O(c^{-2}) . \end{aligned} \tag{2.12}$$

We shall also assume that the coordinate transformation between x^μ and each of the local X^α_A's involves only "slow motions" in the following (weak) sense.

Slow motion assumptions for the N local reference systems. Each Jacobian matrix (2.9) satisfies

$$\begin{aligned} &A^0_0=O(c^{-0}),\quad A^i_0=O(c^{-1}) ,\\ &A^0_a=O(c^{-1}),\quad A^i_a=O(c^{-0}) . \end{aligned} \tag{2.13}$$

In order to simplify the writing we shall sometimes use the notation

$$O(n)\equiv O(c^{-n}) \tag{2.14a}$$

[which is however, dangerous when $n=0$ where it gives $O(0)\equiv O(1)$]. Moreover we shall introduce "multivalued" O symbols for vectorial and tensorial space-time quantities by the definitions

$$a_\mu=O(p,q)\Longleftrightarrow a_0=O(c^{-p}),\quad a_i=O(c^{-q}) , \tag{2.14b}$$

$$\begin{aligned} &B_{\alpha\beta}=O(p,q,r)\Longleftrightarrow B_{00}=O(c^{-p}) ,\\ &B_{0a}=O(c^{-q}),\quad B_{ab}=O(c^{-r}) . \end{aligned} \tag{2.14c}$$

Combining the assumptions (2.12), (2.13) with the tensorial law of transformations of the metric components

$$G_{\alpha\beta}=A^\mu_\alpha A^\nu_\beta g_{\mu\nu} ,$$

it is easy to verify that a necessary and sufficient condition for the compatibility of these two sets of assumptions is that [with the notation (2.14c)]

$$f_{\mu\nu}A^\mu_\alpha A^\nu_\beta-f_{\alpha\beta}=O(2,3,2) . \tag{2.15}$$

Note that any mention of the metric coefficients has disappeared from (2.15), which is just a constraint on the mathematical structure of the coordinate transformation $x^\mu=f^\mu(X^a)$.

Inserting now in Eq. (2.15) the expressions (2.10) for the Jacobian matrix, we get constraints on z^μ, e^μ_α, and ξ^μ. A straightforward study of these constraints allows one to prove our first useful theorem.

Theorem 1. *Under the (weak) assumption that ξ^i admits some c-dependent order when $c^{-1}\rightarrow 0$ [i.e., is $O(f(c))$ for some function $f(c)$], the post-Newtonian assumptions (2.12) and (2.13) are compatible if and only if the $z-e-\xi$ elements of each coordinate transformation (label A omitted) satisfy*

$$e^0_0(T)\equiv\frac{dz^0(T)}{c\,dT}=1+O(2) , \tag{2.16a}$$

$$e^0_a(T)=\frac{1}{c}e^i_a(T)\frac{dz^i(T)}{dT}+O(3) , \tag{2.16b}$$

$$e^i_0(T)\equiv\frac{1}{c}\frac{dz^i(T)}{dT} , \tag{2.16c}$$

$$\delta_{ij}e^i_a(T)e^j_b(T)=\delta_{ab}+O(2) , \tag{2.16d}$$

$$\frac{de^i_a(T)}{dT}=O(2) , \tag{2.16e}$$

$$\xi^0=O(3) , \tag{2.16f}$$

$$\xi^i=O(2) , \tag{2.16g}$$

where we have included for completeness the exact relation (2.10c).

Some comments can be made on the interpretation of the results (2.16). Roughly speaking, they show that the knowledge of the parametrized world line $\mathcal{L}_A$, i.e., of $z^\mu(S)$ (with $S\equiv cT$), and of the 3×3 time-dependent matrix $e^i_a(S)$ determines the other components of the vectorial basis $e^\mu_\alpha(S)$. In turn, the 3×3 matrix $e^i_a(S)$ differs only by $O(1/c^2)$ terms from a slowly changing Euclidean rotation matrix (the time scale for the change tending to infinity like c^{+2}). Moreover, Eq. (2.16f) shows also that the effect of the ξ term on the global coordinate time $t\equiv x^0/c$ is $O(\xi^0/c)=O(c^{-4})$, i.e., of second post-Newtonian order, so that it will not affect any observable quantity at the first post-Newtonian approximation.[43] The only crucial quantity at the first post-Newtonian approximation which is left totally unconstrained by the results (2.16) is the ξ^i term, i.e., the effect of the ξ term on the spatial coordinates x^i, which is $O(\xi^i)=O(c^{-2})$.

D. Constraints on z, e, and ξ from the use of conformally Cartesian spatial coordinates

We just saw that the general post-Newtonian assumptions leave completely unspecified the spatial coordinates at the $O(1/c^2)$ level. To deal with this problem many authors impose *four local* gauge conditions that intend to fix completely the coordinate freedom. The two usual choices are the "harmonic conditions"[3-7]

$$f^{\mu\nu}\left[\partial_\mu g_{\nu\lambda}-\tfrac{1}{2}\partial_\lambda g_{\mu\nu}\right]=O(5,4)\ , \tag{2.17}$$

or the "standard post-Newtonian gauge" ones:[42,44]

$$\partial_j g_{0j}-\tfrac{1}{2}\partial_0 g_{jj}=O(5)\ , \tag{2.18a}$$

$$\partial_j g_{ij}-\tfrac{1}{2}\partial_i(g_{jj}-g_{00})=O(4)\ . \tag{2.18b}$$

We find, however, that the use of such gauge conditions is inconvenient for two seemingly contradictory reasons: (i) the use of *four* gauge conditions is *too restrictive* because it constrains also the choice of the time coordinate at the $O(c^{-4})$ level, while we shall find it very convenient to use a certain "gauge invariance" associated to the $O(c^{-4})$ flexibility in the time coordinate; (ii) the use of *local* gauge conditions is *not restrictive enough* because when using it in one of the local coordinate domains it does not fix completely the choice of the three space coordinates [indeed, the conditions (2.17) or (2.18) fix the coordinate gauge only if some global boundary conditions are further imposed].

Instead of using four local differential conditions such as Eq. (2.17) or (2.18), we shall impose *algebraic conditions* on the structure of the metric coefficients that will conveniently fix the spatial gauge while leaving a useful flexibility in the time gauge at $O(c^{-4})$. To see why such algebraic conditions can be imposed and play a preferred role in post-Newtonian general relativity, let us anticipate on the following sections, and consider the Einstein field equations.

If we introduce, in any coordinate system, the contravariant metric density ("gothic metric"; with $g\equiv-\det g_{\mu\nu}$)

$$\mathbf{g}^{\mu\nu}\equiv\sqrt{g}\ g^{\mu\nu}\ , \tag{2.19}$$

the Einstein tensor can, as was emphasized by Landau and Lifshitz,[45] be expressed as

$$2g\left[R^{\mu\nu}-\tfrac{1}{2}Rg^{\mu\nu}\right]\equiv\partial_{\rho\sigma}(\mathbf{g}^{\mu\nu}\mathbf{g}^{\rho\sigma}-\mathbf{g}^{\mu\rho}\mathbf{g}^{\nu\sigma})+Q^{\mu\nu}(\partial\mathbf{g})\ , \tag{2.20}$$

where $Q^{\mu\nu}$ is a quadratic form in the first derivatives of $\mathbf{g}$ with coefficients rational in $\mathbf{g}$. Under the post-Newtonian assumptions (2.12) for the metric, it is easy to verify that (2.20) yields, for the spatial components $(\mu\nu)=(ij)$,

$$2g(R^{ij}-\tfrac{1}{2}Rg^{ij})\equiv\partial_{kl}[\mathbf{g}^{ij}\mathbf{g}^{kl}-\mathbf{g}^{ik}\mathbf{g}^{jl}]+O(1/c^4)\ , \tag{2.21}$$

where the summation indices k,l run only over $1-3$. Let us now remark that the structure (2.20) is valid in all dimensions, and in particular for a three-dimensional Riemannian manifold, and let us define (in each coordinate system we are using) a three-dimensional (post-Newtonian) metric, say γ_{ij} [from which one can compute its 3×3 determinant $\gamma\equiv\det(\gamma_{ij})$], by

$$\sqrt{\gamma}\ \gamma^{ij}\equiv\mathbf{g}^{ij}\ , \tag{2.22a}$$

which a direct calculation shows to be equivalent to

$$\gamma_{ij}\equiv-g_{00}g_{ij}+g_{0i}g_{0j}\ . \tag{2.22b}$$

Note that, in the absence of a preferred $3+1$ split of the spacetime, the three-metric γ_{ij} depends upon the coordinate system x^μ we started with. It is then clear that

$$g[{}^4R^{ij}(g)-\tfrac{1}{2}{}^4R(g)g^{ij}]=\gamma[{}^3R^{ij}(\gamma)-\tfrac{1}{2}{}^3R(\gamma)\gamma^{ij}]+O(1/c^4)\ , \tag{2.23}$$

where the presuperscripts 4 and 3 remind us that the Einstein tensor on the left-hand side is four-dimensional, while the one on the right hand side is three-dimensional. It is now sufficient to remark that the Einstein field equations

$${}^4R^{\mu\nu}-\tfrac{1}{2}{}^4Rg^{\mu\nu}=\frac{8\pi G}{c^4}T^{\mu\nu}\ , \tag{2.24}$$

together with the usual *post-Newtonian assumptions for the matter,* i.e.,

$$T^{\mu\nu}=O(c^{+2},c^{+1},c^0)\ , \tag{2.25}$$

imply that the left hand side of Eq. (2.23) is of $O(1/c^4)$. We therefore conclude that if $g_{\mu\nu}$ is a post-Newtonian solution of Einstein's equations, its associated three-metric γ_{ij} has a Ricci tensor of *second* post-Newtonian order:

$${}^3R_{ij}(\gamma)=O(c^{-4})\ . \tag{2.26}$$

But, in three dimensions the Ricci tensor is algebraically equivalent to the full Riemann-Christoffel (curvature) tensor; therefore, Eq. (2.26) means that the three-metric γ_{ij} is *flat* to second post-Newtonian order. It is then clear that, among all coordinate systems where the post-Newtonian assumptions hold, there is a preferred subclass of coordinate charts, namely those for which the spatial coordinates are *Cartesian* coordinates for the metric γ_{ij}, i.e.,

$$\gamma_{ij}=\delta_{ij}+O(4)\ . \tag{2.27}$$

In terms of the metric coefficients $g_{\mu\nu}$, this preferred subclass is defined equivalently by

$$\mathbf{g}^{ij}=\delta^{ij}+O(4)\ , \tag{2.28a}$$

$$-g_{00}g_{ij}+g_{0i}g_{0j}=\delta_{ij}+O(4)\ . \tag{2.28b}$$

As $g_{0i}=O(3)$ by the post-Newtonian assumptions (2.12), we see that we can impose, in all coordinate systems, the *spatial isotropy conditions*

$$\begin{aligned}&-g_{00}g_{ij}=\delta_{ij}+O(4)\ ,\\&\forall A,\ -G^A_{00}G^A_{ab}=\delta_{ab}+O(4)\ .\end{aligned} \tag{2.29}$$

When we need to name the coordinates selected by Eqs.

(2.29), we shall say either that the spatial coordinates are "conformally Cartesian" or that they are "isotropic" (a more precise but somewhat barbarian terminology would be from Eq. (2.28a): "gothically Cartesian"). Note that we shall always use the precise isotropy conditions (2.29), and not weaker conditions, requesting only that g_{ij} be proportional to δ_{ij} without specifying the conformal factor. The latter "weak isotropy" condition leads to a larger class of spatial coordinates which can differ from that of the smaller class (2.29) by arbitrary conformal transformations of Euclidean space. But such more general spatial coordinates do not share all the nice properties of the (strongly) isotropic coordinates (2.29) with respect to the Einstein equations that will play an important role in our method.

The definition (2.29) [or (2.27)] ensures the complete fixing of the spatial coordinate grid up to time-dependent isometries of Euclidean three-space (that satisfy the slow-motion assumption (2.13)). More precisely, we can now pursue the work of the last section and study the constraints on z^μ, e_a^μ, and ξ^μ brought about by the conditions (2.29). Using the tensorial law of transformation of the matrix components, and the post-Newtonian assumptions of the previous subsection, it is easy to verify that the spatial isotropy conditions are satisfied if and only if

$$f_{\mu\nu} A_a^\mu A_b^\nu = (2 + f_{\mu\nu} A_0^\mu A_0^\nu)\delta_{ab} + O(4) \; . \tag{2.30}$$

Note that the metric coefficients have disappeared from Eq. (2.30), which is just a constraint on the mathematical structure of the coordinate transformation $x^\mu = f^\mu(X^\alpha)$. In other words we are left with the following purely mathematical question: find in Minkowski spacetime $(f_{\mu\nu})$ the three-parameter families of world lines $\mathcal{L}_{X^a}$ $[x^\mu = f^\mu(S, X^a)$ for fixed $X^a]$ for which the projected spatial metric is (strongly) isotropic.

Inserting in Eq. (2.30) the expressions (2.10) for the Jacobian matrix, we get constraints on z^μ, e_a^μ, and ξ^μ. As, by definition, ξ^μ is at least quadratic in X^a when $X^a \to 0$, we can split these constraints in two sets. First, we get constraints concerning e_a^μ, which can be written simply as

$$f(\mathbf{e}_0, \mathbf{e}_0) f(\mathbf{e}_a, \mathbf{e}_b) = -\delta_{ab} + O(4) \; , \tag{2.31}$$

where

$$f(\mathbf{u}, \mathbf{v}) \equiv f_{\mu\nu} u^\mu u^\nu$$

denotes the "flat" (Minkowskian) scalar product. Second, using the results of the previous sub-section, we get constraints on ξ^i $(i = 1,2,3)$. More precisely, let us define three components Ξ^a by

$$\xi^i \equiv e_a^i \Xi^a \; , \tag{2.32}$$

and an "acceleration three-vector" A_a by

$$A_a \equiv f_{\mu\nu} e_a^\mu \frac{d^2 z^\nu}{d\tau^2} \; , \tag{2.33}$$

where $c^2 d\tau^2 = -f_{\mu\nu} dz^\mu dz^\nu$ is the Minkowskian proper time along the world line $\mathcal{L}$. With this notation the $O(\mathbf{X})$ constraints that we get read

$$\frac{\partial \Xi^b}{\partial X^c} + \frac{\partial \Xi^c}{\partial X^b} = -\frac{2}{c^2}(A_a X^a)\delta_{bc} + O(1/c^4) \; . \tag{2.34}$$

Equation (2.34) means that $\Xi^a(X^b)$ is [mod $O(4)$] *a conformal Killing vector* of Euclidean three-space. Now, it is well known that there are ten independent such vector fields: six of them being the Euclidean Killing vectors, i.e., the translations and rotations $(\Xi^a = C_a + \Omega_{[ab]} X^b)$, one being the dilation (or scaling; $\Xi^a = X^a$), and the remaining three being the "inverted translations" [or proper conformal transformations;[46] $\Xi^a = B^a \mathbf{X}^2 - 2(\mathbf{B}\cdot\mathbf{X})X^a$]. Now, Ξ^a is, by definition (2.32) with (2.5c), at least quadratic in X^a. Therefore, Ξ^a is a pure "inverted translation" uniquely determined by Eq. (2.34) to be

$$\Xi^a = \frac{1}{c^2}[\tfrac{1}{2} A_a \mathbf{X}^2 - X^a(\mathbf{A}\cdot\mathbf{X})] + O(1/c^4) \; , \tag{2.35a}$$

where we are using the usual Euclidean three-space vector notation

$$\mathbf{X}^2 \equiv \delta_{ab} X^a X^b, \quad \mathbf{A}\cdot\mathbf{X} \equiv A_a X^a \; . \tag{2.35b}$$

As for Eq. (2.31), it shows (using $e_0^i \equiv dz^i / c\, dT$) that the 3×3 matrix e_a^i is, modulo $O(4)$, proportional to the space-space part of a general Lorentz transformation (i.e., a boost combined with an arbitrary rotation matrix). Putting together our results so far we get the following.

Theorem 2. *The post-Newtonian assumptions (2.12) and (2.13), and the (strong) spatial isotropy conditions (2.29) imply the following (nearly complete) determination of the z-e-ξ elements of each coordinate transformation (label A omitted): all quantities along the world line $\mathcal{L}(x^\mu = z^\mu(T))$ being parametrized by $T \equiv S/c$, and using the notation $dF(T)/dT \equiv \dot F$, we have*

$$e_0^0(T) \equiv c^{-1}\dot z^0 = 1 + O(2) \; , \tag{2.36a}$$

$$e_0^i(T) \equiv c^{-1}\dot z^i \; , \tag{2.36b}$$

$$e_a^0(T) = c^{-1} e_a^i \dot z^i + O(3) \; , \tag{2.36c}$$

$$e_0^0(T) e_a^i(T) = \left[1 + \frac{1}{2c^2}\mathbf{v}^2\right] \times \left[\delta^{ij} + \frac{1}{2c^2} v^i v^j\right] R_a^j(T) + O(4) \; , \tag{2.36d}$$

$$\xi^0(T, \mathbf{X}) = O(3) \; , \tag{2.36e}$$

$$\xi^i(T, \mathbf{X}) = \frac{1}{c^2} e_a^i(T)[\tfrac{1}{2} A_a \mathbf{X}^2 - X^a(\mathbf{A}\cdot\mathbf{X})] + O(4) \; , \tag{2.36f}$$

where $R_a^i(T)$ is a slowly changing rotation matrix of Euclidean three-space:

$$R_a^i R_a^j = \delta^{ij}, \quad R_a^i R_b^i = \delta_{ab} \; , \tag{2.36g}$$

$$\frac{dR_a^i}{dT} = O(2) \; . \tag{2.36h}$$

The three-velocity appearing in the "boost matrix" $b_{ij} = \delta^{ij} + v^i v^j / 2c^2$ of Eq. (2.36d) needs only to be defined at Newtonian order, so that $v^i = dz^i/dt = dz^i/dT + O(2)$ suffices. The same is true of the three-acceleration appearing in Eq. (2.36f) that we have defined by Eq. (2.33), and which could be simply $A_a = e_a^i d^2 z^i/dt^2 + O(2)$.

In other words we see that our assumptions of post-Newtonian, slow-motion, spatially isotropic coordinates completely determines, at the first post-Newtonian level, the coordinate transformation (2.8) modulo the knowledge (for each local system) of (i) the "central world line" $z^\mu(S)$ endowed with a post-Newtonian parametrization $T \equiv S/c$, (ii) the slow $[O(c^{-2})]$ time dependence of the rotation matrix $R_a^i(T)$, and (iii) the second post-Newtonian time transformation $\delta t = c^{-1}\xi^0 = O(c^{-4})$ [the arbitrary c^{-3} part of e_a^0 will never play any role at the first post-Newtonian (1PN) level, and will be fixed for convenience later].

Finally, let us note that our basic z-e-ξ transformation formula (2.8) can alternatively be written in the z-e-η form

$$x^\mu(X^\alpha) = z^\mu(T) + e_a^\mu(T) Y^a + \eta^\mu \ , \tag{2.37}$$

where

$$\begin{aligned} Y^a(T,\mathbf{X}) &\equiv X^a + \Xi^a \\ &= X^a + \frac{1}{c^2}[\tfrac{1}{2}A_a \mathbf{X}^2 - X^a(\mathbf{A}\cdot\mathbf{X})] \ , \end{aligned} \tag{2.38}$$

and where

$$\eta^0 = O(3), \quad \eta^i = O(4) \tag{2.39}$$

is a post-post-Newtonian order "remainder." Equation (2.37) suggests that the use of $Y^\alpha \equiv (cT, Y^a)$ as local coordinates could be advantageous. And indeed, we shall several times below use Y^α as intermediate quantities; however, their systematic use would spoil the simple structure of the post-Newtonian gravitational field that we shall study in the next section (one might however notice that, because of the fact that Ξ^a is a conformal Killing three-vector, the spatial metric in the Y^a coordinates is still isotropic, or conformally Cartesian, but only in a *weak* sense, the conformal factor differing from g_{00}^{-1} by acceleration effects).

III. POST-NEWTONIAN METRIC

A. Linearization of 1PN field equations by means of a suitable parametrization

The Einstein field equations

$$R^{\mu\nu} = \frac{8\pi G}{c^4}(T^{\mu\nu} - \tfrac{1}{2}g^{\mu\nu}g_{\rho\sigma}T^{\rho\sigma}) \tag{3.1}$$

constitute a complicated system of nonlinear partial differential equations relating the ten components $g_{\mu\nu}$ of the metric tensor (in any coordinate system) to the ten components of the stress-energy tensor $T^{\mu\nu}$ (in the same system). This system simplifies very much if we assume that the gravitational field is everywhere weak and slowly changing [PN assumptions for the metric (2.12)] and, correspondingly, that the material source is nonrelativistic [PN assumptions for the matter (2.25)]. It has been noticed as early as 1916 by Droste[3] that it was then possible to solve explicitly for $g_{\mu\nu}$ (in some adequate coordinate systems) at the first post-Newtonian (1PN) level, i.e., up to errors [using the notation (2.14)]

$$\delta g_{\mu\nu} = O(6,5,4) \ . \tag{3.2}$$

However, the literature on the 1PN approximation to general relativity is full of errors and unnecessary complications. We shall present here what we think is the optimal formulation of Einstein's field equations at the 1PN level. The two basic ingredients of this formulation are (i) the use of a convenient parametrization of the ten metric components by means of a scalar w, a three-vector w_i, and a three-tensor γ_{ij}, and (ii) the systematic use of the contravariant components of the stress-energy tensor without restricting oneself to a particular matter model (like a perfect-fluid model). The (best-motivated) definition of the three-tensor γ_{ij} has already been introduced in Eqs. (2.22) of the previous section. Working consistently at the 1PN level [i.e., modulo errors given by Eq. (3.2)] we shall directly introduce our w-γ parametrization by the representation

$$g_{00} = -\exp\left[-\frac{2}{c^2}w\right] \ , \tag{3.3a}$$

$$g_{0i} = -\frac{4}{c^3}w_i \ , \tag{3.3b}$$

$$g_{ij} = \gamma_{ij}\exp\left[+\frac{2}{c^2}w\right] \ , \tag{3.3c}$$

from which follows

$$\sqrt{g} = \sqrt{\gamma}\exp\left[+\frac{2}{c^2}w\right] + O(4) \ , \tag{3.4}$$

$$g^{00} = -\exp\left[+\frac{2}{c^2}w\right] + O(6) \ , \tag{3.5a}$$

$$g^{0i} = -\frac{4}{c^3}w^i + O(5) \ , \tag{3.5b}$$

$$g^{ij} = \gamma^{ij}\exp\left[-\frac{2}{c^2}w\right] + O(4) \ , \tag{3.5c}$$

where γ^{ij} denotes the inverse matrix of γ_{ij}, γ its determinant, and $w^i \equiv \gamma^{ij}w_j$.

Einstein's equations (3.1) now give (coupled) equations for w, w_i, and γ_{ij}. As anticipated in the previous section, six of the Einstein equations yield Eq. (2.26), which means that the three-metric γ_{ij} is *flat* modulo $O(4)$ terms. Without loss of generality we can always fix the spatial coordinate grid by the condition that

$$\gamma_{ij} = \delta_{ij} + O(4) \ . \tag{3.6}$$

As discussed in the last section we shall use such "conformally Cartesian" spatial coordinates in each reference system under consideration.

The four remaining Einstein equations give four equations for w and w_i. As recently found by Blanchet and Damour[41] and Blanchet, Damour, and Schäfer,[42] the great advantage of the "exponential parametrization" in terms of (w,w_i) is that it *linearizes* the 1PN field equations. Indeed, we find

$$-c^2R^{00}=\Delta w+3\partial_{00}w+\frac{4}{c}\partial_{0i}w_i+O(4)\ , \qquad (3.7a)$$

$$-\frac{c^3}{2}R^{0i}=\Delta w_i-\partial_{ij}w_j-c\partial_{0i}w+O(2)\ , \qquad (3.7b)$$

where $\Delta\equiv\partial_{ii}$ is the ordinary Laplacian defined with respect to the coordinate system used [in general it would be $\gamma^{ij}D_iD_j$ with D_i being the spatial covariant derivative associated with γ_{ij}, similarly for the other spatial gradients in Eqs. (3.7)]. As for the "source terms" in Einstein equations (3.1) they turn out to be very simply expressed in terms of the contravariant components of the stress-energy tensor: namely,

$$T^{00}-\tfrac{1}{2}g^{00}g_{\rho\sigma}T^{\rho\sigma}=\tfrac{1}{2}(T^{00}+T^{ss})[1+O(4)]\ , \qquad (3.8a)$$

$$T^{0i}-\tfrac{1}{2}g^{0i}g_{\rho\sigma}T^{\rho\sigma}=T^{0i}[1+O(2)]\ , \qquad (3.8b)$$

where we recall that we are using the PN assumptions (2.25) for the matter which imply that the T^{ss} term in Eq. (3.8a) is already a $O(c^{-2})$ fractional correction to T^{00}. We then follow Ref. 41 in defining an "active gravitational mass density"

$$\sigma\equiv c^{-2}(T^{00}+T^{ss})\ , \qquad (3.9a)$$

and an "active mass current density"

$$\sigma^i\equiv c^{-1}T^{0i} \qquad (3.9b)$$

(where the adjective "active" refers to the role of these quantities as "sources" of the gravitational field). The powers of c introduced in Eqs. (3.9) ensures that both quantities have a nonzero limit when $c^{-1}\to 0$. The post-Newtonian literature has been plagued by the use of ill-chosen "mass densities" (confusingly denoted ρ) to play the role of the basic Newtonian mass density. By contrast, we shall show how our systematic use of σ, Eq. (3.9a), as a basic mass density drastically simplifies, at once, the field equations, the expression of the metric, and the transformation laws of w when changing the coordinate system. For the time being we shall only note the fact that, through 1PN accuracy, we can also express σ in terms of the mixed components of the stress-energy tensor:

$$\sigma=c^{-2}\sqrt{g}\,(-T_0^0+T_s^s)[1+O(4)]\ . \qquad (3.10)$$

One recognizes in the right-hand side of Eq. (3.10) the integrand of the Tolman mass formula valid for exactly stationary isolated systems.[45] In summary, with the notation just introduced, the 1PN approximation consists of four linear partial differential equations that read ($\partial_t\equiv c\partial_0\equiv\partial/\partial t$)

$$\Delta w+\frac{3}{c^2}\partial_t^2w+\frac{4}{c^2}\partial_t\partial_iw_i=-4\pi G\sigma+O(4)\ , \qquad (3.11a)$$

$$\Delta w_i-\partial_{ij}^2w_j-\partial_t\partial_iw=-4\pi G\sigma^i+O(2)\ . \qquad (3.11b)$$

B. Gauge invariance of 1PN field equations

One checks directly that if $w_\mu\equiv(w,w_i)$ is a solution of Eqs. (3.11) with some given source terms $\sigma^\mu=(\sigma,\sigma^i)$, so is $w'_\mu=(w',w'_i)$ (modulo post-Newtonian error terms) with

$$w'=w-\frac{1}{c^2}\partial_t\lambda\ , \qquad (3.12a)$$

$$w'_i=w_i+\tfrac{1}{4}\partial_i\lambda\ , \qquad (3.12b)$$

where $\lambda(x^\mu)$ is an arbitrary (differentiable) function. This (approximate) gauge invariance is connected with the fact that the left-hand sides of Eqs. (3.11) satisfy an approximate divergence identity, which in turn, implies that σ^μ must be approximately divergence-free:

$$\partial_t\sigma+\partial_i\sigma^i=O(2)\ .$$

This linear "gauge invariance" is similar to the one in Maxwell's theory. The similarity would look closer at the field level (though not at the source level) if we had introduced the "gravitational four potential"

$$a_\mu\equiv(a_0,a_i)\equiv(cw,-4w_i)\ , \qquad (3.13a)$$

for which

$$a'_\mu=a_\mu-\partial_\mu\lambda\ . \qquad (3.13b)$$

However, the use of $a_i\equiv-4w_i\equiv c^3g_{0i}$ instead of w_i as a "vector potential" has as many disadvantages as advantages,[47] so we shall often work directly with $w_\mu\equiv(w,w_i)$. The gauge invariance (3.12) corresponds to a shift of the time variable

$$\delta t=c^{-4}\lambda(x^\mu)\ , \qquad (3.14)$$

which is the only remaining freedom in each of our coordinate systems after having chosen the central world line, a rotation matrix $R_a^i(T)$, and having imposed the use of conformally Cartesian spatial coordinates. Note that the $O(4)$ shift (3.14) affects none of the physical quantities at the 1PN level (because it corresponds to a 2PN change in the equations of motion). It affects, however, the explicit expressions of the g_{00} and g_{0i} metric coefficients. The usual way in the post-Newtonian literature to deal with this gauge freedom is to try to get rid of it by imposing some local "gauge condition." The two most used coordinate conditions are the "harmonic" one and the "standard PN" one.

Let us first note that a consequence of our (strong) spatial isotropy conditions (2.29) is easily seen to be that

$$\Box_g x^i=O(4)\ , \qquad (3.15)$$

where

$$\Box_g\equiv\frac{1}{\sqrt{g}}\partial_\mu(\sqrt{g}\,g^{\mu\nu}\partial_\nu) \qquad (3.16)$$

denotes the covariant d'Alembertian acting on scalar functions. In other words our isotropy conditions (2.29) imply that the three spatial coordinates are "harmonic" modulo O(4). Equation (3.15) is in fact required both by the users of the whole-harmonic gauge, and of the standard PN gauge [as can be seen by comparing the $\lambda=i$ case of Eq. (2.17) with the second Eq. (2.18)]. Note, however, that the converse is not true because the local partial differential equations (3.15) [or (2.18b)] can never fully fix a local coordinate grid. Concerning the constraining of the time gauge the condition that the time coordinate be "harmonic" reads

$$0=\Box_g x^0=-\frac{4}{c^3}(\partial_t w+\partial_i w_i)+O(5)\ , \tag{3.17a}$$

leading to

$$\partial_t w+\partial_i w_i=O(2) \tag{3.17b}$$

[which can also be read off the $\lambda=0$ case of Eq. (2.17)]. The field equations (3.11) then reduce to

$$\Delta w-\frac{1}{c^2}\frac{\partial^2 w}{\partial t^2}=-4\pi G\sigma+O(4)\ , \tag{3.18a}$$

$$\Delta w_i=-4\pi G\sigma^i+O(2)\ . \tag{3.18b}$$

On the other hand, if we constrain the time gauge by means of the standard PN gauge condition, i.e., Eq. (2.18b) or

$$3\partial_t w+4\partial_i w_i=O(2)\ , \tag{3.19}$$

the field equations read

$$\Delta w=-4\pi G\sigma+O(4)\ , \tag{3.20a}$$

$$\Delta w_i-\tfrac{1}{4}\partial_t\partial_i w=-4\pi G\sigma^i+O(2)\ . \tag{3.20b}$$

As we said above, the usual post-Newtonian practice is to choose from the start one of the two gauges: "harmonic," Eqs. (3.18), or "standard PN," Eqs. (3.20). However, part of the flexibility and convenience of the new approach that we propose in this paper is in fact *not to choose* between (3.18) and (3.20) but to keep always the gauge invariance (3.12) [when we need, however, to fix ideas by using a reference gauge we shall use the harmonic one (3.17)]. By analogy with the electromagnetic case, we are then naturally led to introduce some gauge-invariant "electriclike" and "magneticlike" field quantities. To every $w_\mu\equiv(w,w_i)$, or rather, $a_\mu\equiv(cw,-4w_i)$, we associate

$$b_{\mu\nu}\equiv\partial_\mu a_\nu-\partial_\nu a_\mu\ , \tag{3.21a}$$

i.e., explicitly for $e_i\equiv c^{-1}b_{i0}$, and b_{ij},

$$e_i[w]\equiv\partial_i w+\frac{4}{c^2}\partial_t w_i\ , \tag{3.21b}$$

$$b_{ij}[w]\equiv\epsilon_{ijk}b_k[w]\equiv-4[\partial_i w_j-\partial_j w_i]\ . \tag{3.21c}$$

Using vectorial notation (associated with the flat spatial metric δ_{ij}) the gauge-invariant fields (3.21) satisfy the Maxwell-type equations

$$\nabla\cdot\mathbf{b}=0\ , \tag{3.22a}$$

$$\nabla\times\mathbf{e}=-\frac{1}{c^2}\partial_t\mathbf{b}\ , \tag{3.22b}$$

$$\nabla\cdot\mathbf{e}=-\frac{3}{c^2}\partial_t^2 w-4\pi G\sigma+O(4)\ , \tag{3.22c}$$

$$\nabla\times\mathbf{b}=+4\partial_t\mathbf{e}-16\pi G\boldsymbol{\sigma}+O(2)\ . \tag{3.22d}$$

From Eqs. (3.22) we can also deduce the gauge-invariant second-order equations

$$\Box\mathbf{e}=-4\pi G\left[\nabla\sigma+\frac{4}{c^2}\partial_t\boldsymbol{\sigma}\right]+O(4)\ , \tag{3.23a}$$

$$\Delta\mathbf{b}=+16\pi G\nabla\times\boldsymbol{\sigma}+O(2)\ . \tag{3.23b}$$

The "gravitoelectric" and "gravitomagnetic" **e** and **b** fields will play a useful role in our formalism.

C. Structure of the PN metric in the global coordinate system

Let us consider a chunk of the universe made of a finite number of gravitationally interacting spatially compact bodies (e.g., the solar system). Let us describe this entire N-body system by means of a "global" coordinate system x^μ satisfying the (strong) spatial isotropy condition $g_{00}g_{ij}=-\delta_{ij}+O(4)$. The 1PN metric tensor is then fully described by the four quantities $w_\mu\equiv(w,w_i)$ which must satisfy the *linear* partial differential system (3.11) [we could also work with the gauge invariant quantities (3.21) but it is as simple to work directly with the w's]. The linearity of the system (3.11) means that its general solution, w_μ^{general}, can always be written as

$$w_\mu^{\text{general}}=w_\mu^N+\overline{w}_\mu^N\ , \tag{3.24a}$$

where w_μ^N is a particular solution of the inhomogeneous system (3.11), say

$$\mathcal{L}^\mu[w_\nu^N]=-4\pi G\sum_{A=1}^{N}\sigma_A^\mu\ , \tag{3.24b}$$

where σ_A^μ denotes the source contribution of each body of the N-body system, and where $\overline{w}_\mu^N$ is a general solution of the corresponding homogeneous system:

$$\mathcal{L}^\mu[\overline{w}_\nu^N]=0\ . \tag{3.24c}$$

We shall first consider an idealized "isolated" N-body system, i.e., a system which is adequately described by taking

$$\overline{w}_\mu^N=0\ , \tag{3.25}$$

when a suitable w_μ^N is chosen such as the "retarded harmonic" solution of Eq. (3.24b), namely,

$$w_\mu^{N,\text{ret}}=\Box_{x,\text{ret}}^{-1}\left[-4\pi G\sum_{A=1}^{N}\sigma_A^\mu\right]\ . \tag{3.26}$$

In Eq. (3.26) $\Box_{x,\text{ret}}^{-1}$ denotes the "retarded" inverse of the x-coordinate flat-space wave operator, i.e., explicitly

$$\Box_{x,\text{ret}}^{-1}[f(x)]\equiv-\frac{1}{4\pi}\int\frac{d^3x'}{|\mathbf{x}-\mathbf{x}'|}f\left[t-\frac{|\mathbf{x}-\mathbf{x}'|}{c},\mathbf{x}'\right]\ . \tag{3.27}$$

Note, that consistently with the allowed $O(2)$ error term in the "harmonic" field equation for w_i, we have replaced the Laplacian of Eq. (3.18b) by a d'Alembertian.[48] This is done mainly for abbreviating the notation, and at any moment we will allow ourselves to use an inverse Laplacian for $w_i^{\rm ret}$. As for $w_0 \equiv w$, which must be determined modulo $O(4)$, it is necessary, when using the harmonic gauge, to invert a wave operator.

Finally, it is well known that the first physical time-asymmetric effects in the self-interaction of a slow-motion gravitating system belong to the second-and-a-half post-Newtonian (2.5PN) approximation. Therefore, we can, in our 1PN treatment, replace the "retarded" solution (3.26) by the "symmetric" one

$$w_\mu^{N,\rm sym} = \Box^{-1}_{x,\rm sym}\left[-4\pi G \sum_{A=1}^{N} \sigma^\mu_A\right] , \tag{3.28a}$$

where

$$\Box^{-1}_{x,\rm sym} \equiv \tfrac{1}{2}(\Box^{-1}_{x,\rm ret} + \Box^{-1}_{x,\rm adv}) \tag{3.28b}$$

is the half-sum of the retarded and advanced flat-space Green's functions. The use of (3.28) will somewhat simplify our subsequent treatment by suppressing from the start several "gauge terms" that would render our method less transparent.

Summing up, we shall use a starting point for our method the N-body global metric

$$w_\mu^N = \sum_{A=1}^{N} w_\mu^A \tag{3.29a}$$

naturally decomposed as the superposition of contributions generated by each body of the system (written for definiteness in harmonic gauge):

$$\begin{aligned} w_\mu^A(x^\lambda) &= \Box^{-1}_{x,\rm sym}(-4\pi G \sigma^\mu_A) \\ &= G\int_A \frac{d^3x'}{|\mathbf{x}-\mathbf{x}'|}\sigma^\mu\left[t \mp \frac{|\mathbf{x}-\mathbf{x}'|}{c}, \mathbf{x}'\right] . \end{aligned} \tag{3.29b}$$

In Eq. (3.29b) the volume integral extends only over body A, and we have introduced the convenient abbreviated notation of putting a sign ambiguity $\mp$ (or $\pm$) to mean the symmetric half-sum

$$f(\mp) \equiv \tfrac{1}{2}[f(-)+f(+)] . \tag{3.30}$$

D. Local gravitoelectric and gravitomagnetic fields and geometry of the X_A-coordinate congruences

Section III B has shown how a gauge-invariant description of the gravitational field was possible through the use of a "gravitoelectric" and "gravitomagnetic" fields defined by Eqs. (3.21). It is clear from Sec. III A above that these fields will have the same formal properties in any coordinate system where the spatial coordinates are conformally Cartesian, in the sense of Eq. (3.6) above. In particular, this will be the case for the local $\mathbf{E}_A$ and $\mathbf{B}_A$ fields constructed, by the same equations (3.21), from the potentials $W_\alpha^A \equiv (W^A, W_a^A)$ parametrizing the metric $G^A_{\alpha\beta}(X^\gamma_A)$ in each local X_A frame [satisfying the strong spatial isotropy condition (2.29)]. As these local $\mathbf{E}$ and $\mathbf{B}$ fields will play an especially important role in our method (more important than the global frame $\mathbf{e}$ and $\mathbf{b}$ fields introduced above) we shall indicate in this subsection how nicely related they are to the geometry of the coordinate lines belonging to the X_A chart.

Going back to the general abstract setting of Sec. II B, let us consider, in the differentiable manifold V_4, a coordinate chart $P \in V_4 \to X_A^\alpha(P) \in \mathbb{R}^4$ as defining a three-parameter family of world lines (label A omitted):

$$\mathcal{L}_{X^a} = \{P \in V^4,\ X^a(P) = X^a,\ a = 1,2,3\} ,$$

each world line $\mathcal{L}_{X^a}$ being parametrized by $S \equiv X^0$ taken along $\mathcal{L}_{X^a}$. Associated with the chart X^α, there are four vector fields, namely, the natural vectorial basis

$$P \in V^4 \to \epsilon_\alpha(P) \equiv \frac{\partial}{\partial X^\alpha} . \tag{3.31}$$

Among the four vector fields (3.31) we shall, consistently with our world-line vision, privilege the vector field ϵ_0 tangent to the congruence $\mathcal{L}_{X^a}$.

Let us also consider some Riemannian metric g^* defined on V_4. As we shall see, instead of using only one definite metric, it will be convenient to use several of them, hence the $*$ on g to label them. Associated with each choice of the metric g^*, there exists a linear connection, say ∇^* (the unique torsion-free connection such that $\nabla^* g^* = 0$). Moreover, we can also use g^* to define a normalized tangent vector to the congruence $\mathcal{L}_{X^a}$:

$$\mathbf{u}^*(P) \equiv c(-g^*(\epsilon_0,\epsilon_0))^{-1/2}\epsilon_0(P) , \tag{3.32a}$$

such that

$$g^*(\mathbf{u}^*,\mathbf{u}^*) = -c^2 . \tag{3.32b}$$

Let us also assume that the components of g^* with respect to the natural vectorial basis ϵ_α (i.e., the ordinary components of the metric tensor in the X^α coordinates) have our usual spatially isotropic form, so that they can be written as

$$G^*_{00} \equiv g^*(\epsilon_0,\epsilon_0) = -e^{-2W^*/c^2} , \tag{3.33a}$$

$$G^*_{0a} \equiv g^*(\epsilon_0,\epsilon_a) = -\frac{4}{c^3}W^*_a , \tag{3.33b}$$

$$G^*_{ab} \equiv g^*(\epsilon_a,\epsilon_b) = \delta_{ab}e^{+2W^*/c^2} . \tag{3.33c}$$

We deduce in particular from Eq. (3.33a) that the $*$-normalized tangent vector is ($X^0 = cT$)

$$\mathbf{u}^* = ce^{W^*/c^2}\epsilon_0 = e^{W^*/c^2}\frac{\partial}{\partial T} . \tag{3.34}$$

In the usual geometrical study of timelike congruences, one starts from a prescribed metric and a normalized four-velocity $\mathbf{u}$ and defines several geometrical invariants of the $\mathbf{u}$ field: especially the "acceleration" vector field $\nabla_{\mathbf{u}}\mathbf{u}$, and the "rotation" of $\mathbf{u}$ measured by the antisymmetric part of the "spatial" projection of the gradient of $\mathbf{u}$ ("spatial" meaning orthogonal to $\mathbf{u}$). In our setup it is still natural to introduce some normalized four-velocity, namely $\mathbf{u}^*$, but the most natural "spatial" projections are simply with respect to the natural vectors $\epsilon_a \equiv \partial/\partial X^a$. This leads us naturally to consider the ϵ_a-based "acceleration" and "rotation" of the four-velocity $\mathbf{u}^*$. A straightforward calculation of these quantities yields

$$g^*(\epsilon_a, \nabla^*_{\mathbf{u}^*}\mathbf{u}^*) = -E^*_a(P) + O(4) , \tag{3.35a}$$

$$g^*(\epsilon_a, \nabla^*_{\epsilon_b}\mathbf{u}^*) - g^*(\epsilon_b, \nabla^*_{\epsilon_a}\mathbf{u}^*) = -\frac{1}{c^2} B^*_{ab}(P) + O(4) , \tag{3.35b}$$

where, consistently with our previous notation, E^*_a and B^*_{ab} denote the combinations ($\partial_a \equiv \partial/\partial X^a$, $\partial_T \equiv c\partial/\partial X^0$)

$$E^*_a(P) \equiv \partial_a W^* + \frac{4}{c^2}\partial_T W^*_a , \tag{3.36a}$$

$$B^*_{ab}(P) \equiv \partial_a(-4W^*_b) - \partial_b(-4W^*_a) . \tag{3.36b}$$

Equations (3.35) nicely display the geometrical meaning of our gauge-invariant **E** and **B** fields with respect to the three-parameter congruence of world lines $\mathcal{L}_{X^a}$.

In practical calculations it can be a nuisance to normalize first the four-velocity before computing $\mathbf{E}^*$ and $\mathbf{B}^*$ by Eqs. (3.35). In fact, one can check that if $\mathbf{v}(P)$ is an arbitrarily normalized tangent vector field to the same congruence $\mathcal{L}_{X^a}$, such that, however, the relative normalization Λ ($\mathbf{v} = \Lambda \mathbf{u}^*$) varies only by 1PN terms,

$$\Lambda = 1 + O(2) ,$$

then one can still write

$$-E^*_a(P) = c^2 \frac{g^*(\epsilon_a, \nabla^*_{\mathbf{v}}\mathbf{v})}{|g^*(\mathbf{v},\mathbf{v})|} + O\left[\frac{1}{c^4}\right] , \tag{3.37a}$$

$$-\frac{1}{c^2} B^*_{ab}(P) = g^*(\epsilon_a, \nabla^*_{\epsilon_b}\mathbf{v}) - g^*(\epsilon_b, \nabla^*_{\epsilon_a}\mathbf{v}) + O\left[\frac{1}{c^4}\right] . \tag{3.37b}$$

To prevent any ambiguity let us make clear that although the normalization of **v** may differ by 1PN terms, the **E** field calculated by (3.37a) is still accurate up to 2PN terms (on the other hand, all the formulas for **B** are accurate only up to 1PN terms).

The formulas (3.35) have related the components of the connection ∇^* with respect to the vectorial basis $(\mathbf{u}^*, \epsilon_a)$ to the previously introduced **E** and **B** fields. One can also then try to relate the components of the curvature of ∇^* to **E** and **B**. A straightforward calculation yields $[R^*(\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d}) \equiv R^*_{\alpha\beta\gamma\delta} a^\alpha b^\beta c^\gamma d^\delta$, $\partial_a = \partial/\partial X^a$, $\partial_T = c\partial/\partial X^0]$

$$-R^*(\mathbf{u}^*, \epsilon_a, \mathbf{u}^*, \epsilon_b) = \tfrac{1}{2}(\partial_a E^*_b + \partial_b E^*_a) + \frac{1}{c^2}\partial_T^2 W^* \delta_{ab} - \frac{3}{c^2} E^*_a E^*_b + \frac{1}{c^2}\delta_{ab} E^*_c E^*_c + O(4) . \tag{3.38}$$

The left-hand side of Eq. (3.38) can also be written in terms of the usual coordinate components as $+c^2 R^{*0}{}_{a0b}[1+O(4)]$. Tracing each side of Eq. (3.38) by a simple Kronecker δ^{ab} we get (in terms of contravariant natural components)

$$-c^2 R^{*00} = \partial_a E^*_a + \frac{3}{c^2}\partial_T^2 W^* + O(4) , \tag{3.39}$$

which is equivalent to Eq. (3.7a). The really new information in Eq. (3.38) is obtained by taking the (symmetric) trace-free projection (in the usual Euclidean sense) of both sides and reads

$$\mathrm{STF}_{ab}[-R^*(\mathbf{u}^*, \epsilon_a, \mathbf{u}^*, \epsilon_b)] = \partial_{\langle a} E^*_{b\rangle} - \frac{3}{c^2} E^*_{\langle a} E^*_{b\rangle} + O(4) . \tag{3.40}$$

An important lesson can be immediately learned from Eq. (3.38) or (3.40). Indeed, some authors, notably Thorne and Hartle,[24] heuristically motivated by the simple case of a non-self-gravitating test body embedded in some "external" gravitational field, have suggested that, even in the general case of a gravitationally self-interacting and externally interacting body, it might be useful to use the Riemann tensor of some (to be defined) "external metric" to characterize the tidal interaction of the considered body with the other bodies. As we have seen above the important role that the **E** field was playing in *linearizing* (in many senses) the *N*-body problem, we see now that the fact that the curvature components $R^{*0}{}_{a0b}$ are *nonlinear* in $\mathbf{E}^*$ (whatever be the choice of the metric g^*) makes them *ill-adapted* tools already at the 1PN approximation. Similar to the "electric-curvature" equation (3.40) one can check that the following "magnetic-curvature" equation holds:

$$+c^2 \epsilon_{bcd} R^*(\mathbf{u}^*, \epsilon_a, \epsilon_c, \epsilon_d) = \partial_a B^*_b - 2\epsilon_{abc}\partial_T E^*_c + O(2) . \tag{3.41}$$

Tracing each side of Eq. (3.41) gives Eq. (3.22a), while antisymmetrizing them with respect to the indices *ab* gives

$$-2c^3 R^{*0c} = \epsilon_{cab}\partial_a B^*_b - 4\partial_T E^*_c + O(2) , \tag{3.42}$$

which is equivalent to Eq. (3.7b) [see also Eq. (3.22c)]. The new information contained in Eq. (3.41) is obtained, as above, by taking a STF projection, and reads

$$\mathrm{STF}_{ab}[+c^2 \epsilon_{bcd} R^*(\mathbf{u}^*, \epsilon_a, \epsilon_c, \epsilon_d)] = \partial_{\langle a} B^*_{b\rangle} + O(2) .$$

Finally, all the geometrical results of this section apply, *mutatis mutandis*, to the link between the global **e** and **b** fields and geometry of the x^μ-coordinate grid.

IV. TRANSFORMATION PROPERTIES OF GRAVITATIONAL POTENTIALS

A. Definition of "locally generated" and "external" gravitational potentials as seen in a local reference system

In the previous section we have shown how the combined use of spatially isotropic coordinates and of the "exponential" parametrization (3.3) of the metric led to a linearization of the 1PN field equations. The central technical feature of our formalism is to fully take advantage of this linearity by using spatially isotropic coordinates, and an exponential parametrization in all the coordinate systems of interest, i.e., not only in the global coordinates x^μ, but also in each of the local coordinate systems X_A^α that we shall construct. In other words, we are writing not only Eqs. (3.3)–(3.5) with $\gamma_{ij}=\delta_{ij}$ but also, in each local reference frame (labeling index A omitted)

$$G_{00}=-\exp\left[-\frac{2}{c^2}W\right]+O(6)\ , \tag{4.1a}$$

$$G_{0a}=-\frac{4}{c^3}W_a+O(5)\ , \tag{4.1b}$$

$$G_{ab}=\delta_{ab}\exp\left[+\frac{2}{c^2}W\right]+O(4)\ , \tag{4.1c}$$

and therefore

$$\sqrt{-\det G_{\alpha\beta}}=\exp\left[+\frac{2}{c^2}W\right]+O(4)\ , \tag{4.2a}$$

$$G^{00}=-\exp\left[+\frac{2}{c^2}W\right]+O(6)\ , \tag{4.2b}$$

$$G^{0a}=-\frac{4}{c^3}W_a+O(5)\ , \tag{4.2c}$$

$$G^{ab}=\delta_{ab}\exp\left[-\frac{2}{c^2}W\right]+O(4)\ . \tag{4.2d}$$

The main achievement of this section will be to relate the gravitational four-potential $W_\alpha^A\equiv(W^A,W_a^A)$ describing the metric $G_{\alpha\beta}^A$ in the X_A^α local coordinate system to the global potential $w_\mu\equiv(w,w_i)$ describing $g_{\mu\nu}(x^\lambda)$. Let us first notice that W_α^A satisfy the linear system (3.11) written in X_A^α coordinates, i.e.,

$$\Box_X W^A+\frac{4}{c^2}\partial_T(\partial_T W^A+\partial_b W_b^A)=-4\pi G\Sigma_A+O(4)\ , \tag{4.3a}$$

$$\Delta_X W_a^A-\partial_a(\partial_T W^A+\partial_b W_b^A)=-4\pi G\Sigma_A^a+O(2)\ , \tag{4.3b}$$

where $\Box_X=\Delta_X-c^{-2}\partial_T^2$ with $\Delta_X=\partial^2/\partial X^a\partial X^a$, and where the source terms

$$\Sigma_A^\alpha\equiv(\Sigma_A,\Sigma_A^a)\equiv\left[\frac{T_A^{00}+T_A^{aa}}{c^2},\frac{T_A^{0a}}{c}\right] \tag{4.4}$$

are now defined by components of the stress-energy tensor in the X_A^α coordinate system. The only nonzero source terms in the right-hand side of the field equations (4.3) are those that correspond to the body A itself, the neighborhood of which we are studying. Let us consider a particular solution of the inhomogeneous equations (4.3) which is "locally generated" by the source terms Σ_A^α, for instance the local symmetric harmonic solution

$$W_\alpha^{A,\mathrm{loc}}\equiv W_\alpha^{+A}=\Box_{X,\mathrm{sym}}^{-1}(-4\pi G\Sigma_A^\alpha)\ , \tag{4.5}$$

which satisfies Eqs. (4.3) when Σ_A^α is approximately conserved,

$$\partial_T\Sigma_A+\partial_a\Sigma_A^a=O(2)\ , \tag{4.6}$$

a condition that we shall further consider below. In Eq. (4.5) $\Box_{X,\mathrm{sym}}^{-1}$ denotes the time-symmetric inverse of the X-coordinate wave operator [as defined by Eqs. (3.24) and (3.25b) with the replacement $(t,\mathbf{x})\to(T,\mathbf{X})$]. Note that we could also work in a gauge invariant manner by defining "locally generated" **E** and **B** fields as explicit functionals of Σ_A^α.

Having defined the "locally generated" piece of W_A^α by Eq. (4.5), we then define its "external" piece, denoted $W_\alpha^{A,\mathrm{ext}}$ or $\overline{W}_\alpha^A$ (or, in a gauge invariant way, the "external" **E** and **B** fields), by writing

$$W_\alpha^A\equiv W_\alpha^{A,\mathrm{loc}}+W_\alpha^{A,\mathrm{ext}}\equiv W_\alpha^{+A}+\overline{W}_\alpha^A\ . \tag{4.7}$$

In the domain of the local chart X_A^α (i.e., a domain which contains body A and no other body $B\neq A$) the external potentials satisfy homogeneous equations obtained by erasing the source terms Σ_A^α in Eqs. (4.3). These homogeneous equations take a simple form when expressed in terms of the external gauge invariant fields [introduced in Eqs. (3.21)]

$$\overline{E}_a\equiv E_a[\overline{W}]\equiv\partial_a\overline{W}+\frac{4}{c^2}\partial_T\overline{W}_a\ , \tag{4.8a}$$

$$\overline{B}_{ab}\equiv B_{ab}[\overline{W}]\equiv\epsilon_{abc}\overline{B}_c\equiv-4[\partial_a\overline{W}_b-\partial_b\overline{W}_a]\ . \tag{4.8b}$$

The latter quantities satisfy [see Eqs. (3.22)]

$$\nabla\cdot\overline{\mathbf{B}}=0\ , \tag{4.9a}$$

$$\nabla\times\overline{\mathbf{E}}=-\frac{1}{c^2}\partial_T\overline{\mathbf{B}}\ , \tag{4.9b}$$

$$\nabla\cdot\overline{\mathbf{E}}=-\frac{3}{c^2}\partial_T^2\overline{W}+O(4)\ , \tag{4.9c}$$

$$\nabla\times\overline{\mathbf{B}}=4\partial_T\overline{\mathbf{E}}+O(2)\ . \tag{4.9d}$$

B. Transformation laws of gravitational potentials and fields under a change of reference system

As recalled in the introduction an essential property of accelerated frames in Newtonian gravitational theory is that, with respect to such frames, the gravitational potential gets replaced by an "effective gravitational potential"

$$U^{\mathrm{eff}}(\mathbf{X})=U(\mathbf{z}(t)+\mathbf{X})-C(t)-\frac{d^2\mathbf{z}(t)}{dt^2}\cdot\mathbf{X}\ , \tag{4.10}$$

in which $C(t)$ is an arbitrary function of time and the last term represents the inertial forces linked to the acceleration of the origin of the frame $d^2\mathbf{z}/dt^2$. Note that Eq. (4.10) has taken into account the fundamental property of gravitational forces to be proportional to the same mass as the one that appears in Newton's basic law of dynamics. This equivalence of the "gravitational mass" with the "inertial mass" was put by Einstein at the basis of general relativity, and its direct effect in Eq. (4.10) is to lead to an effacement of the external gravitational potential down, essentially, to tidal forces (see Ref. 7 for a fuller discussion of the effacement properties, both in Newtonian and in Einsteinian theories).

In Einsteinian gravitational theory the single scalar potential U gets replaced by the ten components of the metric tensor $g_{\mu\nu}$. However, we have seen how a convenient choice of spatial coordinates (both in global and in local "accelerated" frames) allowed one to "gauge away" six components ($\gamma_{ij}=\delta_{ij}$), leaving us with only a scalar w and a vector w_i in each frame. The relativistic analogue of Eq. (4.10) is then obtained by inserting in the tensorial law of transformation,

$$g^{\mu\nu}(x)=\frac{\partial x^{\mu}(X)}{\partial X^{\alpha}}\frac{\partial x^{\nu}(X)}{\partial X^{\beta}}G^{\alpha\beta}(X) , \tag{4.11}$$

the exponential parametrizations (3.5) and (4.2), and our z-e-ξ form (2.8) with (2.10) and (2.36). A straightforward calculation leads to the following.

Theorem 3. *The local gravitational potentials $W_\alpha(X)\equiv(W,W_a)$ in any local reference system (label A omitted) are related to the global potentials $w_\mu(x)\equiv(w,w_i)$ by the transformation law*

$$w=\left[1+\frac{2}{c^2}V^aV^a\right]W+\frac{4}{c^2}V^aW_a+\frac{c^2}{2}\ln(A^0_0A^0_0-A^0_aA^0_a)+O(4) , \tag{4.12a}$$

$$w_i=v^iW+R^i_aW_a+\frac{c^3}{4}(A^0_0A^i_0-A^0_aA^i_a)+O(2) , \tag{4.12b}$$

where

$$v^i\equiv R^i_aV^a \quad\text{or}\quad V^a\equiv R^i_av^i , \tag{4.12c}$$

are the global and local components of the velocity of the origin of the local frame $v^i=dz^i/dt=dz^i/dT+O(2)$, and where A^μ_α are the components (2.10) of the Jacobian matrix.

We shall write the transformation law above as

$$w_\mu(x)=\mathcal{A}_{\mu\alpha}(T)W_\alpha(X)+\mathcal{B}_\mu(X) , \tag{4.12d}$$

to display its *affine* structure ($y=ax+b$), and clarify the fact that all the coefficients $\mathcal{A}_{\mu\alpha}$ are to be evaluated at $T=X^0/c$ with $X^\alpha\equiv(X^0,X^a)$ and $x^\mu\equiv(x^0,x^i)$ on either side of Eq. (4.12d) denoting the coordinates in two different charts of the same abstract event in spacetime [related by (2.8)].

It is to be remarked that the exponential parametrization of the metric succeeded again in yielding a simple linear property for the w's. In the following we shall also need the inverse of Eqs. (4.12) namely

$$W_\alpha=\mathcal{A}^{-1}_{\alpha\mu}(w_\mu-\mathcal{B}_\mu) , \tag{4.13a}$$

which reads explicitly

$$W=\left[1+\frac{2}{c^2}\mathbf{V}^2\right](w-\mathcal{B})-\frac{4}{c^2}v^i(w_i-\mathcal{B}_i)+O(4) , \tag{4.13b}$$

$$W_a=-V^a(w-\mathcal{B})+R^i_a(w_i-\mathcal{B}_i)+O(2) , \tag{4.13c}$$

where $\mathcal{B}$ [$\mathcal{B}_i$] denotes the inhomogeneous term in the right-hand side of Eqs. (4.12a) [(4.12b)], and where we used the fact that R^i_a is an orthogonal matrix ($R^i_aR^i_b=\delta_{ab}$). We shall see later how the $\mathcal{B}$ terms in Eqs. (4.13) closely resemble the "inertial" terms in Eq. (4.10) [including terms that play the role of the arbitrary function of time $C(t)$]. For the time being, let us complete the transformation law (4.13) for the potentials by the one for the gauge-invariant **E** and **B** fields. From the definitions

$$E_a[W]\equiv\partial_aW+\frac{4}{c^2}\partial_TW_a , \tag{4.14a}$$

$$B_a[W]\equiv\epsilon_{abc}\partial_b(-4W_c) , \tag{4.14b}$$

for the (total) **E** and **B** fields, and the transformation law (4.13) one can split **E** and **B** in two pieces:

$$\mathbf{E}=\mathbf{E}'+\mathbf{E}'',\quad \mathbf{B}=\mathbf{B}'+\mathbf{B}'' , \tag{4.15a}$$

with

$$\mathbf{E}'\equiv\mathbf{E}[\mathcal{A}^{-1}_{\alpha\mu}w_\mu],\quad \mathbf{E}''\equiv\mathbf{E}[-\mathcal{A}^{-1}_{\alpha\mu}\mathcal{B}_\mu],\ \text{etc.} \tag{4.15b}$$

The $\mathcal{B}$ terms ("inertial") fields **E**″ and **B**″ will be studied later; let us concentrate now on the (homogeneous) law of transformation for the local **E**′ and **B**′ fields, in terms of the global **e** and **b** fields defined by Eqs. (3.21) above. The only new element, with respect to Eqs. (4.13), in the calculation of **E**′ and **B**′ is the need for the transformation law of the partial derivatives:

$$\frac{\partial}{\partial T}=cA^\mu_0\frac{\partial}{\partial x^\mu}=ce^\mu_0\frac{\partial}{\partial x^\mu}+O(2) , \tag{4.16a}$$

$$\frac{\partial}{\partial X^a}=A^\mu_a\frac{\partial}{\partial x^\mu}=e^\mu_a\frac{\partial}{\partial x^\mu}+e^\mu_b\frac{\partial\Xi^b}{\partial X^a}\frac{\partial}{\partial x^\mu}+O(4) . \tag{4.16b}$$

Using our previous results and introducing a new time-dependent matrix

$$e'^i_a(T)\equiv e^i_a-\frac{e^0_ae^i_0}{e^0_0}=e^i_a-\frac{v^iV_a}{c^2}+O(4) , \tag{4.17}$$

Eqs. (4.16) can be written as

$$D_T\equiv\frac{\partial}{\partial T}=\partial_t+v^i\partial_i+O(2) , \tag{4.18a}$$

$$D_a\equiv\frac{\partial}{\partial X^a}=e'^i_a\partial_i+\frac{\partial\xi^i}{\partial X^a}\partial_i+\frac{V_a}{c^2}D_T+O(4) , \tag{4.18b}$$

where $\partial_t\equiv\partial/\partial t$, $\partial_i\equiv\partial/\partial x^i$ while $D_T\equiv\partial/\partial T$, $D_a\equiv\partial/\partial X^a$.

Finally, introducing for brevity the *spacetime-dependent* matrix

$$S^i_a(T,\mathbf{X})\equiv\left[1+\frac{2}{c^2}\mathbf{V}^2\right]\left[e'^i_a+\frac{\partial\xi^i}{\partial X^a}\right] , \tag{4.19}$$

we find that the $\mathcal{A}$ part of the gravitoelectric and gravitomagnetic fields in the local frame can be expressed in terms of the corresponding fields in the global frame as

$$E'_a = S^i_a \left[e_i + \frac{1}{c^2} b_{ij} v^j \right] + \frac{1}{c^2} V_a D_T w - \frac{4}{c^2} D_T (V_a w) + O(4) , \quad (4.20a)$$

$$B'_a = S^i_a (b_i - 4\epsilon_{ijk} v^j e_k) + O(2) . \quad (4.20b)$$

The last terms in Eq. (4.20a) which contain the "convective" derivative $D_T = \partial_t + v^i \partial_i + O(2)$ are gauge invariant because the scalar potential w changes only by $O(2)$ under a gauge transformation. In Eq. (4.20b) the matrix S^i_a can be replaced by the simple rotation matrix R^i_a, and we have assumed in deriving it that R^i_a was preserving the spatial orientation [i.e., that $\det(R^i_a) = +1$]. When comparing Eqs. (4.20) with the transformation laws under boosts of the usual electromagnetic fields one notices again the irreducible appearance of a factor 4 in the velocity terms:

$$\mathbf{e} + c^{-2} \mathbf{v} \times \mathbf{b}, \quad \mathbf{b} - 4\mathbf{v} \times \mathbf{e} .$$

One should remember that the left-hand sides of Eqs. (4.20) do not represent the full local fields, but that one must still add to them the $\mathcal{B}$-dependent (or "inertial") contributions $\mathbf{E}''$, $\mathbf{B}''$ that will be studied later.

C. Invariance properties of the time-symmetric Green's function for the d'Alembertian

We shall prove in this sub-section a remarkable technical result that will be very useful for extracting more information from the transformation laws just discussed.

Let us consider the various (flat-space) d'Alembertians (or wave operators) defined in each of our coordinate systems: namely the global-coordinate wave operator

$$\Box_x \equiv f^{\mu\nu} \frac{\partial^2}{\partial x^\mu \partial x^\nu} \equiv \sum_{i=1}^{3} \frac{\partial^2}{(\partial x^i)^2} - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} , \quad (4.21a)$$

and the local-coordinate ones (label A omitted)

$$\Box_X \equiv f^{\alpha\beta} \frac{\partial^2}{\partial X^\alpha \partial X^\beta} \equiv \sum_{a=1}^{3} \frac{\partial^2}{(\partial X^a)^2} - \frac{1}{c^2} \frac{\partial^2}{\partial T^2} . \quad (4.21b)$$

To each one of the wave operators (4.21) corresponds some Green's functions, i.e., some local solutions of

$$\Box_x G_x(x^\mu) = -4\pi \delta^4(x^\mu) , \quad (4.22a)$$

$$\Box_X G_X(X^\alpha) = -4\pi \delta^4(X^\alpha) , \quad (4.22b)$$

where the four-dimensional Dirac distributions are defined in each coordinate system by, e.g.,

$$\int d^4x \, \delta^4(x^\mu) \varphi(x^\mu) = \varphi(0) . \quad (4.23)$$

Note, that we are here working only with a differentiable manifold structure without any metric, or even volume element.

The Green's functions most used in classical physics are the "retarded" ones, e.g.,

$$G_{x,\mathrm{ret}}(x^\mu) \equiv \frac{\delta(x^0 - |\mathbf{x}|)}{|\mathbf{x}|} = 2Y(x^0) \delta(-(x^0)^2 + (\mathbf{x})^2) , \quad (4.24)$$

where Y denotes Heaviside's step function, and $|\mathbf{x}| \equiv \sqrt{(\mathbf{x})^2} \equiv \sqrt{x^i x^i}$. For our present purpose it will be simpler to work with the time-symmetric Green's function:

$$G_{x,\mathrm{sym}}(x) \equiv \tfrac{1}{2}(G_{x,\mathrm{ret}} + G_{x,\mathrm{adv}}) \equiv \frac{1}{2} \left[\frac{\delta(x^0 - |\mathbf{x}|)}{|\mathbf{x}|} + \frac{\delta(x^0 + |\mathbf{x}|)}{|\mathbf{x}|} \right] . \quad (4.25)$$

They can be simply written as one-dimensional Dirac distributions:

$$G_{x,\mathrm{sym}}(x) = \delta(-(x^0)^2 + \mathbf{x}^2) = \delta((x^\mu)^2) , \quad (4.26a)$$

$$G_{X,\mathrm{sym}}(X) = \delta(-(X^0)^2 + \mathbf{X}^2) = \delta((X^\alpha)^2) . \quad (4.26b)$$

The indice x or X on the Green's function (4.26) serve to remind us that the objects so defined depend on which coordinate system is used. For brevity we have also introduced the notation $(x^\mu)^2 \equiv -(x^0)^2 + (x^1)^2 + (x^2)^2 + (x^3)^2$. Let us emphasize again that no Riemannian metric structure is used here.

Let be given a "scalar density" on our differentiable manifold V_4, which is represented in each coordinate system by a scalar function, say $\sigma(x)$ and $\Sigma(X)$, with the transformation law

$$\sigma(x) = \left| \frac{\partial X}{\partial x} \right| \Sigma(X) \equiv J \Sigma(X) , \quad (4.27)$$

where

$$J \equiv \left| \frac{\partial X}{\partial x} \right| \equiv \det \left[\frac{\partial X^\alpha}{\partial x^\mu} \right] \quad (4.28)$$

denotes the Jacobian of the transformation $X^\alpha = F^\alpha(x^\mu)$.

In differential geometric terms Eq. (4.27) means that we are endowing our differential manifold V_4 with an intrinsic four-form

$$\omega = \sigma(x) dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 = \Sigma(X) dX^0 \wedge dX^1 \wedge dX^2 \wedge dX^3 . \quad (4.29)$$

We shall assume that ω has a spatially compact support which is contained in the domains of both the X^α and x^μ charts. The quantities σ and Σ have, *a priori*, nothing to do with what is denoted σ, Σ in the rest of the paper but this notation is used here to convey the meaning that $\omega = \sigma \, d^4x = \Sigma \, d^4X$ represents some kind of "mass distribution" in V_4.

Each "mass distribution" ω intrinsically given on V_4 will "generate" in each coordinate system an, *a priori*, *coordinate-dependent* time-symmetric "field"

$$\varphi_x(x^\mu) \equiv G_{x,\mathrm{sym}} * \omega \equiv \int G_{x,\mathrm{sym}}(x^\mu - x'^\mu) \omega(x') = \int \delta[(x^\mu - x'^\mu)^2] \sigma(x'^\mu) d^4x' , \quad (4.30a)$$

$$\varphi_X(X^\alpha)\equiv G_{X,\text{sym}}*\omega = \int \delta[(X^\alpha-X'^\alpha)^2]\Sigma(X'^\alpha)d^4X' . \quad (4.30b)$$

We can now state the following invariance property.

Theorem 4. *With the notation just introduced, if the "mass distribution" ω is "slowly changing" (in the sense, e.g., $\Sigma(X^\alpha)$ is a smooth function of the variable $T\equiv X^0/c$ when $c\to\infty$) and if the coordinate transformation $x^\mu=f^\mu(X^\alpha)$ satisfies both the slow motion assumptions (2.13) and the conditions (2.30) expressing the conservation of the strong spatial isotropy, then the coordinate-dependent time-symmetric fields generated by $\omega=\sigma d^4x=\Sigma d^4X$ coincide up to terms of second post-Newtonian order*

$$\varphi_x(f^\mu(X^\alpha))=\varphi_X(X^\alpha)[1+O(4)] , \quad (4.31)$$

where $f^\mu(X^\alpha)\equiv x^\mu(X^\alpha)$ and X^α denote the coordinates of any spacetime point in the intersection of the domains of the two charts.

Proof of Theorem 4. We have identically

$$\Sigma(X^0,X^a)=\int d^3Z\int dS\, \Sigma(S,Z^a)\delta^4(X^\alpha-Z^\alpha(S,Z^a)) , \quad (4.32)$$

where $Z^\alpha(S,Z^a)$ denotes the equations of the parametrized congruence of world lines $\mathcal{L}_{Z^a}$ canonically defined by the X coordinate system, i.e.,

$$\mathcal{L}_{Z^a}=\{P\in V_4;\ X^\alpha(P)=Z^\alpha(S,Z^a) \text{ with } S\in\mathbb{R}\} , \quad (4.33a)$$

with $Z^0(S,Z^a)=S$, $Z^a(S,Z^a)=Z^a$. The x-coordinate equations for the intrinsic congruence $\mathcal{L}_{Z^a}$ read

$$\mathcal{L}_{Z^a}=\{P\in V_4; x^\mu(P)=z^\mu(S,Z^a)\}$$

with (4.33b)

$$z^\mu(S,Z^a)\equiv f^\mu(Z^\alpha(S,Z^a)) .$$

Note that this $z^\mu(S,Z^a)$ agrees with the central world line only for $Z^a=0$. The first step of our proof is to notice that the definition of the four-dimensional Dirac distribution as a pure coordinate density, (4.23), ensures that

$$\sigma(x^\mu)=\int d^3Z\int dS\, \Sigma(S,Z^a)\delta^4(x^\mu-z^\mu(S,Z^a)) , \quad (4.34)$$

satisfies the transformation law (4.27) and represents therefore the "mass density" in the x^μ-coordinate system. Let us now prove that the invariance property (4.31) holds for "linear mass distributions" of the type

$$\Sigma_{\mathcal{L}_Z}(X)=\int dS\, F(S)\delta^4(X^\alpha-Z^\alpha(S,\mathbf{Z})) \Longrightarrow \sigma_{\mathcal{L}_Z}(x)=\int dS\, F(S)\delta^4(x^\mu-z^\mu(S,\mathbf{Z})) . \quad (4.35)$$

Because of (4.32) and (4.34), the general property (4.31) will follow by superposition of such linear distributions [i.e., by integration over d^3Z with $F=\Sigma(S,\mathbf{Z})$]. The time-symmetric fields generated by the linear distributions (4.35) will be denoted by $\Phi_{x,F,\mathcal{L}}$ and $\Phi_{X,F,\mathcal{L}}$. Using Eqs. (4.26) they can be written as

$$\Phi_{x,F,\mathcal{L}_Z}(x^\mu)=\int dS\, F(S)\delta([x^\mu-z^\mu(S,\mathbf{Z})]^2) , \quad (4.36a)$$

$$\Phi_{X,F,\mathcal{L}_Z}(X^\alpha)=\int dS\, F(S)\delta([X^\alpha-Z^\alpha(S,\mathbf{Z})]^2) , \quad (4.36b)$$

where we recall that the squares in Eqs. (4.36) denote the (coordinate) Minkowskian square

$$(a^\mu)^2\equiv f_{\mu\nu}a^\mu a^\nu\equiv -(a^0)^2+(a^1)^2+(a^2)^2+(a^3)^2 . \quad (4.36c)$$

Because of the simpler (linear) dependence of $Z^\alpha(S,\mathbf{Z})$ on S [see Eq. (4.33a)], Eq. (4.36b) is immediately integrated [see Eq. (4.25)] and yields the simple exact result

$$\Phi_X(X^\alpha)=\frac{F(T\mp R/c)}{R} , \quad (4.37)$$

where we have put

$$T\equiv X^0/c,\quad R\equiv|\mathbf{X}-\mathbf{Z}| , \quad (4.38)$$

and we have used the abbreviated notation (3.30) for the symmetric half-sum (ret+adv)/2. Note that by a slight inconsistency of notation we are denoting by $F(T)$ in Eq. (4.37) what was denoted $F(cT)$ in Eqs. (4.36). By the "slow changing" assumption of Theorem 4 we can expand (4.37) in powers of c^{-1} and get the 2PN approximate result

$$\Phi_X(X^\alpha)=\frac{F(T)}{R}+\frac{1}{2c^2}\frac{d^2F(T)}{dT^2}R+O\left[\frac{1}{c^4}\right] . \quad (4.39)$$

As for the time-symmetric field generated in the x-coordinate system its exact expression is obtained from Eq. (4.36a) by using the formula

$$\delta[f(S)]=\sum_n\frac{\delta(S-S_n)}{|df/dS|_n} , \quad (4.40)$$

in which S_n $(n=1,2,\dots)$ labels all the zeros of $f(S)$: $f(S_n)=0$. In the present case there are two zeros of $f(S)=[x^\mu-z^\mu(S,\mathbf{Z})]^2$ that we shall denote by $S_\pm$ (no half-sum over $+/-$):

$$S_\pm(x^\mu,Z^a):\ f_{\mu\nu}[x^\mu-z^\mu(S_\pm,\mathbf{Z})][x^\nu-z^\nu(S_\pm,\mathbf{Z})]=0 . \quad (4.41)$$

Geometrically S_+ (S_-) is the parameter of the intersection of the future (past) x-coordinate Minkowskian cone emitted by x^μ with the world line $\mathcal{L}_{Z^a}$. S_+ and S_- are functions of x^μ and Z^a. Introducing also the notation

$$\rho(x^\mu,S,Z^a)\equiv\left|f_{\mu\nu}(x^\mu-z^\mu(S,Z^a))\frac{\partial z^\nu(S,Z^a)}{\partial S}\right| \quad (4.42)$$

for the Minkowskian scalar product between $x^\mu-z^\mu$ and the S-tangent vector to $\mathcal{L}_Z$: $u^\nu=\partial z^\nu/\partial S$, we can write the following exact expression for the time-symmetric x-coordinate field generated by (4.35):

$$\Phi_x(x^\mu)=\left[\frac{F(S)}{\rho(x,S)}\right]_\pm\equiv\frac{1}{2}\left[\frac{F(S_+)}{\rho(x,S_+)}+\frac{F(S_-)}{\rho(x,S_-)}\right] . \quad (4.43)$$

This type of Minkowski-invariant representation of retarded or advanced fields generated by a relativistic point mass is well known and dates back to formulas due to Liénard and Wiechert.[45] In order to compare the exact expression (4.43) with its exact X-system correspondent (4.37) we need to perform some kind of expansion. There are two expansion procedures we could use: (i) the Minkowski-covariant expansion in the curvature of the world line $\mathcal{L}$ which expands around the Minkowski-orthogonal projection of x^μ on $\mathcal{L}$, i.e., $z_\perp^\mu \in \mathcal{L}$ such that $f_{\mu\nu}(x^\mu - z_\perp^\mu)\partial z_\perp^\nu/\partial S = 0$; or (ii) a direct "Lagrange expansion" in powers of c^{-1}. The first type of expansion was used by many authors studying pointlike sources in Poincaré-invariant theories, including the gravitational case treated by post-Minkowskian expansions.[49–53] The second type of expansion, named after a famous theorem of Lagrange,[54] is directly related with the type of slow-motion expansions used for retarded integrals of continuous sources. We shall here take advantage of both approaches and devise a convenient Minkowski-motivated form of Lagrange expansion.

The Lagrange expansions of the advanced or retarded field ($\epsilon = +1$ or -1), is simply obtained from the usual non-covariant form (4.24) of the Green's function,

$$\Phi_{x,\epsilon}(x) = \int dS\, F(S) \frac{\delta(x^0 - z^0(S) + \epsilon|\mathbf{x} - \mathbf{z}(S)|)}{|\mathbf{x} - \mathbf{z}(S)|} , \tag{4.44}$$

by formally expanding the δ function in powers of ϵ, and integrating over dz^0 instead of over dS. This leads to

$$\Phi_{x,\epsilon}(t,\mathbf{x}) = \sum_{n=0}^{\infty} \frac{\epsilon^n}{n!c^n} \frac{d^n}{dt^n} [f(t)|\mathbf{x} - \mathbf{z}(t)|^{n-1}] , \tag{4.45}$$

where

$$f(t) \equiv \left[F(S(z^0)) \frac{dS}{dz^0} \right]_{z^0 = ct} . \tag{4.46}$$

The time-symmetric solution is then simply

$$\Phi_x(t,\mathbf{x}) = \frac{f(t)}{|\mathbf{x} - \mathbf{z}(t)|} + \frac{1}{2c^2} \frac{d^2}{dt^2} [f(t)|\mathbf{x} - \mathbf{z}(t)|] + O(1/c^4) . \tag{4.47}$$

Note, that the expansions (4.45), (4.46) are performed around the point on the world line which is $\mathbf{x}$-coordinate simultaneous with the field point x^μ: i.e., such that $x^0 = z^0$. We can now take advantage of the fact that the exact field Φ_x, Eq. (4.43), is Poincaré invariant (with respect to the fictitious x-coordinate Poincaré transformations) to choose, for any fixed field point, a Lorentz frame in which the world-line velocity $\mathbf{v}(t) = d\mathbf{z}/dt$ is zero at $t = x^0/c$ (we are here joining the first expansion method around a Minkowski orthogonal $z_\perp^\mu$ projection of x^μ onto $\mathcal{L}$). This simplifies the explicit form of Eq. (4.47) which becomes, denoting the acceleration $d^2\mathbf{z}/dt^2 \equiv \mathbf{a}(t)$,

$$\Phi_{x,\perp}(t,\mathbf{x}) = \frac{f(t)}{|\mathbf{x} - \mathbf{z}(t)|} - \frac{1}{2c^2} f(t)\mathbf{a}(t) \cdot \frac{\mathbf{x} - \mathbf{z}(t)}{|\mathbf{x} - \mathbf{z}(t)|} + \frac{1}{2c^2} \frac{d^2 f}{dt^2} |\mathbf{x} - \mathbf{z}(t)| + O(1/c^4) . \tag{4.48}$$

Using now the explicit coordinate transformation formula (2.37):

$$x^\mu(X^\alpha) = f^\mu(X^\alpha) = z^\mu(X^0) + e_a^\mu(X^0) Y^a(X^0, X^b) + O(3,4) , \tag{4.49}$$

we first see that the x^μ rest-frame condition $\mathbf{v} = 0$ ensures $e_a^0 = O(3)$ and therefore the fact that the corresponding events in the X-coordinate system are also simultaneous $[X^0 = cT = S + O(4)]$. Let us then put

$$R^a \equiv X^a - Z^a \quad \text{and} \quad S^a \equiv Y^a(X^0, X^a) - Y^a(X^0, Z^a) ,$$

so that

$$x^i - z^i(t) = f^i(X^0, X^a) - f^i(X^0, Z^a) = e_a^i(X^0) S^a ,$$

where we recall that we are considering the world-line $\mathcal{L}_{Z^a}$ image of $X^0 = S$, $X^a = Z^a$ under $x^\mu = f^\mu(X^\alpha)$, i.e., $z^\mu(S, Z^a) = f^\mu(S, Z^a)$.

Because of Eq. (2.36d) we have $e_0^0(T) e_a^i(T) = R_a^i + O(4)$ when $v^i = 0$, so that $e_0^0 |\mathbf{x} - \mathbf{z}(t)| = |\mathbf{S}|$. On the other hand, a simple calculation from the definition (2.38) of $Y^a(T, \mathbf{X})$ gives

$$|\mathbf{S}| = \left[1 - \frac{1}{2c^2} \mathbf{A} \cdot \mathbf{R} - \frac{1}{c^2} \mathbf{A} \cdot \mathbf{Z} \right] |\mathbf{R}| + O(4) .$$

Moreover, the definition (4.46) of $f(t)$, taken along the world line $\mathcal{L}_{Z^a}$ where

$$\frac{dz^0}{dS} = \frac{\partial f^0(S, Z^a)}{\partial S} = e_0^0 + \frac{de_a^0}{dS} Y^a(S, Z^b) + O(4) ,$$

gives

$$f(t) = \frac{F(T)}{e_0^0(T)(1 + c^{-2} \mathbf{A} \cdot \mathbf{Z})} + O(4) .$$

Putting together these results we see that the first two terms in the right-hand-side of Eq. (4.48) simplify to

$$\frac{f(t)(1 - \mathbf{A} \cdot \mathbf{R}/2c^2)}{|\mathbf{x} - \mathbf{z}|} = \frac{F(T)(1 - \mathbf{A} \cdot \mathbf{R}/2c^2)}{|\mathbf{S}|(1 + \mathbf{A} \cdot \mathbf{Z}/c^2)} = \frac{F(T)}{|\mathbf{R}|} + O(4)$$

which is the first term in the right-hand side of the c^{-1} expansion of Φ_X, Eq. (4.39). The remaining terms

$$\frac{1}{2c^2} \ddot{f} |\mathbf{x} - \mathbf{z}| = \frac{1}{2c^2} R \ddot{F} + O(4)$$

match clearly also modulo $O(4)$. We have thereby proven that the time-symmetric fields generated by pointlike source distributions coincide,

$$\Phi_{x,F,\mathcal{L}_Z}(x^\mu(X^\alpha)) = \Phi_{X,F,\mathcal{L}_Z}(X^\alpha) + O(4) , \tag{4.50}$$

independently of the choice of the Green's function (global coordinate $\Box_x^{-1}$ or local coordinate $\Box_X^{-1}$). As explained above, by integration over Z^a, with $F(S) = \Sigma(S, Z^a)$ the "pointlike case" implies the general "continuous case" (4.31). This completes the proof of Theorem 4.

D. "Detailed" transformation laws of gravitational potentials under a change of reference system

We have seen in Sec. IV B that the complete gravitational potentials w_μ and W_α (which include the effects of all the bodies in the system, and for W_α also the "inertial" effects of the change of frame) were very simply related to the affine transformation law (4.12). The technical result of the last subsection will allow us now to split the transformation law (4.12) in two more detailed transformation laws concerning the "locally generated" and the "external" potentials, which constitute the heart of our approach.

Theorem 5. *For each local reference system X_A^α the "locally generated" X_A-system potential (harmonic gauge)*

$$W_\alpha^{A,\mathrm{loc}} \equiv W_\alpha^{+A} \equiv \Box^{-1}_{X,\mathrm{sym}}(-4\pi G \Sigma_\alpha^A) \,, \tag{4.51}$$

is homogeneously related to the A-generated piece of the global potential,

$$w_\mu^A \equiv \Box^{-1}_{x,\mathrm{sym}}(-4\pi G \sigma_\mu^A) \,, \tag{4.52}$$

through

$$w_\mu^A(x) = \mathcal{A}^A_{\mu\alpha}(X^0) W_\alpha^{+A}(X) + O(4,2) \tag{4.53}$$

while the "external" X_A-system potential

$$W_\alpha^{A,\mathrm{ext}} \equiv \overline{W}{}_\alpha^A \equiv W_\alpha^A - W_\alpha^{+A} \tag{4.54}$$

is inhomogeneously related to the part of the global potential generated by all the external bodies $B \neq A$:

$$\sum_{B\neq A} w_\mu^B(x) = \mathcal{A}^A_{\mu\alpha}(X^0)\overline{W}{}_\alpha^A(X) + \mathcal{B}_\mu^A(X) + O(4,2) \,. \tag{4.55}$$

The A-transformation coefficients $\mathcal{A}^A_{\mu\alpha}$, $\mathcal{B}^A_\mu$ are the ones defined in Eqs. (4.12).

Proof of Theorem 5. Using Eq. (3.10) for the mass density σ, written in the form

$$\sigma = c^{-2}\sqrt{g}\,(-T + 2T_i^i) \,,$$

where $T \equiv g_{\mu\nu}T^{\mu\nu}$ is the covariant trace of the global-coordinate stress-energy tensor components $T^{\mu\nu}$, together with the tensorial transformation law of the components of $T^{\mu\nu}$ between the x and the X coordinates it is easy to get the transformation law for the mass density:

$$\sigma = \left|\frac{\partial X}{\partial x}\right| \left[\left[1 + \frac{2}{c^2}\mathbf{V}^2\right]\Sigma + \frac{4}{c^2}V_a\Sigma^a\right] + O(4) \,. \tag{4.56a}$$

It is even simpler to check that the mass-current density transforms as

$$\sigma^i = v^i\Sigma + R_a^i\Sigma^a + O(2) \,. \tag{4.56b}$$

Comparing with Eqs. (4.12), and using $|\partial X/\partial x| = 1 + O(2)$, this shows that

$$\sigma_\mu^A(x) = \left|\frac{\partial X}{\partial x}\right| \mathcal{A}^A_{\mu\alpha}(X^0)\Sigma_\alpha^A(X) + O(4,2) \,. \tag{4.56c}$$

The invariance property of the Green's function for the wave operator proven in the previous subsection allows one to deduce from Eq. (4.56c) that

$$\Box^{-1}_{x,\mathrm{sym}}(\sigma_\mu^A) = \Box^{-1}_{X,\mathrm{sym}}[\mathcal{A}^A_{\mu\alpha}(T)\Sigma_\alpha^A] + O(4,2) \,. \tag{4.57}$$

Making now use of the fact that the time-derivatives of $\mathcal{A}_{\mu\alpha}$ are of order $O(c^{-2})$ when $\mu=0$, and $O(c^0)$ when $\mu=i$, the slow-motion expansion of $G_{X,\mathrm{sym}} = -4\pi\Box^{-1}_{X,\mathrm{sym}}$,

$$-4\pi\Box^{-1}_{X,\mathrm{sym}}[F(X^\alpha)] = \int d^3X' \frac{F(T,\mathbf{X}')}{|\mathbf{X}-\mathbf{X}'|} + \frac{1}{2c^2}\int d^3X' \frac{\partial^2 F(T,\mathbf{X}')}{\partial T^2}|\mathbf{X}-\mathbf{X}'| + O(1/c^4) \,, \tag{4.58}$$

shows that one can, modulo $O(4,2)$, factor $\mathcal{A}^A_{\mu\alpha}(T)$ out of Eq. (4.57) which leads directly to the detailed transformation law (4.53). The remaining law (4.55) follows then by differencing with the complete law (4.12).

As a comment, let us emphasize that although it has been convenient to define both w_μ^A and W_α^{+A} in a particular (harmonic) gauge we could also have worked entirely in a gauge-invariant manner by applying Theorem 4 to the propagation equations for the gauge-invariant **e** and **b** fields [Eqs. (3.23) above].

Theorem 5 is central in our method, and its consequences will be explored in the following sections.

V. GENERAL FORMULATION OF THE METHOD

As recalled in the introduction the central difficulty in general-relativistic celestial mechanics is to deal simultaneously, and with equal accuracy, with $N+1$ problems: (i) the external problem (roughly speaking "motion of the centers of mass" of the N bodies), and (ii) the N internal problems ("motion of each body in its center-of-mass frame"). The results that we have obtained in the preceding sections will allow us to define now a new exact approach to post-Newtonian celestial mechanics in which these $N+1$ problems are formulated simultaneously, each one being formulated in its natural reference frame. In order to clarify the logic of our approach we shall first state explicitly which structures we need in our abstract differentiable manifold to set up the problem.

A. World-line data

Let us start by assuming that the following structures are given in our originally structureless differentiable manifold V_4.

Datum 0. N (abstract) world lines $\mathcal{L}_A$.

Datum 1. The global x^μ-coordinate representation of these world lines, i.e., either $x^\mu = z_A^\mu(\tau_A)$ with $c\,d\tau_A = (-f_{\mu\nu}dz_A^\mu dz_A^\nu)^{1/2}$ or $x^0 = ct$, $x^i = z_A^i(t)$.

Datum 2. A special parametrization of each world line by a parameter S_A, i.e., $x^\mu = z_A^\mu(S_A)$.

Datum 3. Three time-dependent quantities $\epsilon_a^A(S_A)$ along each $\mathcal{L}_A$.

Datum 4. A slowly changing (special) orthogonal matrix along $\mathcal{L}_A$:

$$R^i_{Aa}(S_A):\ R^i_{Aa}R^i_{Ab} = \delta_{ab} \,,$$

$$\det(R^i_{Aa}) = +1, \quad \frac{dR^i_{Aa}(S_A)}{c^{-1}dS_A} = O(c^{-2}) \,.$$

Datum 5. One function of four variables associated with each $\mathcal{L}_A$, $\xi_A(X^0,X^1,X^2,X^3)$, which is at least quadratic in the last three variables X^a when $X^a \to 0$.

We shall symbolize the set 0–5 of these world-line data by $\mathcal{D}_A$. The structures $\mathcal{D}_A$ being given we can uniquely define N local X_A^α-coordinate systems by means of the transformation formula (label A omitted)

$$x^\mu(X^\alpha)=f^\mu_{\mathcal{D}}(X^\alpha)=z^\mu(X^0)+e^\mu_a(X^0)X^a+\xi^\mu(X^0,X^a) , \tag{5.1a}$$

where $z^\mu(X^0)$ is the value of the S-parametrized function $z^\mu(S)$ at $S=X^0$, $e^i_a(S)$ is defined by Eq. (2.36d) with $e^0_0=dz^0/dS$ and $v^i \equiv ce^i_0 \equiv c\, dz^i/dS$, $e^0_a(S)$ is defined by

$$e^0_a(S)\equiv e^i_a \frac{dz^i}{dS}+\frac{1}{c^3}\epsilon_a(S) , \tag{5.1b}$$

while

$$\xi^i(X^0,X^a)\equiv\frac{1}{c^2}e^i_a(X^0)[\tfrac{1}{2}A_a(X^0)X^bX^b-A_b(X^0)X^bX^a] , \tag{5.1c}$$

$$\xi^0(X^0,X^a)\equiv\frac{1}{c^3}\xi(X^0,X^a) , \tag{5.1d}$$

where

$$A_a(S)\equiv f_{\mu\nu}e^\mu_a(S)\frac{d^2z^\nu}{d\tau^2} , \tag{5.1e}$$

with $c^2d\tau^2=-f_{\mu\nu}dz^\mu dz^\nu$. From Eqs. (5.1) one can then compute, for each body A, the transformation coefficients $\mathcal{A}^A_{\mu\alpha}(T)$ and $\mathcal{B}^A_\mu(T,\mathbf{X})$ [using Eqs. (2.10) for the Jacobian matrix elements].

Having explicated the way to connect the X_A-coordinate descriptions to the "common-view" x^μ-coordinate one, let us consider the evolution equations for the material distributions of each body as seen in its own local frame.

B. Energy-momentum evolution equations in each local reference system

Let us show how one can express in terms of the objects introduced in the previous sections the partial differential equations that represent the exchange of energy and momentum between each volume element of the material system and the gravitational field that it experiences. The general form of these equations is well known to be

$$0=\nabla_\nu T^{\mu\nu}=\partial_\nu T^{\mu\nu}+\Gamma^\mu_{\lambda\nu}T^{\lambda\nu}+\Gamma^\nu_{\lambda\nu}T^{\mu\lambda} , \tag{5.2a}$$

where

$$\Gamma^\lambda_{\mu\nu}=\tfrac{1}{2}g^{\lambda\sigma}(\partial_\mu g_{\nu\sigma}+\partial_\nu g_{\mu\sigma}-\partial_\sigma g_{\mu\nu}) \tag{5.2b}$$

denote the Christoffel symbols, i.e., the components of the Levi-Civita connection ∇ with respect to the coordinate frame ∂_μ. Let us first emphasize the fact that the contravariant components of the stress-energy tensor $T^{\mu\nu}$ are 1PN gauge invariant. Indeed, one checks easily that, under a 2PN change of the time variable,

$$x'^0=x^0+\xi^0(x^\mu) \text{ with } \xi^0=O(c^{-3}) ,$$

the Lie derivative of $T^{\mu\nu}$ is of order

$$\mathcal{L}_\xi T^{\mu\nu}=O\left[\frac{1}{c^2},\frac{1}{c^3},\frac{1}{c^4}\right]=O\left[\frac{1}{c^4}\right]O(c^2,c^1,c^0) ,$$

i.e., of 2PN fractional order [remembering the PN assumptions (2.25) for $T^{\mu\nu}$]. This property of gauge invariance does not hold for the mixed components T^μ_ν, $\mathcal{L}_\xi T^0_i$ being of 1PN fractional order only. Consistently with our notation above, let us define, in each local X frame (label A omitted),

$$\Sigma(T,\mathbf{X})\equiv\left[\frac{T^{00}+T^{bb}}{c^2}\right]_{\text{in } X \text{ frame}} , \tag{5.3a}$$

$$\Sigma^a(T,\mathbf{X})\equiv\left[\frac{T^{0a}}{c}\right]_{\text{in } X \text{ frame}} , \tag{5.3b}$$

$$E_a[W]\equiv\partial_a W+\frac{4}{c^2}\partial_T W_a , \tag{5.4a}$$

$$B_{ab}[W]\equiv\epsilon_{abc}B_c\equiv\partial_a(-4W_b)-\partial_b(-4W_a) , \tag{5.4b}$$

as well as the new notation

$$F^a[W]\equiv\Sigma E_a[W]+\frac{1}{c^2}B_{ab}[W]\Sigma^b , \tag{5.5a}$$

i.e., in vectorial notation,

$$\mathbf{F}\equiv\Sigma\mathbf{E}+\frac{1}{c^2}\mathbf{\Sigma}\times\mathbf{B} . \tag{5.5b}$$

The quantity $\mathbf{F}$ has precisely the form of the usual Laplace-Lorentz force density in electromagnetic theory. It can be thought of as playing also the role of a "gravitational force density" at the 1PN approximation of general relativity because an explicit calculation of Eqs. (5.2) yields the following 1PN evolution equations for (Σ,Σ^a):

$$\frac{\partial}{\partial T}\left[\left[1+\frac{4}{c^2}W\right]\Sigma^a\right]+\frac{\partial}{\partial X^b}\left[\left[1+\frac{4}{c^2}W\right]T^{ab}\right]=F^a(T,\mathbf{X})+O(4) , \tag{5.6a}$$

$$\frac{\partial}{\partial T}\Sigma+\frac{\partial}{\partial X^a}\Sigma^a=\frac{1}{c^2}\frac{\partial}{\partial T}T^{bb}-\frac{1}{c^2}\Sigma\frac{\partial}{\partial T}W+O(4) . \tag{5.6b}$$

In the particular case of an *isentropic* perfect fluid (with given equation of state) Eqs. (5.6) constitute, when the W potentials are given, a complete set of evolution equations for the matter variables. In more general cases (nonisentropic perfect fluid, elastic material,. . .) one may need to add some other material evolution (and/or constitutive) equations (e.g., some entropy evolution law) to get a (formally) complete system of evolution equations for the material system in the gravitational potentials W_α. We shall symbolically write such a material evolution system as

$$\frac{\partial\Sigma^\alpha_A}{\partial T_A}=\mathcal{F}^\alpha_A[W^A_\beta] . \tag{5.7}$$

The material evolution system (as seen in the local X_A-frame) (5.7) needs to be completed by equations determining the simultaneous evolution of the gravitational variables.

C. A formally closed evolution system for the $\Sigma_\alpha^A(T_A,\mathbf{X}_A)$'s

The gravitational potentials that are felt by each body A in its own frame [i.e., the ones that appear in Eq. (5.7)] have been found to be equal to (Theorem 5) the sum of a locally generated potential, W_α^{+A}, Eq. (4.51), and of $\overline{W}{}_\alpha^A$ obtained by solving Eq. (4.55), i.e.,

$$\overline{W}{}_\alpha^A=\mathcal{A}_{\alpha\mu}^{A(-1)}\left[\sum_{B\neq A} w_\mu^B(x)-\mathcal{B}_\mu^A(X)\right] .$$

Moreover, each $w_\mu^B(x)$ is generated by the matter currents of body B according to (in harmonic gauge)

$$w_\mu^B(x)=\Box_{x,\mathrm{sym}}^{-1}[-4\pi G\sigma_\mu^B(x)] .$$

We can also relate the x^μ-frame description of the material content of body B to its own local B-frame description through [see Eq. (4.56)]

$$\sigma_\mu^B(x)=\left|\frac{\partial X_B}{\partial x}\right|\mathcal{A}_{\mu\beta}^B(X_B^0)\Sigma_\beta^B(X_B) .$$

Putting together these results, we reach the conclusion that W_α^A can finally be expressed entirely as a functional of the $\Sigma_\alpha^A(X_A)$ and of the $\Sigma_\beta^B(X_B)$. Namely, we can write

$$W_\alpha^A(X_A)=W_\alpha^{+A}(X_A)+\overline{W}{}_\alpha^A(X_A)+O(4,2) , \quad (5.8a)$$

with

$$W_\alpha^{+A}(X_A)=\Box_{X,\mathrm{sym}}^{-1}[-4\pi G\Sigma_\alpha^A(X_A)] , \quad (5.8b)$$

$$\overline{W}{}_\alpha^A(X_A)=\sum_{B\neq A} W_\alpha^{B/A}(x^\mu(X_A))+W_\alpha^{\prime\prime A}(X_A) , \quad (5.8c)$$

$$W_\alpha^{B/A}(x^\mu)\equiv\Box_{x,\mathrm{sym}}^{-1}\left[-4\pi G\left|\frac{\partial X_B(x)}{\partial x}\right|\mathcal{A}_{\alpha\mu}^{A(-1)}(T_B)\times\mathcal{A}_{\mu\beta}^B(T_B)\Sigma_\beta^B(X_B(x))\right] , \quad (5.8d)$$

$$W_\alpha^{\prime\prime A}(X_A)\equiv-\mathcal{A}_{\alpha\mu}^{A(-1)}(T_A)\mathcal{B}_\mu^A(X_A) . \quad (5.8e)$$

In Eq. (5.8d) the 1PN dependence of $\mathcal{A}_{0\mu}^{A(-1)}$ on T_A, and the fact that all times $(t\equiv x^0/c,T_A,T_B)$ differ only by 1PN terms has allowed us to bring in $\mathcal{A}_{\alpha\mu}^{A(-1)}$ and to replace T_A by T_B. In other words, if we insert Eqs. (5.8) in Eqs. (5.7) we get a formally closed integro-differential evolution system for the material distributions of the N bodies, each one being described in its own frame, of the form

$$\frac{\partial\Sigma_\alpha^A(T_A,\mathbf{X}_A)}{\partial T_A}=\Phi_\alpha^A[\Sigma^A(T_A',\mathbf{X}_A'),\Sigma^B(T_B',\mathbf{X}_B'),\mathcal{D}_A,\mathcal{D}_B] , \quad (5.9a)$$

$$\frac{\partial\Sigma_\beta^B(T_B,\mathbf{X}_B)}{\partial T_B}=\Phi_\beta^B[\Sigma^B(T_B',\mathbf{X}_B'),\Sigma^A(T_A',\mathbf{X}_A'),\mathcal{D}_B,\mathcal{D}_A] , \quad (5.9b)$$

The various primes on the space-time variables in Eqs. (5.9) remind us of the fact that the evolution equations for, say, $\Sigma_\alpha^A(T_A,\mathbf{X}_A)$ are, *a priori,* nonlocal both in time and in space (because of the Green's functions $\Box^{-1}$).

D. Outline of the definition of a closed evolution system for both the Σ_α^A's and the $\mathcal{D}_A$'s

Note that in the evolution system (5.9) we are assuming that the world-line data $\mathcal{D}_A, \ldots,$ are given. This is certainly allowed because there is no logical necessity to connect in a definite way the world-line data to the actual motion of the bodies. For instance if we were interested in the quasicircular motions of a binary star, we could define in advance two world lines representing two exactly circular motions, together with some convenient rotation matrices and study the solutions of the system (5.9), i.e., the motion of each star with respect to the prescribed circular motion. Such an approach can, and has been, employed in a Newtonian context too. However, it is often convenient, in Newtonian gravity, to relate in a definite way the motion of the origin of the local A frame to the actual motion of body A, namely by identifying it with the Newtonian center of mass of A. However, there is no consensus on what is a good relativistic definition of a "center of mass" of a body A, member of an N-body system, and we shall need to develop new tools to find a definition which fits nicely within the present approach. Anticipating on what will be discussed in detail below, let us only quote the definition we shall use of a post-Newtonian "center-of-mass frame."

Definition of a local center-of-mass frame. A local X_A coordinate system around body A will be said to have its spatial origin coinciding (for all T_A times) with the center of mass of body A if and only if

$$0=M_a^A(T_A)\equiv\int_A d^3X_A X_A^a\Sigma(T_A,\mathbf{X}_A)+\frac{1}{10c^2}\frac{d^2}{dT_A^2}\int_A d^3X_A X_A^a\mathbf{X}_A^2\Sigma(T_A,\mathbf{X}_A)-\frac{6}{5c^2}\frac{d}{dT_A}\int_A d^3X_A(X_A^aX_A^b-\tfrac{1}{3}\delta^{ab}\mathbf{X}_A^2)\times\Sigma^b(T_A,\mathbf{X}_A) , \quad (5.10)$$

where the spatial integrations extend only over the volume of body A.

Note that (contrary to most of the post-Newtonian center-of-mass definitions in the literature) the definition (5.10) is expressed entirely in terms of quantities referred to a local frame and is given by a well-defined compact-support integral. It is intuitively clear that Eq. (5.10), which expresses that the abstract world line $\mathcal{L}_A$ is constrained to follow, in a precise way, the motion of the matter within body A, will imply some "equations of motion" for the global-coordinate representation of $\mathcal{L}_A$: $x^\mu=z_A^\mu(\tau_A)$. These equations of motion will be discussed in detail in a subsequent publication (see also Sec. VII below) and we shall, for the time being, just assume that they follow from the condition (5.10).

In order to complete the general outline of our method, we need to discuss the role of the other world-line data (apart from Datum 1 that we just discussed).

Datum 2 (i.e., the special S parametrization of the world line, which amounts to a precise post-Newtonian definition of the "local time scale" T_A) and Datum 3 (post-Newtonian fixing of the e_a^0 components) will be fixed by a requirement of *effacement.* We have discussed in the introduction the double role that the accelerated Newtonian center-of-mass frames were playing: both as "comoving" frames, and as frames in which the external gravitational field is strongly "effaced." These roles are closely related (because of the equivalence principle, i.e., the "universality of free fall") but they are still slightly independent. Indeed, a Newtonian frame comoving with an extended body has a different acceleration than a Newtonian frame freely falling (i.e., accelerated like test bodies) in the (Newtonianly well-defined) external gravitational field. The difference in acceleration is due to quadrupole, and higher-multipole couplings, and is

$$a^i_{\text{comoving}} - a^i_{\text{free-fall}} = \frac{1}{2M} Q_{jk}\partial_{ijk} U^{\text{ext}} + \text{higher multipoles} , \tag{5.11}$$

where M denotes the mass and

$$Q_{jk} = \int d^3X \rho (X^j X^k - \tfrac{1}{3}\mathbf{X}^2 \delta^{jk})$$

the Newtonian quadrupole moment (see, e.g., Ref. 7, Sec. 6.4, for a derivation of this purely Newtonian effect).

In a relativistic context the difference between the concepts of "comoving frames" versus "external-field-effacing frames" is bigger, because of the ambiguity of the concept of "frame." For instance, Thorne and Hartle[24] have advocated the use of relativistic frames that are "freely falling" in some (undefined) "external" gravitational field (in the sense that their origin follows a geodesic of the "external" universe), and have generalized and computed the dominant relativistic corrections to the deviation effect (5.11). By contrast, we will find it very convenient to use mainly "comoving" frames, in the precise relativistic sense of Eq. (5.10). This choice leaves us however some freedom in the precise definition of the frame, i.e., of the full coordinate system X_A^α around the now "center-of-mass world line" $\mathcal{L}_A$. We shall take advantage of this freedom to *efface,* at the origin of our local frames, the "external" relativistic gravitational potentials, in the following definite sense.

Weak effacement of post-Newtonian external gravitational potentials in local frames. Using the definition (5.8c) of the "external" PN potentials $\overline{W}{}^A_\alpha(T_A,\mathbf{X}_A)$, as seen in a local X_A-coordinate system we shall say that they are (weakly) effaced in the local A frame if they vanish, for all T_A times, at the origin of the frame: i.e.,

$$\forall T_A, \quad \overline{W}{}^A_\alpha(T_A,0,0,0) = 0 . \tag{5.12}$$

Because of the presence of the "inertial" contribution $W''^A_\alpha(X)$, Eq. (5.8e), in $\overline{W}{}^A_\alpha$, which depends on $\mathcal{B}_\mu$, and thereby on $e_0^0 \equiv dz^0/dS$ and $e_a^0 \equiv e_a^i dz^i/dS + c^{-3}\epsilon_a(S)$, the values of the $\overline{W}{}^A_\alpha$ at the origin depend on $e_0^0(S)$ and $\epsilon_a(S)$. It is easy then to check that the four conditions of effacement (5.12) can always be enforced by a suitable choice of $e_2(S) \equiv c^2(e_0^0(S)-1)$ and $\epsilon_a(S)$ $(a=1,2,3)$, and that this choice is unique. In other words, the weak effacement condition (5.12) ties down in a unique way the world-line Data 2 and 3, to the other variables of the scheme.

Let us now consider Datum 4, i.e., the fixing of the slowly changing local rotation matrices $R^i_{Aa}(T_A)$. For this rotational degree of freedom arises the same dichotomy that was present for the translational degree of freedom: namely, to choose between a "body-based" way of fixing it ("comoving" condition in the translational case), and a "gravitational-field-based" one ("efface" $\partial_i U^{\text{ext}}$ by going to a freely falling frame). Many authors, notably Brumberg and Kopejkin[33] advocate the use of a "gravitational-field-based" criterion for fixing the time-dependence of the rotational matrices. Indeed, there is a natural criterion that we shall call *effacement of post-Newtonian Coriolis effects in local frames.*

Using the definition (5.8c) of the "external" PN potentials $\overline{W}{}^A_\alpha(T_A,\mathbf{X}_A)$, as seen in a local X_A-coordinate system we shall say that the post-Newtonian Coriolis effects are effaced in the local A-frame, if the *external* "gravitomagnetic" $\mathbf{B}$ field vanishes, for all T_A times, at the origin of the frame, i.e.,

$$\forall T_A, \quad \overline{B}{}^A_a(T_A,0) \equiv [\epsilon_{abc}\partial_b(-4\overline{W}{}^A_c)]_{\mathbf{X}=0} = 0 . \tag{5.13}$$

Because of the presence of the "inertial" contributions $W''^A_a(X)$, Eq. (5.8e), in $\overline{W}{}^A_a$, which depends on $\mathcal{B}_\mu$, and thereby on dR^i_a/dT, it can be checked that the three-conditions (5.13) fix uniquely the vectorial angular velocity $\Omega_a(T)$ which determines the time evolution of the orthogonal matrix $R^i_a(T)$ (this will be clear from the formulas given below).

The name "Coriolis" is here meant to indicate the space-independent part of the $\overline{\mathbf{B}}$ field, which, as is already clear from Eq. (5.5b), exerts on the matter forces $\propto \mathbf{\Omega}\times\mathbf{v}$. This issue will be examined in more details in a subsequent publication, as well as the meaning of Eq. (5.13) in terms of a Fermi-Walker transport in some "external" metric [defined in Eq. (5.17) below].

In spite of the naturalness of the rotational effacement criterion (5.13), we want to emphasize that there is no logical necessity in choosing it. On the contrary, although we essentially leave to the reader the choice of the "best" way of fixing Datum 4, we wish to emphasize that there is in fact an equally natural and technically simpler (and far more convenient in many practical applications), alternative criterion, namely through a *global fixing of local rotation matrices*

$$\forall A, \; \forall T_A, \quad R^i_{Aa}(T_A) = \delta^i_a . \tag{5.14}$$

Equation (5.14) is a drastic way of tying the rotational degrees of freedom to the rest of the scheme which is "body based" instead of being "gravitational-field" based. It is the rotational analogue of our "comoving" condition (5.10), while the gravitational-field-based analogue of (5.10) would have been the *effacement of post-Newtonian external + inertial acceleration effects*

$$\forall T_A, \quad \overline{\mathbf{E}}(T_A,\mathbf{0})=0 \ . \tag{5.15}$$

We shall use (5.10) [and never (5.15)] to fix the translational state of our local frames, and leave open the flexibility of choosing (5.13) or (5.14) [with a technical preference for (5.14)] for fixing the rotational data.

Finally, as far as Datum 5 is concerned [i.e., the fixing of the time gauge at the 2PN level, $\delta t \equiv c^{-1}\xi^0 = c^{-4}\xi(T,\mathbf{X})$] we have already seen several times that it is often more convenient to leave open the corresponding time-gauge freedom, and to work systematically with the gauge-invariant **E** and **B** fields. Even in cases where it is, provisionally, useful to choose a definite time gauge (generally of the time-harmonic family) we do not need to compute explicitly to which choice of ξ^0 it corresponds. We shall return to this issue in a subsequent publication.

In conclusion, we have just outlined how one can tie in a definite way the world-line data, $\mathcal{D}_A$, to the other dynamical variables of the problem. When this is done, it defines a formally closed evolution system, that we shall symbolize as

$$\frac{\partial \Sigma_A(T_A)}{\partial T_A} = \Phi_A[\Sigma_B, \mathcal{D}_C] \ , \tag{5.16a}$$

$$\frac{\partial \mathcal{D}_A(T_A)}{\partial T_A} = \Psi_A[\Sigma_B, \mathcal{D}_C] \ . \tag{5.16b}$$

Up to this section we have stayed on a general level where each body was always fully represented (both internally and externally) as an extended object. In the following sections, we shall go to a second level where one "skeletizes" both the locally generated gravitational fields (W_α^{+A}) and the externally felt ones ($\overline{W}_\alpha^A$) by (infinite) sequences of relativistic multipole moments.

E. The vectorial basis $\mathbf{e}_\alpha^A(S)$ as a tetrad in a well-defined external gravitational field

The transformation properties of the post-Newtonian gravitational potentials gave a special role to the decomposition of the W_α^A potentials, in the local A frame, in a locally generated part W_α^{+A} and an external part $\overline{W}_\alpha^A$. According to Eq. (5.8c), the latter part is related to the sum of the global-frame gravitational potentials generated by all the bodies external to the A frame, $\sum_{B\neq A} w_\mu^B(x)$, by the general transformation formula, Eqs. (4.12), valid for the "real" potentials ($W = W^+ + \overline{W}$). Let us now define an external metric.

Definition of the external metric, with respect to body A. To each body A we associate an external metric

$$d\overline{s}_A^{\,2} = \overline{g}_{\mu\nu}^{\,A}(x)dx^\mu dx^\nu = \overline{G}_{\alpha\beta}^{\,A}(X)dX^\alpha dX^\beta \tag{5.17}$$

defined by

$$\overline{g}_{00}^{\,A}(x) \equiv -e^{-2\overline{w}^A(x)/c^2} \ , \tag{5.17a}$$

$$\overline{g}_{0i}^{\,A}(x) \equiv -\frac{4}{c^3}\overline{w}_i^A(x) \ , \tag{5.17b}$$

$$\overline{g}_{ij}^{\,A}(x) \equiv \delta_{ij} e^{+2\overline{w}^A(x)/c^2} \ , \tag{5.17c}$$

where

$$\overline{w}_\mu^A(x) \equiv \sum_{B\neq A} w_\mu^B(x) \ . \tag{5.17d}$$

We can reexpress the detailed transformation result (4.55) by saying that the local-frame components of the external metric, $\overline{G}_{\alpha\beta}^{\,A}(X)$, are given in terms of the external local-frame potentials, as defined by Eq. (4.54) by our usual exponential formulas:

$$\overline{G}_{00}^{\,A}(X) = -e^{-2\overline{W}^A(X)/c^2} + O(6) \ , \tag{5.18a}$$

$$\overline{G}_{0a}^{\,A}(X) = -\frac{4}{c^3}\overline{W}_a^A(X) + O(5) \ , \tag{5.18b}$$

$$\overline{G}_{ab}^{\,A}(X) = \delta_{ab} e^{+2\overline{W}^A(X)/c^2} + O(4) \ . \tag{5.18c}$$

From Eqs. (5.18) we see that the weak effacement condition (5.12) implies that $\overline{G}_{00}^{\,A}(T,\mathbf{0}) = -1 + O(6)$, $\overline{G}_{0a}^{\,A}(T,\mathbf{0}) = 0 + O(5)$, and $\overline{G}_{ab}^{\,A}(T,\mathbf{0}) = \delta_{ab} + O(4)$. But the values of the Jacobian matrix elements $\partial x^\mu/\partial X^\alpha$ at the origin of the local system are nothing but the global components $e_{A\alpha}^\mu$ of the vectorial basis along $\mathcal{L}_A$ introduced in Sec. II B. Hence, we conclude that, under our general hypotheses, the weak effacement condition (5.12) implies, along $\mathcal{L}_A$,

$$\overline{G}_{\alpha\beta}^{\,A} = \overline{g}_{\mu\nu}^{\,A} e_{A\alpha}^\mu e_{A\beta}^\nu = f_{\alpha\beta} + O(6,5,4) \ . \tag{5.19}$$

In other words, Eq. (5.19) says that the vectorial basis $\mathbf{e}_\alpha^A$ along $\mathcal{L}_A$ is, with the precision indicated, an orthonormalized tetrad with respect to the external metric (5.17). In particular, the 00 component of Eq. (5.19) [i.e., the $\alpha=0$ component of the weak effacement condition (5.12)] means that the special parametrization S of the central world line $\mathcal{L}_A$ [see Eq. (2.3)] is the proper distance along $\mathcal{L}_A$, as recorded with the external metric

$$S = \int_{\mathcal{L}_A} d\overline{s}_A = \int_{\mathcal{L}_A} \sqrt{-\overline{g}_{\mu\nu}^{\,A}(x)dx^\mu dx^\nu} \ . \tag{5.20}$$

If we define

$$v_A^i \equiv c\frac{dz_A^i}{dz_A^0} \quad \text{so that } e_{A0}^i \equiv c^{-1} e_{A0}^0 v_A^i \tag{5.21a}$$

we deduce, from Eqs. (5.19),

$$e_{A0}^0 = 1 + \frac{1}{c^2}\left(\tfrac{1}{2}\mathbf{v}_A^2 + \overline{w}^A\right) + \frac{1}{c^4}\left[\tfrac{3}{8}\mathbf{v}_A^4 + \tfrac{1}{2}(\overline{w}^A)^2 + \tfrac{5}{2}\overline{w}^A\mathbf{v}_A^2 - 4\overline{w}_i^A v_A^i\right] + O(6) \ , \tag{5.21b}$$

$$e_{Aa}^0 = R_a^i\left[\frac{v_A^i}{c}\left[1 + \frac{1}{c^2}\left[\frac{1}{2}\mathbf{v}_A^2 + 3\overline{w}^A\right]\right] - \frac{4}{c^3}\overline{w}_i^A\right] + O(5) \ , \tag{5.21c}$$

$$e_{Aa}^i = \left[1 - \frac{1}{c^2}\overline{w}^A\right]\left[\delta^{ij} + \frac{1}{2c^2}v_A^i v_A^j\right]R_a^j + O(4) \ , \tag{5.21d}$$

in which all the external potentials must be evaluated on $\mathcal{L}_A$.

VI. RELATIVISTIC TIDAL AND MULTIPOLE EXPANSIONS

A. Newtonian tidal and multipole expansions

In Newtonian celestial mechanics, the exact equations for the motion of the barycenter $z_A^i(t)$ of one body A member of a N-body system are

$$M_A \frac{d^2 z_A^i(t)}{dt^2} = \int_A d^3X_A \rho_A(\mathbf{X}_A,t)\partial_i U_A^{\text{ext}}(\mathbf{z}_A(t)+\mathbf{X}_A,t) \ , \tag{6.1}$$

where $\rho_A(\mathbf{X}_A)$ is the mass density of body A expressed in terms of the relative coordinate $\mathbf{X}_A = \mathbf{x} - \mathbf{z}_A(t)$, and the external Newtonian gravitational potential is

$$U_A^{\text{ext}}(\mathbf{x},t) = \sum_{B\neq A} U^B(\mathbf{x},t) \ , \tag{6.2a}$$

$$U^B(\mathbf{x},t) = G\int_B d^3X_B \frac{\rho_B(\mathbf{X}_B,t)}{|\mathbf{x}-\mathbf{z}_B(t)-\mathbf{X}_B|} \ . \tag{6.2b}$$

Moreover, the *effective* local external potential whose gradient governs the motion of mass elements in the accelerated A-barycentric frame is

$$U_A^{\text{eff}}(\mathbf{X}_A) \equiv U_A^{\text{ext}}(\mathbf{z}_A+\mathbf{X}_A) - U_A^{\text{ext}}(\mathbf{z}_A) - \frac{d^2\mathbf{z}_A}{dt^2}\cdot\mathbf{X}_A \ , \tag{6.3}$$

where the arbitrary function $C(t)$ of Eq. (4.10) has been chosen for convenience to be $U_A^{\text{ext}}(\mathbf{z}_A(t))$. The system (6.1) and (6.2) is an integro-differential system. In order to reduce it, formally, to a system of ordinary differential equations for the N barycenters, one makes two simultaneous expansions.

(i) The effective local potential, and thereby also $U^{\text{ext}} \equiv U^{\text{eff}} + U_A^{\text{ext}}(\mathbf{z}_A) + d^2\mathbf{z}_A/dt^2\cdot\mathbf{X}_A$, is expanded in a "tidal expansion", i.e., a Taylor series in powers of $\mathbf{X}_A$,

$$U_A^{\text{eff}}(\mathbf{X}_A) = G_i^A X_A^i + \frac{1}{2!}G_{ij}^A X_A^i X_A^j + \cdots + \frac{1}{l!}G_{i_1\cdots i_l}^A X_A^{i_1}\cdots X_A^{i_l} + \cdots \ , \tag{6.4a}$$

where the Newtonian "tidal moments," or "gravitational Gradients," felt by body A are ($\partial_i \equiv \partial/\partial x^i$)

$$G_i^A(t) \equiv \partial_i U_A^{\text{ext}}(\mathbf{z}_A) - \frac{d^2 z_A^i}{dt^2} \ , \tag{6.4b}$$

$$l \geq 2 \rightarrow G_{i_1\cdots i_l}^A(t) \equiv \partial_{i_1\cdots i_l} U_A^{\text{ext}}(\mathbf{z}_A) \ . \tag{6.4c}$$

(ii) the potential generated at some field-point x by each body B is expanded in a "multipole expansion," simply obtained by Taylor expanding the $r^{-1} \equiv |\mathbf{x}-\mathbf{z}_B-\mathbf{X}_B|^{-1}$ factor in Eq. (6.2b) in powers of $\mathbf{X}_B$ ($\partial_i \equiv \partial/\partial x^i$):

$$U^B(\mathbf{x},t) = \frac{GM^B}{|\mathbf{x}-\mathbf{z}_B|} - \partial_i\left[\frac{GM_i^B}{|\mathbf{x}-\mathbf{z}_B|}\right] + \frac{1}{2!}\partial_{ij}\left[\frac{GM_{ij}^B}{|\mathbf{x}-\mathbf{z}_B|}\right] + \cdots + \frac{(-)^l}{l!}\partial_{i_1i_2\cdots i_l}\left[\frac{GM_{i_1i_2\cdots i_l}^B}{|\mathbf{x}-\mathbf{z}_B|}\right] + \cdots \tag{6.5a}$$

with the Newtonian "multipole moments" of body B

$$M_{i_1i_2\cdots i_l}^B(t) = \text{STF}_{i_1\cdots i_l}\int_B d^3X_B X_B^{i_1}\cdots X_B^{i_l}\rho_B(\mathbf{X}_B,t) \ . \qquad (l\geq 0) \tag{6.5b}$$

We have kept in Eq. (6.5b) only the symmetric-trace-free projection of the full mass moment $I_L \equiv \int d^3X_B X_B^L \rho_B$ ($L\equiv i_1\cdots i_l$) because the trace terms of I_L do not contribute to Eq. (6.5a) on account of the vanishing of the Laplacian of $|\mathbf{x}-\mathbf{z}_B|^{-1}$. On the other hand, all tensorial coefficients $G_{i_1\cdots i_l}^A$ in Eq. (6.4) are automatically symmetric and trace-free ($L\equiv i_1\cdots i_l$):

$$G_L^A \equiv \text{STF}_L[\partial_L U_A^{\text{ext}}(\mathbf{z}_A)] \quad (l\geq 2)$$

(for the same basic reason that $\Delta U^B = 0$ outside local body B). Both expansions (6.4) and (6.5) are equivalent to expansions in scalar spherical harmonics, respectively in $r^l Y_{lm}(\theta,\phi)$ and $r^{-(l+1)}Y_{lm}(\theta,\phi)$, but the use of irreducible Cartesian tensors (i.e., STF tensors $G_{\langle L\rangle}$ or $M_{\langle L\rangle}$) renders more transparent (especially when a suitable condensed multi-index notation is used, like $L\equiv i_1 i_2\cdots i_l$) the use of these expansions in many algebraic operations, and, moreover, is definitively simpler when one needs to "tidal", or "multipole", expand vector fields or tensor fields, instead of simply scalars.

An example of the algebraic usefulness of the STF-tensor expansions is that, by inserting Eqs. (6.4) and (6.5) into Eqs. (6.1) [with Eq. (6.3)], the exact equations of motion (6.1) can be written in the form

$$M^A\left[\frac{d^2 z_A^i}{dt^2} - \partial_i U_A^{\text{ext}}(\mathbf{z}_A)\right] \equiv -M^A G_i^A = \sum_{l\geq 1}\frac{1}{l!}M_L^A G_{iL}^A \ , \tag{6.6}$$

which yields the explicit double series

$$M^A \frac{d^2 z_A^i}{dt^2} = G\sum_{B\neq A}\sum_{l,k\geq 0}\frac{(-)^k}{l!k!}M_L^A M_K^B \times \partial_{iLK}^A\left[\frac{1}{|\mathbf{z}_A-\mathbf{z}_B|}\right] , \tag{6.7}$$

in which $L\equiv i_1\cdots i_l$, $K\equiv j_1\cdots j_k$ and $\partial_i^A \equiv \partial/\partial z_i^A$. Both in Eqs. (6.6) and (6.7) the $l=1$ (and $k=1$) terms are actually zero because of the Newtonian definition of the center of mass which, remembering $X_A^i \equiv x^i - z_A^i$, is nothing but

$$0 = M_i^A(t) \equiv \int_A d^3X_A X_A^i \rho_A(\mathbf{X}_A,t) \ . \tag{6.8}$$

B. Post-Newtonian multipole expansions

It has been shown by many authors[55] that, in the case of exactly stationary relativistic gravitational fields that fall off at spatial infinity (i.e., in the case of a stationary and isolated material system), there existed a "good" (and essentially unique) generalization to the full general-relativistic context of the multipole expansion (6.5a) of the gravitational field outside the material source. This stationary relativistic multipole expansion of the metric field $g_{\mu\nu}$ contains two sets of multipole moments: some "mass moments" [which reduce to those appearing in Eq. (6.5a) in the nonrelativistic limit] together with some "spin moments" (analogue to the magnetic type multipole moments in electromagnetism[45]). However, even in this simple case, no exact analogue of Eq. (6.5b) exists. In other words the stationary mass and spin relativistic moments are only "field multipole moments," and not, as in Eq. (6.5b) "source multipole moments." Moreover, if one drops the very restrictive assumption of exact stationarity the situation becomes much more intricate, and the uniqueness, and even the existence (for generic time-dependent field) of exact relativistic time-dependent multipole moments is dubious. Under some weak assumptions about the asymptotic fall-off of the metric field in null directions, it is possible to introduce a concept of (relativistic) "radiative multipole moments" (see, in particular, Thorne[36]) as a set of irreducible Cartesian tensors that parametrize the angular pattern of the $[O(r^{-1})]$ gravitational-wave zone field. On the other hand, under some stronger asymptotic fall-off assumptions, one can show that other definitions of asymptotic multipole moments [based on the subdominant $O(r^{-n})$, $n>1$ terms in the wave zone expansion] are possible and are actually inequivalent to the $O(r^{-1})$-based definition.[56]

But the main problems are, anyway, that there exist (i) neither an analogue of Eq. (6.5a), i.e., the representation of the field at a *finite* distance from the source (ii) nor an analogue of Eq. (6.5b), expressing the moments in terms of the material distribution. In an attempt to cure the latter problem, Thorne,[36] generalizing previous work of Epstein and Wagoner,[57] has derived a *formal* expansion in powers of c^{-1} of the radiative multipole moments as a series of (undefined because divergent) infinite-support integrals over some "effective stress-energy tensor" (which itself contains the unknown metric field).

The situation is much better if one considers only the post-Newtonian approximation to general relativity (in the improved sense of considering $O(c^{-2})$ corrections to the leading terms both in the near-zone, the intermediate zone, and the wave-zone gravitational field). In that case, Blanchet and Damour[41] have recently shown that the situation is as good as in the Newtonian case, in the sense that there exist useful analogues of *both* Eqs. (6.5a) and (6.5b). The main emphasis of their work was to obtain a well-defined integral representation in terms of the material source of the radiative multipole moments parametrizing the asymptotic gravitational wave field emitted by any post-Newtonian gravitationally interacting system. However, we shall show here how a slight generalization of their work leads to very useful constructs, even when considering the post-Newtonian gravitational field *in between* the bodies of an N-body system (while their original expansions were valid only outside a sphere enclosing the full N-body system).

A first generalization of their work consists in working in an arbitrary, not necessarily harmonic, gauge simply by adding the gradient of a general function λ (and in considering time-symmetric, rather than retarded potentials). A second, physically much more important generalization consists in remarking that, thanks to the linearity of the field equations for the w potentials [Eqs. (3.11)], we can apply, step by step, their method to the case of the w potentials generated by only one body, say A, selected from an N-body system.[58] Moreover, their arguments apply both in the global x^μ-coordinate system, and in the local X^α_A one. In keeping with the general spirit of our method outlined in Sec. V C the most useful generalization is to consider the "locally generated" W^{+A}_α potential, as seen in the X_A system. This leads to the following theorem.

Theorem 6. *In any local system, say X^α_A, the locally generated post-Newtonian potentials, $W^{+A}_\alpha(X^\beta)$ [defined in the harmonic gauge by Eq. (4.51)], admit, everywhere outside body A, the following multipole expansion (with label A omitted for readability on the local coordinates $T_A \equiv X^0_A/c, X^a_A$):*

$$W^{+A}(T,\mathbf{X})=G\sum_{l\geq 0}\frac{(-)^l}{l!}\partial_L(R^{-1}M^A_L(T\pm R/c))+\frac{1}{c^2}\partial_T(\Lambda^A-\lambda)+O(4)\ , \tag{6.9a}$$

$$W^{+A}_a(T,\mathbf{X})=-G\sum_{l\geq 1}\frac{(-)^l}{l!}\left[\partial_{L-1}\left[R^{-1}\frac{d}{dT}M^A_{aL-1}\right]+\frac{l}{l+1}\epsilon_{abc}\partial_{bL-1}(R^{-1}S^A_{cL-1})\right]-\tfrac{1}{4}\partial_a(\Lambda^A-\lambda)+O(2)\ , \tag{6.9b}$$

where

$$\Lambda^A\equiv 4G\sum_{l\geq 0}\frac{(-)^l}{(l+1)!}\frac{2l+1}{2l+3}\partial_L(R^{-1}\mu^A_L(T\pm R/c))\ , \tag{6.10a}$$

$$\mu^A_L(T)\equiv\int_A d^3X\,\hat{X}^{bL}\Sigma^b(T,X)\ , \tag{6.10b}$$

and where [arguments (T,X) omitted for the local integrands $\Sigma^\alpha(T,X)$]

$$M_L^A(T)\equiv \int_A d^3X\,\hat{X}^L\Sigma + \frac{1}{2(2l+3)c^2}\frac{d^2}{dT^2}\left[\int_A d^3X\,\hat{X}^L\mathbf{X}^2\Sigma\right] - \frac{4(2l+1)}{(l+1)(2l+3)c^2}\frac{d}{dT}\left[\int_A d^3X\,\hat{X}^{aL}\Sigma^a\right]\quad (l\geq 0) \tag{6.11a}$$

$$S_L^A(T)\equiv \int_A d^3X\,\epsilon^{ab\langle c_l}\hat{X}^{L-1\rangle a}\Sigma^b\ , \quad (l\geq 1) \tag{6.11b}$$

Following the convention (3.30) the $\pm$ sign in Eqs. (6.9), (6.10) denotes a time-symmetric average.

The function $\lambda(T,\mathbf{X})$ denotes an arbitrary gauge transformation. The original extended-body-harmonic-gauge solution (4.51) is obtained when $\lambda=0$. Note, however, that the gauge with $\lambda=\Lambda$ is also harmonic ("skeletonized-body-harmonic-gauge").

We see that the symmetric and trace-free Cartesian tensors defined by Eqs. (6.11) play, at the post-Newtonian level, both the role of "field multipole moments" [Eqs. (6.9)] and of "source multipole moments" [Eqs. (6.11)].[59] They constitute one of the essential tools of our approach, and we shall refer to them in the following as the (local) BD moments of body A. The moments $M_L^A(T)$ will be referred to as the "mass" moments, by contrast to the "spin" moments $S_L^A(T)$. Note from the $O(4,2)\ c^{-n}$ error terms in W_a^A Eqs. (6.9) that only $O(4)$ uncertainty could be accepted in M_L, while S_L could admit a bigger $O(2)$ uncertainty (i.e., S_L needs to be defined only at Newtonian accuracy). The problem of generalizing the BD results so as to reach also a post-Newtonian accuracy [$O(4)$ errors] for S_L has been recently solved,[60] and we shall return to it in a subsequent paper.

C. Post-Newtonian tidal expansions

Having shown how to generalize Eqs. (6.5) to the post-Newtonian level, let us now turn to the problem of finding a "good" (i.e., useful) PN generalization of the Newtonian tidal expansions (6.4). The usefulness of the Newtonian tidal moments (6.4b) and (6.4c) rested on two features: (i) they were irreducible Cartesian tensors and (ii) they constituted a skeletonized representation of the effective external gravitational forces felt by mass elements in a local center-of-mass frame. Now, we have seen in Eqs. (5.6a) above that in the post-Newtonian momentum density evolution equation, written in a local X_A system, the role of the effective gravitational force density $\rho\nabla U^{\rm eff}$ was played by the "Lorentz" force density $\mathbf{F}=\Sigma\mathbf{E}+c^{-2}\boldsymbol{\Sigma}\times\mathbf{B}$, which, being *linear* in $\mathbf{E}$ and $\mathbf{B}$, is naturally decomposed in a "self"-force density $(\Sigma\mathbf{E}^{+}+c^{-2}\boldsymbol{\Sigma}\times\mathbf{B}^{+})$ and an "external" one $(\Sigma\overline{\mathbf{E}}+c^{-2}\boldsymbol{\Sigma}\times\overline{\mathbf{B}})$. This points out clearly at using the external gauge-invariant fields $\mathbf{E}[\overline{W}_A]$, $\mathbf{B}[\overline{W}_A]$ as post-Newtonian analogues of $\nabla U^{\rm eff}$. And for the analogues of the multigradients of $U^{\rm eff}$, we shall not take simply the multigradients of $\overline{\mathbf{E}}$ and $\overline{\mathbf{B}}$, because they are not irreducible Cartesian tensors, but, very naturally, the symmetric-and-trace-free projections of the latter. Hence we get our *definition of post-Newtonian tidal moments.*

Let, with $\partial_a\equiv\partial/\partial X_A^a$, $\partial_T\equiv c\partial/\partial X_A^0$ in some local A frame,

$$\overline{E}{}_a^A(T,\mathbf{X})\equiv\partial_a\overline{W}{}^A+\frac{4}{c^2}\partial_T\overline{W}{}_a^A\ , \tag{6.12a}$$

$$\overline{B}{}_a^A(T,\mathbf{X})\equiv\epsilon_{abc}\partial_b(-4\overline{W}{}_c^A)\ , \tag{6.12b}$$

denote the external gauge-invariant fields. We skeletonize them by defining two corresponding (gravitoelectric and gravitomagnetic) sets of post-Newtonian tidal moments:

$$G_L^A(T)\equiv[\partial_{\langle L-1}\overline{E}{}_{a_l\rangle}^A(T,\mathbf{X})]_{X^a=0}\quad (l\geq 1)\ , \tag{6.13a}$$

$$H_L^A(T)\equiv[\partial_{\langle L-1}\overline{B}{}_{a_l\rangle}^A(T,\mathbf{X})]_{X^a=0}\quad (l\geq 1)\ . \tag{6.13b}$$

Note that the moments so defined are of order $l\geq 1$ [see Eq. (6.16) below for the definition of a gravitoelectric monopole tidal moment].

Before studying the properties of these PN tidal moments, let us say that Thorne and Hartle[24] have pointed out that the general solution to the vacuum Einstein equations which is regular near a timelike geodesic can be fully parametrized (up to coordinate changes) by two sets of STF tensors. Their work has been extended and refined by Suen[40] and Zhang.[39] At the *linearized* gravity level the moments defined by these authors differ only by some normalization factors from ours. However, a notable difference arises when nonlinear effects come into play (as they do at the 1PN approximation). Indeed, Thorne and Hartle[24] and Zhang[39] insist on relating their moments to the curvature tensor of some (undefined) "external metric," while we have seen in Sec. III D that, whatever be the choice of external metric g^*, its curvature tensor will differ from $G_{ab}^*=\partial_{\langle a}E_{b\rangle}^*$ by nonlinear terms. This appearance of nonlinearities in $\mathbf{E}^*$ spoils the nice linear properties of the *W-E-B* formulation of PN gravity. Moreover, another notable advantage of our scheme is that it furnishes an unambiguous definition of a (useful) "external metric," namely the metric $\overline{G}_{\alpha\beta}(X^\gamma)$ defined by Eqs. (5.17) and (5.18).

The *external* gauge-invariant fields (6.12) satisfy the homogeneous equations ($\nabla=\partial/\partial X^a$)

$$\nabla\times\overline{\mathbf{E}}=-\frac{1}{c^2}\partial_T\overline{\mathbf{B}}\ , \tag{6.14a}$$

$$\nabla\times\overline{\mathbf{B}}=4\partial_T\overline{\mathbf{E}}+O(2)\ , \tag{6.14b}$$

$$\nabla\cdot\overline{\mathbf{E}}=-\frac{3}{c^2}\partial_T^2\overline{W}+O(4)\ , \tag{6.14c}$$

$$\nabla\cdot\overline{\mathbf{B}}=0\ , \tag{6.14d}$$

and

$$\Delta\overline{\mathbf{E}}=\frac{1}{c^2}\partial_T^2\overline{\mathbf{E}}+O(4)\ , \tag{6.15a}$$

$$\Delta\overline{\mathbf{B}}=0+O(2)\ . \tag{6.15b}$$

It is easy to see, by induction, from these equations that the knowledge of all the STF projections of the spatial derivations of $\overline{\mathbf{E}}$ and $\overline{\mathbf{B}}$ at the origin [i.e., our tidal moments (6.13)], if it is augmented by the knowledge of

$$G(T)\equiv\overline{W}(T,\mathbf{0})+O(2)\ , \tag{6.16}$$

determines completely [up to a usual $O(4,2)$ uncertainty] all the components of the spatial derivatives of $\overline{\mathbf{E}}$ and $\overline{\mathbf{B}}$. The additional datum $G(T)$, Eq. (6.16), plays the role of a "monopole tidal moment", and should be, in general, added to our definitions (6.13) [note that $G(T)$ is gauge invariant within the precision, $O(2)$, with which it is defined]. However, we have seen in Sec. V above that this datum can be gauged away by a proper normalization of the S parametrization of the central world line. In the applications of our formalism we shall generally assume that we have chosen "world-line data" such that the weak effacement conditions of Sec. V, i.e.,

$$\overline{W}(T,\mathbf{0})=0\ , \tag{6.17a}$$

$$\overline{W}_a(T,\mathbf{0})=0\ , \tag{6.17b}$$

are satisfied. However, in the present section it will be more convenient (for reasons that will appear in subsection E below) to stay fully general by assuming nothing about the datum (6.16).

The tidal moments (6.13), (6.16) determine uniquely all the spatial derivatives $\partial_L\overline{E}_a$, $\partial_L\overline{B}_a$. In order to perform explicitly the calculation of these derivatives, it is convenient to remark first that (as follows easily from the equations given in Sec. II A above)

$$X^L=\hat{X}^L+\frac{l(l-1)}{2(2l-1)}\mathbf{X}^2\hat{X}^{(L-2}\delta^{a_{l-1}a_l)}+O(\delta\delta)\ , \tag{6.18}$$

where $O(\delta\delta)$ denotes any term that contains two (uncontracted) Kronecker deltas, and where we recall that the caret above $\hat{X}^L$ means a STF projection of the multi-index $L\equiv a_1\cdots a_l$. From Eq. (6.18) follows immediately the fact that the Taylor expansion (with respect to the X^a's) of any field $\varphi(\mathbf{X})$ reads

$$\varphi(\mathbf{X})=\sum_{l\geq 0}\frac{1}{l!}X^L\partial_L\varphi\Big|_{\mathbf{X}=0}$$
$$=\sum_{l\geq 0}\frac{1}{l!}\left[\hat{X}^L\hat{\partial}_L\varphi+\frac{\mathbf{X}^2\hat{X}^L}{2(2l+3)}\hat{\partial}_L\Delta\varphi+O(\Delta^2\varphi)\right]_{\mathbf{X}=0}. \tag{6.19}$$

Equation (6.19) can be applied to each component $\overline{E}_a$ or $\overline{B}_a$, for which we see from Eqs. (6.15) that $\Delta^2\overline{E}_a=O(4)$ and $\Delta\overline{B}_a=O(2)$, so that ($\cdot=d/dT$).

$$\overline{E}_a=\sum_{l\geq 0}\frac{1}{l!}\left[\hat{X}^L\hat{\partial}_L\overline{E}_a+\frac{1}{2(2l+3)c^2}\mathbf{X}^2\hat{X}^L\hat{\partial}_L\ddot{\overline{E}}_a\right]_{\mathbf{X}=0}+O(4)\ , \tag{6.20a}$$

$$\overline{B}_a=\sum_{l\geq 0}\frac{1}{l!}\hat{X}^L\hat{\partial}_L\overline{B}_a\Big|_{\mathbf{X}=0}+O(2)\ . \tag{6.20b}$$

The problem is thereby reduced to computing $\partial_{\langle L\rangle}\overline{E}_a$ and $\partial_{\langle L\rangle}\overline{B}_a$ at the origin.

We can now make use of the STF form of the Clebsch-Gordan reduction for the multiplication of two irreducible representations of the rotational group, in the case $D_1\otimes D_l=D_{l+1}\oplus D_l\oplus D_{l-1}$ (see Appendix A of Ref. 37). Namely, if $T_{a\langle L\rangle}$ is a (reducible) tensor of order $l+1$ that is STF only with respect to the multi-index L, one can decompose it into three algebraic pieces:

$$T_{a\langle L\rangle}=\hat{T}^{(+1)}_{aL}+\epsilon_{ca\langle a_l}\hat{T}^{(0)}_{L-1\rangle c}+\delta_{a\langle a_l}\hat{T}^{(-1)}_{L-1\rangle}\ , \tag{6.21a}$$

in which each $\hat{T}^{(\pm1,0)}$ is STF (i.e., irreducible):

$$\hat{T}^{(+1)}_{L+1}\equiv\mathrm{STF}_{L+1}(T_{a_{l+1}L})\ , \tag{6.21b}$$

$$\hat{T}^{(0)}_{L}\equiv\mathrm{STF}_L\left[\frac{l}{l+1}\epsilon_{a_lbc}T_{bcL-1}\right]\ , \tag{6.21c}$$

$$\hat{T}^{(-1)}_{L-1}\equiv\frac{2l-1}{2l+1}T_{ccL-1}\ . \tag{6.21d}$$

Applying Eqs. (6.21) to $\partial_{\langle L\rangle}\overline{E}_a$ and $\partial_{\langle L\rangle}\overline{B}_a$, and using Eqs. (6.14), one can derive

$$\partial_{\langle L\rangle}\overline{E}_a|_{\mathbf{X}=0}=G_{La}+\frac{l}{(l+1)c^2}\epsilon_{ca\langle a_l}\dot{H}_{L-1\rangle c}-\frac{7l-4}{(2l+1)c^2}\delta_{a\langle a_l}\ddot{G}_{L-1\rangle}+O(4)\ , \tag{6.22a}$$

$$\partial_{\langle L\rangle}\overline{B}_a|_{\mathbf{X}=0}=H_{La}-\frac{4l}{l+1}\epsilon_{ca\langle a_l}\dot{G}_{L-1\rangle c}+O(2)\ . \tag{6.22b}$$

Hence, we get finally the following tidal expansions for $\overline{\mathbf{E}}^A$ and $\overline{\mathbf{B}}^A$ (overall label A omitted):

$$\overline{E}_a(T,\mathbf{X})=\sum_{l\geq 0}\frac{1}{l!}\left[\hat{X}^LG_{aL}(T)+\frac{1}{2(2l+3)c^2}\mathbf{X}^2\hat{X}^L\frac{d^2}{dT^2}G_{aL}(T)-\frac{7l-4}{(2l+1)c^2}\hat{X}^{aL-1}\frac{d^2}{dT^2}G_{L-1}(T)+\frac{l}{(l+1)c^2}\epsilon_{abc}\hat{X}^{bL-1}\frac{d}{dT}H_{cL-1}(T)\right]+O(4)\ , \tag{6.23a}$$

$$\bar{B}_a(T,\mathbf{X})=\sum_{l\geq 0}\frac{1}{l!}\left[\hat{X}^L H_{aL}(T)-\frac{4l}{l+1}\epsilon_{abc}\hat{X}^{bL-1}\frac{d}{dT}G_{cL-1}(T)\right]+O(2)\ . \tag{6.23b}$$

By convention, we are assuming in Eqs. (6.22) and (6.23) that any term which contains an undefined tidal moment (or a meaningless expression) is to be replaced by zero, e.g., the term containing the factor $(7l-4)/(2l+1)$ is absent when $l=0$. Note, however, that the latter term is present when $l=1$, in which case it represents the sole contribution of the monopole tidal moment $G(T)$ to the gauge invariant fields.

D. General structure of the post-Newtonian tidal moments

We have defined our tidal moments G_L^A and H_L^A by Eqs. (6.13) and (6.16). Therefore, they are *linear* in the external potentials $\bar{W}_\alpha^A$ and we have seen above, Eq. (5.8c), that the external potentials were themselves a *linear* superposition of $N+1$ terms:

$$\bar{W}_\alpha^A=\sum_{B\neq A}W_\alpha^{B/A}+\bar{W}_\alpha^{A''}\ , \tag{6.24}$$

N of them being generated by the N separate bodies, and the last one being an "inertial" contribution arising because of the "accelerated" frame transformation $x^\mu\to X_A^\alpha$. Consequently, the tidal moments can also be decomposed into $N+1$ contributions, parallel to Eq. (6.24).

Moreover, the results that we have presented above concerning the transformation laws of the gravitational potentials, together with the fact that each locally generated piece, W_α^{+A}, is completely expressible (modulo a gauge transformation) in terms of the BD multipole moments [see Eq. (6.9)] imply that each of the (gauge-invariant) body-generated contributions to G_L^A and H_L^A can be expressed in terms of the BD moments of body B, and of the coefficients of the transformation between the A and B frames: $\mathcal{A}_{\alpha\mu}^{A(-1)}\mathcal{A}_{\mu\beta}^B$. As for the "inertial" contributions to G_L^A and H_L^A, say $G_L^{A''}$, $H_L^{A''}$, they depend only on the world-line data $\mathcal{D}_A$ of $\mathcal{L}_A$. Hence we shall have the following structure for the post-Newtonian tidal moments of the A frame:

$$G_L^A=\sum_{B\neq A}G_L^{B/A}[M_K^B,S_K^B,\mathcal{D}_A,\mathcal{D}_B]+G_L^{A''}[\mathcal{D}_A]\ , \tag{6.25a}$$

$$H_L^A=\sum_{B\neq A}H_L^{B/A}[M_K^B,S_K^B,\mathcal{D}_A,\mathcal{D}_B]+H_L^{A''}[\mathcal{D}_A]\ . \tag{6.25b}$$

The derivation of the explicit expressions for the body-generated contributions to the tidal moments will be left to a subsequent publication. We shall here study only the inertial contributions.

E. Inertial contributions to the post-Newtonian tidal moments

Thanks to the tools we have introduced above, there are two ways in which we can compute the explicit expressions of the inertial contributions $G_L^{A''}$, $H_L^{A''}$ to the tidal moments.

A direct method would consist of computing

$$\bar{W}_\alpha^{A''}\equiv-\mathcal{A}_{\alpha\mu}^{A(-1)}\mathcal{B}_\mu^A\ , \tag{6.26}$$

from Eqs. (4.13), with the $\mathcal{B}_\mu$'s read off from Eqs. (4.12), and the Jacobian matrix elements A_α^μ taken from Eqs. (2.10). Then, the $G_L^{A''}$ and $H_L^{A''}$ are obtained by differentiating $\mathbf{E}''=\mathbf{E}[W'']$, etc. A more elegant, and more instructive method consists of using the geometrical properties of the $\mathbf{E}$ and $\mathbf{B}$ fields discussed in Sec. III D. Note that we are here placing ourselves again in the general setting of Sec. II B, within which one is given an abstract vectorial basis $\mathbf{e}_\alpha=e_\alpha^\mu(S)\partial/\partial x^\mu$ which is not restricted beyond the results of Theorem 2 [Eqs. (2.36)]. In other words, we are not assuming here that Data 2 and 3 of Sec. V A are fixed by the orthonormality conditions (5.19) [i.e., equivalently we do not impose here the weak effacement conditions (5.12)].

The key remark for using the geometrical results of Sec. III D is that, because of the affine nature of the transformation law,

$$w_\mu=\mathcal{A}_{\mu\alpha}W_\alpha+\mathcal{B}_\mu\ ,$$

the inertial W's (6.26) formally are the transforms of $w_\mu=0$, i.e., of flat space: $g_{\mu\nu}[w=0]=f_{\mu\nu}$. In other words, the $\mathbf{E}''$ and $\mathbf{B}''$ inertial fields are obtained from the formulas of section III D by taking simply

$$ds^{*2}=g_{\mu\nu}^*(x^\lambda)dx^\mu dx^\nu=f_{\mu\nu}dx^\mu dx^\nu\ . \tag{6.27}$$

Therefore, the curvature tensor $R^*(\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d})$ is identically zero, and we learn from combining Eqs. (3.40), (3.43) with the tidal expansions (6.23) that the only nonvanishing inertial tidal moments are simply G'', G_a'', G_{ab}'', and H_a'' and that

$$G_{ab}''=\frac{3}{c^2}G_{\langle a}''G_{b\rangle}''\ . \tag{6.28}$$

The dipole inertial tidal moments G_a'' and H_a'' are very simply obtained from Eqs. (3.37) in terms of the (Minkowskian) acceleration of $\mathcal{L}_A$ and of the (Minkowskian) rotation of the $\mathbf{e}_a$ triad along $\mathcal{L}_A$. If we define precisely [as in Eq. (2.33) above] the $\mathbf{e}_a$ component of the Minkowskian acceleration A_a of $\mathcal{L}_A$ as

$$A_a\equiv f_{\mu\nu}e_a^\mu\frac{d^2z^\nu}{d\tau_f^2}\ , \tag{6.29}$$

where $d\tau_f^2\equiv-c^{-2}f_{\mu\nu}dz^\mu dz^\nu$ is the Minkowskian (globally flat) proper time along the world line $\mathcal{L}$, we have, for each world line,

$$G_a''=-A_a+O(4)\ , \tag{6.30a}$$

$$G_{ab}''=\frac{3}{c^2}A_{\langle a}A_{b\rangle}+O(4)\ , \tag{6.30b}$$

$$H_a''=\epsilon_{abc}\left[V_bA_c+c^2\frac{dR_b^i}{dT}R_c^i\right]+O(2)\ , \tag{6.30c}$$

where $V_b=R_b^i dz^i/d\tau_f+O(2)$. Equation (6.30a) is the post-Newtonian generalization of the second term in the right-hand side of Eq. (6.4b). The second term in the right-hand side of Eq. (6.30c) is just the Newtonian Coriolis effect while the first term is a special-relativistic addition to the Coriolis effect induced by the Thomas precession. As for the monopole inertial tidal moment, $G''(T) \equiv \overline{W}''(T,0)+O(2)$, one sees from Eq. (3.34) that it measures simply the relative scaling, along each world line, between the special parametrization $S \equiv cT$ and the Minkowskian proper time τ_f:

$$G''(T)=c^2\ln\left[\frac{dT}{d\tau_f}\right]+O(2) .$$

Finally, the tidal expressions (6.23) of the inertial fields read simply

$$\overline{E}''_a(T,\mathbf{X})=G''_a+G''_{ab}X^b-\frac{1}{c^2}X^a\frac{d^2G''}{dT^2}+\frac{1}{2c^2}\epsilon_{abc}X^b\frac{dH''_c}{dT}+\frac{1}{6c^2}\mathbf{X}^2\frac{d^2G''_a}{dT^2}-\frac{1}{c^2}\hat{X}^{ab}\frac{d^2G''_b}{dT^2}+O(4) , \quad (6.31a)$$

$$\overline{B}''_a(T,\mathbf{X})=H''_a-2\epsilon_{abc}X^b\frac{dG''_c}{dT}+O(2) . \quad (6.31b)$$

Equation (6.31a) is the post-Newtonian generalization of the gradient of the last term in the right-hand side of Eq. (6.3). We have checked that the direct method leads to the same results. Finally, let us emphasize that, unless otherwise indicated, in all subsequent applications of our formalism (including next section), we shall choose the T parametrization of the world lines so that the weak effacement condition (6.17a) holds for the total external potentials. This condition will ensure that the total monopole tidal moment, $G=G''+\sum_{B\neq A}G^{B/A}$, vanishes for all times. As a consequence, all monopole tidal contributions will disappear from the formalism (e.g., the term $-c^{-2}X^a\ddot{G}''$ in the "inertial" gravitoelectric field (6.31a) will be canceled by a corresponding term in the "externally generated" field), and it will be sufficient to use the sets (6.13) of tidal moments.

VII. RELATIVISTIC MULTIPOLE-EXPANDED CELESTIAL MECHANICS

A. Structure of the post-Newtonian mass, barycenter, and spin equations of motion in each local system.

In each local X^α_A system, the stress-energy tensor of body A must satisfy the evolution equations (5.6). In the linearized gravity case, it is well known that the four energy-momentum local conservation laws imply only a finite number of constraints on the time variation of the irreducible mass and spin moments (of an isolated system): namely,

$$\frac{dM^{(\text{lin})}}{dT}=0, \quad \frac{d^2M_a^{(\text{lin})}}{dT^2}=0, \quad \frac{dS_a^{(\text{lin})}}{dT}=0 .$$

By continuity, one expects that the only equation-of-state-independent constraints on the BD multipole moments will concern the time-evolution of the three lowest multipole moments, M, M_a, and S_a. We shall show in a subsequent publication that this is indeed the case, and that a theorem of the following form holds.

Theorem7. *The energy-momentum-conservation equations in the local X_A frame, Eqs. (5.6), imply constraints on the time evolution of the three lowest BD multipole moments of the form*

$$\frac{dM^A}{dT}=\frac{1}{c^2}\mathcal{E}^{(1\text{PN})}(M_L^{(p)A},G_{L'}^{(p')A})+O(4) , \quad (7.1a)$$

$$\frac{d^2M_a^A}{dT^2}=\sum_{l\geq 0}\frac{1}{l!}M_L^AG_{aL}^A+\frac{1}{c^2}\mathcal{F}_a^{(1\text{PN})}(M_L^{(p)A},S_L^{(q)A};G_{L'}^{(p')A},H_{L'}^{(q')A})+O(4) , \quad (7.1b)$$

$$\frac{dS_a^A}{dT}=\sum_{l\geq 0}\frac{1}{l!}\epsilon_{abc}M_{bL}^AG_{cL}^A+\frac{1}{c^2}\mathcal{G}_a^{(1\text{PN})}(M_L^{(p)A},S_L^{(q)A};G_{L'}^{(p')A},H_{L'}^{(q')A})+O(2/4) , \quad (7.1c)$$

where all the right-hand sides of Eqs. (7.1) are bilinear in the BD multipole moments and in the above-introduced tidal moments, and their time derivatives

$$M_L^{(p)}\equiv\frac{d^p}{dT^p}M_L, \quad \text{etc} .$$

More explicitly, the right-hand sides of Eqs. (7.1) consist of an infinite series of terms, each having the form $M^{(p)}G^{(q)}$, $M^{(p)}H^{(q)}$, $S^{(p)}G^{(q)}$, or $S^{(p)}H^{(q)}$. The special notation $O(2/4)$ in Eq. (7.1c) means that, when one is working strictly within the 1PN approximation, it is sufficient to know S_a at the Newtonian accuracy and therefore the explicitly written Newtonian torque is enough. However, we shall show in a separate paper that it is possible to define a local spin vector for body A [differing from the Newtonian spin moment (6.11b) by $O(c^{-2})$ additional terms] whose time evolution is given, modulo $O(4)$, by an equation of the form (7.1c).

A very satisfactory feature of Eqs. (7.1) is their "closed" structure, i.e., the fact that the right-hand sides depend only on our "good" PN moments, and not on the many possible other multipolelike terms that could appear (and do appear in intermediate calculations) such as

$$\hat{N}_L^A\equiv\int_A d^3X\,\mathbf{X}^2\hat{X}^L\Sigma , \quad (7.2a)$$

$$\hat{P}_L^A\equiv\int_A d^3X\,\hat{X}^{aL}\Sigma^a, \quad \text{etc.} . \quad (7.2b)$$

The derivation of Eqs. (7.1), and a detailed discussion of their meaning is left to a subsequent paper.

B. Multipole-expanded post-Newtonian equations of motion in the global coordinate system.

So far we did not need to attach the N central world lines $\mathcal{L}^A$ to the matter distribution of the nominal corresponding bodies. As we anticipated in Sec. V D, the most natural way to do so is to require that each local X_A frame be "mass centered" in the precise sense that the *BD* dipole moment vanishes for all times, as expressed by Eq. (5.10). This attachment of the origin of the X_A-frame to the material body A entails the fact that the *local-frame* time-evolution constraint (7.1b) yields *global-frame* equations of motion for the center-of-mass world line $\mathcal{L}_A$. The precise way in which this arises is easily seen by comparing the post-Newtonian result (7.1b) (with $M_a=0$) with the Newtonian one (6.6), remembering that the post-Newtonian G_a^A contains the inertial contribution $G_a''=-A_a$.

We see therefore that, by inserting the Eqs. (6.25) into Eq. (7.1b) (with $M_a=0$), we shall derive global-frame equations for the motion of the centers of mass of the form

$$M^A A_a^A = \sum_{B\neq A}\left[M^A G_a^{B/A}[M_K^B,S_K^B]+\sum_{l\geq 2}\frac{1}{l!}M_L^A G_{aL}^{B/A}[M_K^B,S_K^B]\right] + \frac{1}{c^2}\mathcal{F}_a^{(1\mathrm{PN})}(M_L^A,S_L^A;G_{L'}^A[M_K^B,S_K^B],H_{L'}^A[M_K^B,S_K^B])+O(1/c^4)\ , \tag{7.3}$$

where the dependence of the right-hand on the world-line data has not been indicated. The global equations of motion (7.3) represent the skeletonized version of our symbolic Eq. (5.16b). Let us emphasize again that the remarkable feature of Eq. (7.3) is that it succeeds in expressing the global-frame motion of an N-body system in terms of a set of locally measurable multipole moments for each body (in precisely the same physical sense as the one in which one measures by means of satellites the multipole moments of the Earth).

C. Application to an improved derivation of the Lorentz-Droste-Einstein-Infeld-Hoffmann equations of motion

We relegate to a subsequent paper the explicit derivation of the full post-Newtonian multipole-extended equations of motion (7.3). We wish to remark here that these equations (for $A=1,\ldots,N$), considered by themselves, do not form a closed evolution system because the time evolution of the higher ($l\geq 2$) multipole moments is left unspecified. This is because, in the language of Sec. V D above, Eqs. (7.3) represent only the skeletonized version of the world-line-data evolution equations (5.16b), which represent only a half of the closed system (5.16). There are several ways in which Eqs. (7.3) can be completed by additional equations.

(i) The exact way consists of adding the full Eqs. (5.16a), for instance in *tidal*-expanded form: i.e., to write down Eqs. (5.6) with all the external potentials and fields being tidal-expanded (and the tidal moments being expressed in terms of the multipole moments of the other bodies).

(ii) The approximate ways consist of defining some "models" that do not intend to be the first term of an asymptotic approximation to reality, but only to be able to "save the phenomena" with an acceptable accuracy and in a logically consistent manner. In a subsequent publication we shall consider both some "rigid models," as well as some "truncated models."

Here we shall consider only the simplest example of a consistent, and closed, truncated model, namely the "monopole model" by which each body's gravitational structure is skeletonized by only one parameter, its BD mass. Let us first discuss the consistency of this truncation. This consistency is not *a priori* evident (especially at the 1PN level) because, whatever be the internal structure of the bodies, Eqs. (7.1) must be satisfied. In Eqs. (7.1) the tidal moments are no longer free variables because they are all computable from the multipole moments via Eqs. (6.25). However, the mere bilinear structure of the right-hand sides of Eqs. (7.1a) and (7.1c) [Eq. (7.1b) playing the different role of determining the translational equations of motion] together with various basic necessary algebraic requirements (index-structure, dimensional analysis, . . .) can be easily checked to imply that the *Ansätze*

$$\forall A=1,\ldots,N \quad l\geq 1 \rightarrow M_L^A=S_L^A=0\ , \tag{7.4}$$

are not only consistent with the general constraints (7.1) but that Eq. (7.1a) implies the necessary constraint

$$\frac{dM^A}{dT}=O(4)\ . \tag{7.5}$$

We shall therefore define our monopole model by taking *constant* BD masses for each body, and zero higher mass and spin moments.

Let us now show how the tools we have introduced above allow us to compute explicitly, in a quite elegant manner, the global-frame post-Newtonian equations of motion for such a monopole-truncated N-body system.

The orbital equations of motion are obtained from Eq. (7.1b) and read very simply

$$G_a^A=0 \quad (\text{monopole model})\ . \tag{7.6}$$

Instead of decomposing G_a^A, following Eqs. (6.25), in an external-bodies-generated piece and an inertial piece [like we did to derive the general structure (7.3)] let us remark that, by the definition (6.13a), Eq. (7.6) means that the gravitoelectric external field $\overline{\mathbf{E}}_A(P)$ *vanishes* all along the central world line $\mathcal{L}_A$. Let us now return to the geometric formulation of the general $\mathbf{E}^*$ fields discussed in Sec. III D, and *define*, as in Sec. V E, for each body A, an *external metric* $\bar{g}_A$ by choosing $W^*\equiv\overline{W}_A$ in Eqs. (3.33). In other words, using our results on the transformation of potentials, this means that the global-frame natural components of this external metric,

$$-c^2 d\bar{\tau}_A^2 = d\bar{s}_A^2 = \bar{g}^A_{\mu\nu}(x^\lambda) dx^\mu dx^\nu , \tag{7.7}$$

are *by definition* [see also Eqs. (5.17)] given by

$$\bar{g}^A_{00}(x) \equiv -\exp\left[-\frac{2}{c^2}\sum_{B\neq A} w^B(x)\right] , \tag{7.7a}$$

$$\bar{g}^A_{0i}(x) \equiv -\frac{4}{c^3}\sum_{B\neq A} w_i^B(x) , \tag{7.7b}$$

$$\bar{g}^A_{ij}(x) \equiv \delta_{ij}\exp\left[+\frac{2}{c^2}\sum_{B\neq A} w^B(x)\right] . \tag{7.7c}$$

With this definition, we can now conclude from Eq. (3.35a), that the external-metric-normalized four-velocity of $\mathcal{L}_A$,

$$\bar{u}^\mu_A \equiv \frac{dz^\mu_A}{d\bar{\tau}_A} , \tag{7.8}$$

must satisfy

$$\bar{\nabla}^A_{\bar{\mathbf{u}}_A}\bar{\mathbf{u}}_A = O(4) \ \text{(monopole model)} , \tag{7.9a}$$

i.e., explicitly

$$\frac{d\bar{u}^\lambda_A}{d\bar{\tau}_A} + \bar{\Gamma}^\lambda_{A\mu\nu}(\mathbf{z}_A)\bar{u}^\mu_A\bar{u}^\nu_A = O(4) \ \text{(monopole model)} , \tag{7.9b}$$

where the $\bar{\Gamma}_A$ denote the Christoffel symbols of $\bar{g}^A_{\mu\nu}$:

$$\bar{\Gamma}^\lambda_{A\mu\nu} = \tfrac{1}{2}\bar{g}^{\lambda\sigma}_A(\partial_\mu\bar{g}^A_{\nu\sigma} + \partial_\nu\bar{g}^A_{\mu\sigma} - \partial_\sigma\bar{g}^A_{\mu\nu}) . \tag{7.10}$$

It is to be noted that, contrarily to several existing "derivations" which remained essentially heuristic, we have here proven (as a consequence of Theorem 7 that we shall prove explicitly in a subsequent paper) that the, well-defined, BD-barycentric world lines are, in a consistent monopole-truncated model, *geodesics* of a well-defined external metric (7.7).

In order to compute explicitly the equations of motion (7.9) we need to express each $w^B_\mu(x)$ potential of Eqs. (7.7) in terms of the BD mass of body B. Our above-introduced tools allow us to do so in the following manner. Equation (4.53) tells us that

$$w^B_\mu(x) = \mathcal{A}^B_{\mu\alpha}(X^0) W^{+B}_\alpha(X) + O(4,2) , \tag{7.11}$$

while Eqs. (6.9) give, in the skeletonized-body harmonic gauge, and in the monopole model

$$W^{+B} = \frac{GM_B}{R_B} + O(4) , \tag{7.12a}$$

$$W^{+B}_a = 0 + O(2) . \tag{7.12b}$$

Hence

$$w^B_\mu = G\mathcal{A}^B_{\mu 0}\frac{M_B}{R_B} + O(4,2) ,$$

with

$$\mathcal{A}^B_{\mu 0} = \left[1 + \frac{2}{c^2}\mathbf{v}_B^2, v_B^i\right] .$$

It remains to express R_B^{-1} in global coordinates, and this follows from Eq. (4.50), which gives, for the simplest case, $F(S)=1$,

$$\frac{1}{R} = \left[\frac{1}{\rho}\right]_\pm \equiv \frac{1}{2}\left[\left[\frac{1}{\rho}\right]_+ + \left[\frac{1}{\rho}\right]_-\right] , \tag{7.13}$$

where ρ is defined by Eq. (4.42) (with $Z^a=0$; central world line). Finally we have

$$w^B(x) = \left[\frac{GM_B(1+2\mathbf{v}_B^2/c^2)}{\rho_B}\right]_\pm , \tag{7.14a}$$

$$w_i^B(x) = \left[\frac{GM_B v_B^i}{\rho_B}\right]_\pm . \tag{7.14b}$$

The complete 1PN metric is obtained by summing Eqs. (7.14) over all B's $1 \le B \le N$, and using the usual exponential parametrization, while the A-external 1PN metric is obtained from summing over all $B \neq A$ [see Eqs. (7.7)]. As introduced above, we shall denote for brevity

$$\bar{w}^A_\mu(x) = \sum_{B\neq A} w^B_\mu(x) . \tag{7.15}$$

The equations of motion (7.9), when written in terms of the global-frame coordinate time, $t = z^0_A/c$, read

$$\frac{d^2 z^\lambda_A}{dt^2} + \bar{\Gamma}^\lambda_{A\mu\nu}\frac{dz^\mu_A}{dt}\frac{dz^\nu_A}{dt} = \frac{1}{c}\frac{dz^\lambda_A}{dt}\bar{\Gamma}^0_{A\mu\nu}\frac{dz^\mu_A}{dt}\frac{dz^\nu_A}{dt} . \tag{7.16}$$

Using Eqs. (7.7), (7.15) we get more explicitly

$$\frac{d^2 z^i_A}{dt^2} = \left[1 - \frac{4}{c^2}\bar{w}_A + \frac{1}{c^2}\mathbf{v}_A^2\right]\partial_i\bar{w}_A + \frac{4}{c^2}\partial_t\bar{w}^A_i - \frac{4}{c^2}(\partial_i\bar{w}^A_j - \partial_j\bar{w}^A_i)v^j_A - \frac{1}{c^2}(3\partial_t\bar{w}_A + 4v^j_A\partial_j\bar{w}_A)v^i_A + O(4) , \tag{7.17}$$

in which one will note the appearance of the gauge-invariant **e** and **b** fields. In order to get a fully explicit quasi-Newtonian form of these equations of motion one needs the explicit expressions of $(1/\rho_B)_\pm$ in terms of instantaneous positions and velocities. This explicit expression follows immediately from the Lagrange expansion (4.47) with [using Eqs. (5.21)]

$$f(t) = \frac{dS}{dz^0} = (e^0_{B0})^{-1} = 1 - \frac{1}{c^2}[\tfrac{1}{2}\mathbf{v}_B^2 + \bar{w}_B(\mathbf{z}_B)] + O(4) ; \tag{7.18}$$

it reads

$$\left[\frac{1}{\rho_B}\right]_{\pm}=\frac{1-c^{-2}[\mathbf{v}_B^2/2+\overline{w}_B(\mathbf{z}_B)]}{|\mathbf{x}-\mathbf{z}_B(t)|}+\frac{1}{2c^2}\frac{d^2}{dt^2}|\mathbf{x}-\mathbf{z}_B(t)|+O(4)\ . \tag{7.19a}$$

More explicitly, with $r_B^i(t)\equiv x^i-z_B^i(t)$, $n_B^i(t)\equiv r_B^i(t)/|\mathbf{r}_B(t)|$, $a_B^i\equiv d^2z_B^i/dt^2$ we get

$$(\rho_B^{-1})_{\pm}=R_B^{-1}=r_B^{-1}\left[1-\frac{1}{c^2}\overline{w}_B(\mathbf{z}_B)-\frac{1}{2c^2}(\mathbf{n}_B\cdot\mathbf{v}_B)^2-\frac{1}{2c^2}\mathbf{a}_B\cdot\mathbf{r}_B\right]+O(4)\ . \tag{7.19b}$$

A straightforward calculation of the right-hand side of Eq. (7.17) using Eqs. (7.14) and (7.19) yields finally

$$\frac{d^2z_A^i}{dt^2}=a_A^{i(\mathrm{LD})}(\mathbf{z}_B,\mathbf{v}_B)+O(4)\ , \tag{7.20a}$$

$$\begin{aligned}\mathbf{a}_A^{(\mathrm{LD})}=-\sum_{B\neq A}\frac{GM_B}{r_{AB}^2}\mathbf{n}_{AB}\Bigg[1+\frac{1}{c^2}\Bigg[&\mathbf{v}_A^2+2\mathbf{v}_B^2-4\mathbf{v}_A\cdot\mathbf{v}_B-\frac{3}{2}(\mathbf{n}_{AB}\cdot\mathbf{v}_B)^2\\ &-4\sum_{C\neq A}\frac{GM_C}{c^2r_{AC}}-\sum_{C\neq B}\frac{GM_C}{c^2r_{BC}}\left[1+\frac{1}{2}\frac{r_{AB}}{r_{CB}}\mathbf{n}_{AB}\cdot\mathbf{n}_{CB}\right]\Bigg]\Bigg]\\ -\frac{7}{2}\sum_{B\neq A}\sum_{C\neq B}\mathbf{n}_{BC}\frac{G^2M_BM_C}{c^2r_{AB}r_{BC}^2}+\sum_{B\neq A}&(\mathbf{v}_A-\mathbf{v}_B)\frac{GM_B}{c^2r_{AB}^2}(4\mathbf{n}_{AB}\cdot\mathbf{v}_A-3\mathbf{n}_{AB}\cdot\mathbf{v}_B)\ ,\end{aligned} \tag{7.20b}$$

where

$$r_{AB}\equiv|\mathbf{z}_A(t)-\mathbf{z}_B(t)|,\quad \mathbf{n}_{AB}\equiv[\mathbf{z}_A(t)-\mathbf{z}_B(t)]/r_{AB}\ . \tag{7.21}$$

One can check directly that the equations of motion (7.20) can be derived from the Lagrangian

$$\begin{aligned}L^{(\mathrm{LD})}(\mathbf{z}_A,\mathbf{v}_A)=&\sum_A\frac{1}{2}M_A\mathbf{v}_A^2+\sum_A\sum_{B\neq A}\frac{GM_AM_B}{2r_{AB}}+\sum_A\frac{1}{8c^2}M_A\mathbf{v}_A^4+\sum_A\sum_{B\neq A}\frac{3GM_AM_B\mathbf{v}_A^2}{2c^2r_{AB}}\\ &-\sum_A\sum_{B\neq A}\frac{GM_AM_B}{4c^2r_{AB}}[7\mathbf{v}_A\cdot\mathbf{v}_B+(\mathbf{n}_{AB}\cdot\mathbf{v}_A)(\mathbf{n}_{AB}\cdot\mathbf{v}_B)]-\sum_A\sum_{B\neq A}\sum_{C\neq A}\frac{G^2M_AM_BM_C}{2c^2r_{AB}r_{AC}}\ .\end{aligned} \tag{7.22}$$

The Lagrangian (7.22) can, e.g., be obtained[45] by starting from the fact that the motion of each body derives from the indidivual (geodesic) action

$$S_A=-M_Ac^2\int_{L_A}d\overline{\tau}_A=\int L_A(\mathbf{z}_A,\mathbf{v}_A)dt\ , \tag{7.23}$$

with

$$\begin{aligned}L_A&=-M_Ac\left[-\overline{g}^{\,A}_{\mu\nu}(\mathbf{z}_A)\frac{dz_A^\mu}{dt}\frac{dz_A^\nu}{dt}\right]^{1/2}\\ &=-M_Ac^2+\frac{1}{2}M_A\mathbf{v}_A^2\left[1+\frac{1}{4c^2}\mathbf{v}_A^2+\frac{3}{c^2}\overline{w}_A\right]\\ &\quad+M_A\left[\overline{w}_A-\frac{1}{2c^2}(\overline{w}_A)^2-\frac{4}{c^2}\overline{w}_i^{\,A}v_A^i\right]\ ,\end{aligned} \tag{7.24}$$

and then by symmetrizing, over the body labels, the explicit expression (in terms of the $\mathbf{z}_B$'s and $\mathbf{v}_B$'s) of the individual Lagrangians.[61]

The equations of motion (7.20) and the Lagrangian (7.22) were first obtained (for the general N-body case) by Lorentz and Droste[5] as early as 1917 (thereby correcting the even earlier results of Droste and de Sitter). This explains our label LD in Eqs. (7.20) and (7.22). However, the method used by Lorentz and Droste assumed from the start a very peculiar matter model (incompressible fluid balls), and assumed, purely by analogy with the Newtonian case, that one could neglect the mutual "tidal" influences between the N bodies. In 1938, Einstein, Infeld and Hoffmann,[19] dissatisfied by the practice of assuming specific matter models when treating the relativistic problem of motion introduced a new approach in which the information concerning the internal structure of bodies was replaced by assumptions about the structure of the exterior gravitational field near the bodies. They obtained the equations of motion (7.20), in the particular two-body case (because of that, and the fact that Ref. 5 has been long forgotten the Eqs. (7.20) are often named EIH, after the authors of Ref. 19). However, there were some flaws in the method of Ref. 19 (replaced by other flaws in subsequent papers of Einstein and Infeld), and, moreover, it was not clear to what kind of physical bodies their results could be applied. In the theory of the motion of strongly self-gravitating bodies (black holes or neutron stars) it proved very useful to devise methods that were somewhat related to the EIH one, but which completed it by techniques of asymptotic matching, allowing the transfer of information between the internal structure of the bodies and their exterior gravitational field. Examples of such methods are the ones of D'Eath[25] (for the motion of black holes) and the one of Damour[26] (for the motion of black holes or neutron stars). These works proved that the dynamics (7.20) (7.22) described the motion of strongly self-gravitating bodies if the coefficients M_A appearing in them denoted the "Schwarzschild" masses of the various bodies (see Refs. 7 and 26 for reviews of other contributions to the problem of the motion of gravitationally condensed bodies).

Concerning the Lorentz-Droste method, and the motion of weakly self-gravitating bodies, it has been refined and generalized by many subsequent works (notably Fock, and quite recently by Grishchuk and Kopejkin[62]). However, even in the most recent works developing this approach the treatment of "tidal influences" remains nearly as primitive and heuristic as in 1917. As discussed in Ref. 7 (Sec. 6.13) this leads even to an inconsistency of the derivation of higher-order relativistic corrections to Eqs. (7.20). This situation has improved only quite recently in the works of Brumberg and Kopejkin, and in the present work. We have obtained here the dynamics (7.20) as the simplest example of a consistent truncation of our scheme, and we think that already as such it improves over its previous derivations. A rather definitive improvement in the derivation and meaning of Eqs. (7.20) will follow from our subsequent papers where we shall derive the full post-Newtonian equations of motion, containing all the (relativistic) multipole moments.

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[1]F. Tisserand, *Traité de Mécanique Céleste* (reprint of the first edition of 1889) (Gauthier-Villars, Paris, 1960), Vol. I.

[2]A. Einstein, Preuss. Akad. Wiss. Berlin, Sitzber., 831 (1915).

[3]J. Droste, Versl. K. Akad. Wet. Amsterdam **19**, 447 (1916).

[4]W. de Sitter, Mon. Not. R. Astron. Soc. **76**, 699 (1916); **77**, 155 (1916).

[5]H. A. Lorentz and J. Droste, Versl. K. Akad. Wet. Amsterdam **26**, 392 (1917) (part I); **26**, 649 (1917) (part II), English translation in H. A. Lorentz, *Collected Papers,* edited by P. Zeeman and A. D. Fokker (Nijhoff, The Hague, 1937), Vol. V, pp. 330–355.

[6]V. Fock, *The Theory of Space, Time and Gravitation* (Pergamon, Oxford, 1959).

[7]T. Damour, in *300 Years of Gravitation,* edited by S. W. Hawking and W. Israel (Cambridge University Press, Cambridge, England, 1987), pp. 128–198.

[8]M. H. Soffel, *Relativity in Astrometry, Celestial Mechanics and Geodesy* (Springer, Heidelberg, 1989).

[9]J. Eisenstaedt, Arch. Hist. Exact Sci. **35**, 115 (1986).

[10]C. F. Martin, M. H. Torrence, and C. W. Misner, J. Geophys. Res. **90**, 9403 (1985).

[11]R. W. Hellings, Astron. J. **91**, 650 (1986).

[12]J. L. Synge, *Relativity: The General Theory* (North-Holland, Amsterdam, 1960).

[13]C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (Freeman, San Francisco, 1973).

[14]H. Weyl, *Raum, Zeit, Materie,* 4th ed. (Springer, Berlin, 1921).

[15]F. K. Manasse, J. Math. Phys. **4**, 746 (1963).

[16]P.D. D'Eath, Phys. Rev. D **11**, 1387 (1975).

[17]R. E. Kates, Phys. Rev. D **22**, 1853 (1980).

[18]B. Bertotti, Nuovo Cimento **12**, 226 (1954).

[19]A. Einstein, L. Infeld, and B. Hoffmann, Ann. Math. **39**, 65 (1938).

[20]C. M. Will, in *Experimental Gravitation,* 56th Enrico Fermi School, Varenna, Italy, 1972, edited by B. Bertotti (Academic, New York, 1974), pp. 1–110.

[21]C. M. Will and D. M. Eardley, Astrophys. J. **212**, L91 (1977).

[22]B. Mashhoon, Gen. Relativ. Gravit. **16**, 311 (1984).

[23]B. Mashhoon, Found. Phys. **15**, 497 (1985).

[24]K. S. Thorne and J. B. Hartle, Phys. Rev. D **31**, 1815 (1985).

[25]P. D. D'Eath, Phys. Rev. D **11**, 2183 (1975).

[26]T. Damour, in *Gravitational Radiation,* Proceedings of the Les Houches Summer School, Les Houches, France, 1982, edited by N. Deruelle and T. Piran (North-Holland, Amsterdam, 1983), pp. 59–144.

[27]N. Ashby and B. Bertotti, Phys. Rev. Lett. **52**, 485 (1984).

[28]N. Ashby and B. Bertotti, Phys. Rev. D **34**, 2246 (1986).

[29]M. H. Soffel, H. Ruder, and M. Schneider, Astron. Astrophys. **157**, 357 (1986).

[30]*Relativity in Celestial Mechanics and Astronomy,* Proceedings of the Symposium, Leningrad, USSR, 1985, edited by J. Kovalevsky and V. A. Brumberg, IAU Symposium No. 114, (Reidel, Dordrecht, 1986).

[31]V. A. Brumberg and S. M. Kopejkin, in *Reference Systems,* edited by J. Kovalevsky, I. I. Mueller, and B. Kolaczek (Reidel, Dordrecht, 1988).

[32]S. M. Kopejkin, Celest. Mech. **44**, 87 (1988).

[33]V. A. Brumberg and S. M. Kopejkin, Nuovo Cimento B **103**, 63 (1988).

[34]V. A. Brumberg, *Essential Relativistic Celestial Mechanics* (Hilger, Bristol, in press).

[35]A. V. Voinov, Celest. Mech. **42**, 293 (1988).

[36]K. S. Thorne, Rev. Mod. Phys. **52**, 299 (1980).

[37]L. Blanchet and T. Damour, Philos. Trans. R. Soc. London **A320**, 379 (1986).

[38]T. Damour, in *Gravitation in Astrophysics (Cargèse 1986),* proceedings of the NATO Advanced Study Institute, Cargese, France, 1986, edited by B. Carter and J. B. Hartle, NATO ASI Series B: Physics, Vol. 156 (Plenum, New York, 1987), pp. 3-62.

[39]X.-H. Zhang, Phys. Rev. D **34**, 991 (1986).

[40]W. M. Suen, Phys. Rev. D **34**, 3617 (1986).

[41]L. Blanchet and T. Damour, Ann. Inst. Henri Poincaré, **50**, 377 (1989).

[42]L. Blanchet, T. Damour, and G. Schäfer, Mon. Not. R. Astron. Soc. **242**, 289 (1990).

[43]Let us recall that the "nth post-Newtonian level" traditionally refers to the terms in the metric which contribute $1/c^{2n}$ corrections to the Newtonian equations of motion; this means the orders $1/c^{2n+2}$ in g_{00}, $1/c^{2n+1}$ in g_{0i}, and $1/c^{2n}$ in g_{ij}.

[44]C. Will, *Theory and Experiment in Gravitational Physics* (Cambridge University Press, Cambridge, England, 1981).

[45]L. D. Landau and E. M. Lifshitz, *The Classical Theory of*

Fields (Pergamon, Oxford, 1971).

[46]Y. Choquet-Bruhat, C. DeWitt-Morette, and M. Dillard-Bleick, *Analysis, Manifolds and Physics* (North-Holland, Amsterdam, 1982), pp. 352–354.

[47]The fact that we are dealing with a spin-2 field instead of a spin-1 field introduces irreducibly some factors 4 with respect to Maxwell's familiar equations. By changing the notation these factors 4 can change place, but they never disappear. This is why it is finally useful not to try to hide them, but on the contrary to make them explicit in the formalism, as the choice of w_i as a basic vector variable is doing.

[48]Actually Eq. (3.18b) is more accurate than one could think. Indeed, when using the isotropy condition the terms linear in w_i give Δw_i and not $\Box w_i$. This is due to the fact that the "isotropic gauge" is not exactly harmonic, but that the $O(4)$ error term in Eq. (3.15) contains even a term *linear* in w_i.

[49]P. A. M. Dirac, Proc. R. Soc. London **A167**, 148 (1938).

[50]S. T. Ma, Phys. Rev. **71**, 787 (1947).

[51]T. Damour, Thèse 3[ème] cycle, Paris, 1974.

[52]T. Damour, Nuovo Cimento B **26**, 157 (1975).

[53]L. Bel, T. Damour, N. Deruelle, J. Ibanez, and J. Martin, Gen. Relativ. Gravit. **13**, 963 (1981).

[54]J. L. Lagrange, Mémoires de l'Académie Royale des Sciences et Belles Lettres de Berlin, **24** (1770); see also E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis,* 4th ed. (Cambridge University Press, Cambridge, England, 1978), p. 132.

[55]R. Geroch, J. Math. Phys. **11**, 2580 (1970); R. O. Hansen, *ibid* **15**, 46 (1974); Thorne (Ref. 36); W. Simon and R. Beig, J. Math. Phys. **24**, 1163 (1983).

[56]T. Damour, in *Gravitatonal Collapse and Relativity,* edited by H. Sato and T. Nakamura (World Scientific, Singapore, 1986), pp. 63–73.

[57]R. Epstein and R. V. Wagoner, Astrophys. J. **197**, 717 (1975).

[58]The only place where there could be a difficulty in generalizing the method of Ref. 41 to a body arbitrarily abstracted from a gravitationally *interacting* system is when the local energy-momentum balance equations ($\nabla_\beta T^{\alpha\beta}=0$) are used. However, this happens only in the derivation of Eq. (2.29) of Ref. 41, and it makes use only of the *Newtonian level* mass-conservation equation, which does not contain any gravitational interaction term.

[59]The quantities defined by Eqs. (6.11) do not need any further justification for deserving their name of "multipole moments,"' but we can quote two of their additional properties in the special case where "body A" (which can always, in our approach, be a composite body) is really isolated (and $X_A^\alpha \equiv x^\mu$): (i) they coincide modulo $O(3,2)$ (note the c^{-3} for the mass moments) with the asymptotic radiative multipole moments, and (ii) for the particular case of stationary systems they coincide modulo $O(4,2)$ with the exact relativistic multipole moments.

[60]T. Damour and B. R. Iyer, Ann. Inst. Henri Poincaré **54**, 115 (1991).

[61]When applying this method, it is convenient to introduce a gauge transformation $\bar{\lambda}^A = \sum_{B\neq A} \frac{1}{2} G M_B \partial_t |\mathbf{x}-\mathbf{z}_B(t)|$, which gets rid of the second time derivatives in $\bar{w}^A$ brought about by the second term in the right-hand side of Eq. (7.19a).

[62]L. P. Grishchuk and S. M. Kopejkin, in *Relativity in Celestial Mechanics and Astrometry* (Ref. 30), pp. 19–34. See also T. Damour, L. P. Grishchuk, S. M. Kopejkin, and G. Schäfer, in *Fifth Marcel Grossmann Meeting,* proceedings, Perth, Australia, 1988, edited by D. Blair and M. J. Buckingham (World Scientific, Singapore, 1989), Part B, pp. 1311–1313.