Dissipative Boltzmann-Robertson-Walker cosmologies

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The equations governing a flat Robertson-Walker cosmological model containing a dissipative Boltzmann gas are integrated numerically. The bulk viscous stress is modeled using the Eckart and Israel-Stewart theories of dissipative relativistic fluids; the resulting cosmologies are compared and contrasted. The Eckart models are shown to always differ in a significant quantitative way from the Israel-Stewart models. It thus appears inappropriate to use the pathological (nonhyperbolic) Eckart theory for cosmological applications. For large bulk viscosities, both cosmological models approach asymptotic nonequilibrium states; in the Eckart model the total pressure is negative, while in the Israel-Stewart model the total pressure is asymptotically zero. The Eckart model also expands more rapidly than the Israel-Stewart models. These results suggest that "bulk-viscous" inflation may be an artifact of using a pathological fluid theory such as the Eckart theory.

I. INTRODUCTION

The first attempts at creating a theory of relativistic dissipative fluids were those of Eckart[1] and Landau and Lifshitz,[2] which are presented in many texts on relativistic physics.[2–4] These theories are now known to be pathological in several respects. Regardless of the choice of equation of state, all equilibrium states in these theories are unstable;[5] in addition, signals may be propagated through the fluid at velocities exceeding the speed of light.[6,7] These theories are thus unacceptable even in principle, since they violate the basic principles of special relativity.

A number of workers have more recently developed theories closely related to the Grad approximation method in nonrelativistic kinetic theory; the most comprehensive early treatment of such theories was that of Israel and Stewart.[8–11] In the Israel-Stewart theories, it can be shown that stable equilibria are possible, and further, that perturbative signals will propagate causally via hyperbolic equations if the fluid equilibria are stable.[12,13]

Despite the apparent superiority of the Israel-Stewart class of theories of dissipative relativistic fluids, many calculations involving fluid dissipation in a relativistic context (e.g., in astrophysics, cosmology, or heavy-ion collisions) still are performed using the Eckart and/or Landau-Lifshitz theories. This is largely due to two reasons. First, the Eckart and Landau-Lifshitz theories are much simpler to deal with; in these theories, a fluid possesses only five degrees of freedom (three velocity, two thermodynamic), while in the Israel-Stewart-type extended theory, the deviations from equilibrium (bulk stress, heat flow, and shear stress) are treated as independent dynamical variables, leading to a total of 14 dynamical fluid variables to be determined. Second, it is widely believed that the unstable, noncausal behavior associated with the simpler theories always occurs in a wildly nonphysical domain (e.g., the growth time scale for an Eckart instability in room-temperature water is 10^{-34} sec), so that it will be easy (in general) to identify unphysical spurious modes and discard them by hand.

The purpose of this paper is compare and contrast the predictions of the Eckart and Israel-Stewart theories in a physically interesting context: to examine the full nonlinear evolution of a dissipative relativistic fluid in an expanding isotropic universe. Since the detailed behavior of a dissipative relativistic fluid depends strongly on the thermodynamic properties of the fluid, which are generally inadequately known for realistic materials, we are forced to use the Boltzmann gas as the cosmological fluid in our models. Only for the Boltzmann gas are the dissipation coefficients and second-order coefficients known precisely over the entire range of fluid densities and temperatures encountered in a cosmology. Thus, while a Boltzmann gas is not a realistic model for the cosmological fluid in a realistic universe, it is the only fluid for which the thermodynamic properties are well enough established by relativistic kinetic theory to allow us to construct precise models. Since our goal is to compare the predictions of alternative fluid theories rather than construct realistic cosmological models, this restriction is not too burdensome.

The cosmological model is taken to be described by a spatially flat Robertson-Walker metric. The combined gravitational and fluid equations are derived and numerically integrated for both the Eckart and Israel-Stewart theories (the predictions of the Landau-Lifshitz theory are identical to those of the Eckart theory in this case). The resulting numerically integrated cosmological models can be properly referred to as dissipative Boltzmann-Robertson-Walker models.

The only previous comparisons which have been made for nonlinear evolution of a dissipative relativistic fluid are for plane-symmetric motions of an inviscid fluid (i.e., the only allowed dissipation was a nonzero heat flow).[14,15] In contrast, in an isotropic cosmological model, the high degree of symmetry guarantees that the heat-flow vector and shear-stress tensor vanish. The only nonzero deviation from equilibrium possible is then a nonequilibrium contribution to the isotropic pressure, i.e., a bulk stress.

The effects of a bulk stress (or the existence of a nonzero bulk viscosity) have often been ignored in fluid dynamics, largely because of the curious circumstance that the bulk viscosity coefficient vanishes in both the Newtonian and ultrarelativistic limits for a Boltzmann gas (though it is nonzero at all finite temperatures greater than zero[16]). It is, however, not generally valid to set the bulk viscosity to zero as an approximation in the Israel-Stewart theory, since the bulk stress constitutes a dynamic degree of freedom. There are situations (such as in the study of shock waves[17]) where the very existence of a nonzero bulk viscosity can have important effects on the physics of the fluid, regardless of the magnitude of the bulk-viscosity coefficient.

The effects of a nonzero bulk viscosity on isotropic cosmological models have been previously considered in several studies. It has long been realized that an expanding Robertson-Walker universe does not allow equilibrium solutions of the relativistic Boltzmann equation for massive particles (i.e., the expansion of the universe *must* be nonadiabatic). Weinberg[18] performed the first serious study of the effects of radiative bulk viscosity in an expanding universe; his study was the first to show that radiative bulk-viscous effects could in no way account for the large entropy of the present Universe. A number of studies by Bernstein and his co-workers have examined in detail relativistic kinetic theory in isotropic cosmological models.[19] Studies have also considered whether viscous effects might sufficiently violate the energy conditions to allow the removal of the initial singularity.[20–22] More recently, bulk viscosity has been suggested by a number of workers as a possible driving force for an inflationary epoch in the early Universe.[23–27] These models have been criticized[28] on the basis that kinetic theory does not allow the physical pressure (equilibrium pressure plus bulk stress) of a radiative gas to become negative, a necessary condition of inflationary expansion. In rebuttal, it has been pointed out[29] that kinetic theory cannot be expected to describe adequately a dense gas; the difficulty of proceeding further and accurately describing the bulk viscosity of the early Universe is that one quickly encounters conditions under which we have only the most rudimentary knowledge of the equation of state, much less the detailed form of the viscosity coefficients. All of these studies (with the exception of Ref. 29) have used the Eckart theory to describe the dissipative fluids within the expanding Universe.

There have also been a number of works which have examined various aspects of dissipative fluid cosmology within the framework of an extended, Israel-Stewart-type fluid description.[29–32] In these studies, various approximations have been made in evaluating the bulk-viscosity coefficient and the second-order coefficient β_0 (essentially the relaxation time for the bulk stress). Further, in all of these cases, the authors have used an approximation to the full Israel-Stewart theory in which β_0/T (where T is the temperature) is assumed to have zero time derivative. We believe the present work is the first to faithfully incorporate the complete Israel-Stewart fluid dynamics in an isotropic cosmological model.

Since we use the physically inadequate Boltzmann gas model for the cosmological fluid at all temperatures and densities, we do not expect our results to yield new insights into cosmology, but rather into the question of whether the simpler (yet pathological) Eckart theory will yield reliable results in an appropriate physical application. Our primary conclusion is that the Eckart theory does shows its pathology in a cosmological setting: There are always significant quantitative differences between the Eckart and Israel-Steward fluid model cosmologies. The differences are most dramatic for large bulk viscosities: There both theories yield "runaway" solutions in which the bulk stress grows to become comparable to the equilibrium pressure and remains large asymptotically into the future. In the Eckart theory, the "runaway" has a negative total pressure (bulk stress larger than equilibrium pressure), and the universe expands asymptotically as $t^{7/9}$; as it expands, the universe heats up rather than cooling off. For comparison, in the Israel-Stewart "runaway" solutions, the total pressure is asymptotically zero (bulk stress equal to equilibrium pressure), the universe expands as $t^{2/3}$, and the universe tends to an asymptotically constant temperature. Neither of these solutions should have any special relevance to cosmology: It is notable, however, that the Israel-Stewart "runaway" solution, while fairly bizarre as a cosmological model, is significantly less so than the Eckart model (e.g., the total pressure remains non-negative in the Israel-Stewart theory). Further, it is interesting to note that the Eckart "runaway" bears some resemblance to the supposed "bulk-viscous" inflationary models; we are strongly suspicious that bulk-viscous inflation may be a consequence of using an inadequate theory of fluid dissipation, such as the Eckart theory. This reinforces the need to use a stable, hyperbolic theory of dissipative fluids to achieve trustworthy results.

II. DISSIPATIVE RELATIVISTIC FLUID MECHANICS

The fundamental variables of a theory of relativistic dissipative fluids are the stress-energy tensor T^{ab} and the particle number current N^a. These fields obey the conservation equations

$$\nabla_a T^{ab}=0 \tag{1}$$

and

$$\nabla_a N^a=0 \; . \tag{2}$$

The derivative operators appearing in Eqs. (1) and (2) are four-dimensional covariant derivatives. The stress-energy tensor and particle number current can be decomposed as

$$T^{ab}=\rho u^a u^b+(p+\tau)q^{ab}+q^a u^b+q^b u^a+\tau^{ab} \tag{3}$$

and

$$N^a=nu^a \; , \tag{4}$$

where u^a is the four-velocity of the fluid, ρ is the energy density (as measured by an observer comoving with the fluid), p is the equilibrium pressure, n is the number density, and q^{ab} is the projection tensor orthogonal to u^a. The fields τ, q^a, and τ^{ab} describe the deviation from equilibrium in the fluid; τ is the bulk stress, q^a is the heat-flow vector, and τ^{ab} is the shear-stress tensor. While there is a unique definition of the four-velocity for a fluid in equilibrium, there are an infinite number of choices which may be made off equilibrium. In our decomposition in Eqs. (3) and (4), we have chosen to use the "Eckart" definition of four-velocity, in which the four-velocity is chosen to be parallel to the particle number current vector. In order for the decomposition in Eq. (3) to be unique, it is necessary that the fields q^a and τ^{ab} satisfy the constraints

$$q^a u_a = \tau^{ab} u_b = \tau^a{}_a = \tau^{ab} - \tau^{ba} = 0 \ . \tag{5}$$

The pressure p is defined in terms of ρ and n by using the equilibrium equation of state for the fluid; the bulk stress τ is then the difference between the physical isotropic spatial stress $(T^{ab} q_{ac} q_b{}^c/3)$ and the thermodynamically defined equilibrium pressure p.

The equations defining the deviations from equilibrium (τ, q^a, τ^{ab}) are obtained by imposing the second law of thermodynamics. The entropy current s^a, written in the most general form through quadratic order in the deviations from equilibrium, takes the form

$$\begin{aligned} s^a = {} & snu^a + q^a/T \\ & -\tfrac{1}{2}(\beta_0 \tau^2 + \beta_1 q^b q_b + \beta_2 \tau^{ab}\tau_{ab}) u^a/T \\ & + \alpha_0 \tau q^a/T + \alpha_1 \tau^a{}_b q^b/T \ , \end{aligned} \tag{6}$$

where T is the temperature and s is the entropy per particle. The three thermodynamic coefficients β_i model the deviations of the physical entropy density from the thermodynamic entropy density sn. The other two coefficients α_i describe couplings between the heat flow and viscous deviations from equilibrium. The expression for s^a given in Eq. (6) contains all possible terms through second order in the deviations from equilibrium; choosing s^a to have this form yields a "second-order" theory, of the sort developed by Israel and Stewart. If the α_i and β_i are taken to be identically zero, then the first-order theory of Eckart results (it is worth noting that relativistic kinetic theory shows that these coefficients are nonzero for simple gases).

The second law of thermodynamics is embodied in the requirement that[8]

$$\nabla_a s^a \geq 0 \ . \tag{7}$$

The divergence of the entropy current defined in Eq. (6) may be computed and simplified using the conservation equations [Eqs. (1) and (2)]. The divergence of s^a may then be forced into the following manifestly non-negative form:

$$T\nabla_a s^a = \frac{\tau^2}{\zeta} + \frac{q_a q^a}{\kappa T} + \frac{\tau_{ab}\tau^{ab}}{2\eta} \ . \tag{8}$$

The three (positive) dissipation coefficients may be identified as the bulk viscosity ζ, the thermal conductivity κ, and the shear viscosity η. In the Israel-Stewart theory, the resulting expressions for τ, q^a, and τ^{ab} are

$$\tau = -\zeta \left[\nabla_a u^a + \beta_0 u^a \nabla_a \tau - \alpha_0 \nabla_a q^a - \gamma_0 T q^a \nabla_a \left[\frac{\alpha_0}{T} \right] + \frac{1}{2} \tau T \nabla_a \left[\frac{\beta_0}{T} u^a \right] \right] , \tag{9}$$

$$\begin{aligned} q^a = -\kappa T q^{ab} \Bigg[& \frac{1}{T}\nabla_b T + u^c \nabla_c u_b + \beta_1 u^c \nabla_c q_b - \alpha_0 \nabla_b \tau - \alpha_1 \nabla_c \tau^c{}_b + \frac{1}{2} T q_b \nabla_c \left[\frac{\beta_1}{T} u^c \right] \\ & -(1-\gamma_0)\tau T \nabla_b \left[\frac{\alpha_0}{T} \right] - (1-\gamma_1) T \tau_b{}^c \nabla_c \left[\frac{\alpha_1}{T} \right] + \gamma_2 \nabla_{[b} u_{c]} q^c \Bigg] , \end{aligned} \tag{10}$$

$$\tau^{ab} = -2\eta \left\langle \nabla^a u^b + \beta_2 u^c \nabla_c \tau^{ab} - \alpha_1 \nabla^a q^b + \frac{1}{2} T \tau^{ab} \nabla_c \left[\frac{\beta_2}{T} u^c \right] - \gamma_1 T q^a \nabla^b \left[\frac{\alpha_1}{T} \right] + \gamma_3 \nabla^{[a} u^{c]} \tau_c{}^b \right\rangle , \tag{11}$$

where the brackets which appear in Eq. (11) are defined by

$$\langle A^{ab} \rangle = \tfrac{1}{2} q^a{}_c q^b{}_d (A^{cd} + A^{dc}) - \tfrac{1}{3} q^{ab} q_{cd} A^{cd} \ . \tag{12}$$

The expressions for τ, q^a, and τ^{ab} in the Eckart theory may be obtained from Eqs. (9)–(11) by setting all the α_i, β_i, and γ_i equal to zero. Note that in the Eckart theory, the equations defining τ, q^a, and τ^{ab} are algebraic rather than differential equations; in the first-order theories, the deviations from equilibrium are not additional degrees of freedom. The first-order theories contain only five degrees of freedom (three components of the four-velocity and two thermodynamic variables), exactly as in the perfect-fluid case. In the second-order theories, there are an additional nine degrees of freedom, representing the free components of τ, q^a, and τ^{ab}, whose evolution is governed by Eqs. (9)–(11).

Equations (1), (2), and (9)–(11) form the complete set of equations for the second-order Israel-Stewart theory, with the Eckart choice of four-velocity (in the "Eckart frame"). Gravitational interactions are included by adding the Einstein equations to this set:

$$G_{ab} = 8\pi T_{ab} \ . \tag{13}$$

III. ROBERTSON-WALKER COSMOLOGICAL MODELS

The Robertson-Walker metric[33,34] is the most general spatially homogeneous metric which is in addition isotropic about every point. The metric may be written as

$$ds^2=-dt^2+a^2(t)[(1-Kr^2)^{-1}(dx^2+dy^2+dz^2)] , \quad (14)$$

where $r^2=x^2+y^2+z^2$, and K is the (constant) spatial curvature. The spatial three-geometry is spherical, flat, or hyperbolic as K is chosen to be positive, zero, or negative, respectively.

In the present study, we are interested in applying the models of relativistic dissipative fluids described in Sec. II to isotropic cosmological models. The spatial isotropy of the Robertson-Walker metric requires that the four-velocity of the fluid must be taken to be purely along the time axis; in the coordinate system of Eq. (14), $u^a=(1,0,0,0)$. The heat-flow vector and shear-stress tensor are then purely spatial, by the constraints of Eq. (5). The isotropy of the model then further implies that both q^a and τ^{ab} must be identically zero. The heat flow must be zero, since if it were not, it would pick out a preferred direction in space (being a purely spatial vector field). Isotropy implies that the matrix of components of τ^{ab} must be invariant under SO(3) rotations at every point in space; the only matrix which is so invariant is a constant multiplied by the unit matrix; since τ^{ab} is traceless, the multiplying constant must be zero. Thus the only form of dissipation which can be accommodated in a Robertson-Walker cosmological model is a bulk stress τ. With these restrictions on the deviations from equilibrium, the defining equation for τ [Eq. (9)] takes on the simpler form

$$\tau=-\zeta\left[\nabla_a u^a+\beta_0 u^a\nabla_a\tau+\frac{1}{2}\tau T\nabla_a\left[\frac{\beta_0}{T}u^a\right]\right] . \quad (15)$$

This will be our fundamental equation describing the evolution of the bulk stress. The bulk-stress equation for the Eckart theory is obtained by setting β_0 equal to zero. Note again that if β_0 is zero, then Eq. (15) is an algebraic equation defining τ, rather than a differential equation. The final term on the right-hand side of Eq. (15) has been neglected in most studies applying the second-order theory to cosmology;[29-31] however, as we shall show, this term has large effects on the evolution of the cosmological model.

Isotropy and homogeneity thus force the stress-energy tensor to be of the simple diagonal form

$$T^{ab}=\rho u^a u^b+(p+\tau)q^{ab} . \quad (16)$$

We shall consider only the case of zero spatial curvature, $K=0$, in order to simplify the resulting cosmological models. The metric then reduces to the familiar form

$$ds^2=-dt^2+a^2(t)(dx^2+dy^2+dz^2) . \quad (17)$$

The Einstein equations for the metric given in Eq. (17), with the stress-energy tensor of Eq. (16), may be written as

$$\left[\frac{\partial a}{\partial t}\right]^2=\frac{8\pi}{3}\rho a^2 \quad (18)$$

and

$$2a\frac{\partial^2 a}{\partial t^2}=-\left[\frac{\partial a}{\partial t}\right]^2-8\pi(p+\tau)a^2 . \quad (19)$$

The time component of the conservation of stress-energy equation ($\nabla_a T^{ab}=0$) for the stress-energy tensor of Eq. (16) has the form

$$\frac{\partial}{\partial t}(\rho a^3)=-(p+\tau)\frac{\partial}{\partial t}(a^3) . \quad (20)$$

The conservation of stress-energy equation (20) of course follows directly from the Einstein equations (18) and (19) as a consequence of the Bianchi identities; thus only two of the three equations (18)–(20) are independent.

The conservation equation for the particle number current [Eq. (1)] has the form

$$\frac{\partial}{\partial t}(na^3)=0 , \quad (21)$$

when the metric is taken to be of the Robertson-Walker (RW) form.

In order to completely integrate a spatially flat RW cosmological model from initial conditions, it is necessary to be able to determine the dynamical evolution of three variables: $a(t)$, $\rho(t)$, and $\tau(t)$. The three necessary equations may be chosen as the Einstein equation [Eq. (18)], the conservation of stress-energy equation [Eq. (20)], and the evolution equation for the bulk stress [Eq. (15)]. Additional equations of state are needed in order to relate the various thermodynamic variables to one another; specifically, it is necessary to know the form of $p=p(\rho,n)$, $\zeta(\rho,n)$, and $\beta_0(\rho,n)$. The number density n is not an independent variable, but is determined by an initial condition along with Eq. (21). The form of these additional equations of state for the relativistic Boltzmann gas is discussed in the next section.

IV. RELATIVISTIC BOLTZMANN GAS

In order to integrate successfully the equations describing a dissipative fluid isotropic cosmological model, it is necessary to have a detailed knowledge of the thermodynamic properties of the fluid. In particular, it is necessary to know the equations of state of the fluid [which may be represented by the function $s(n,\rho)$] and the equation defining the bulk viscosity as a function of the thermodynamic state [i.e., $\zeta(n,\rho)$]. If the fluid theory is a second-order theory, such as the Israel-Stewart theory, then it is also necessary to know the form of the thermodynamic coefficient $\beta_0(n,\rho)$. In principle, this knowledge is obtainable either from a phenomenological analysis of experimental data or by derivation from a microscopic model of the fluid (e.g., kinetic theory).

The thermodynamic properties of a relativistic Boltzmann gas may be obtained by solving the Boltzmann equation in an appropriate approximation. A number of authors have determined the fundamental thermodynamic equations of state using the relativistic version of the Grad method of moments.[35–37] This method also allows for the calculation of the second-order coefficients[9–11] and the dissipation coefficients.[35,38]

While a Boltzmann gas is not a physically realistic model of the contents of the Universe at all temperatures and densities (among other problems, it ignores the creation of particle-antiparticle pairs at relativistic temperatures), it is the only precise model we can treat at this time within the context of the second-order Israel-Stewart fluid theory. The "second-order coefficients" (α_i, β_i, and γ_i) have been evaluated using relativistic kinetic theory only for simple gases.[11] Closed-form expressions for these coefficients are known only for the Boltzmann[11] and degenerate Fermi[39] gases. While the Boltzmann gas is not an adequate model of the real material content of the Universe, it would appear that the degenerate Fermi gas is even less appropriate. Within the current development of relativistic kinetic theory, it then appears that the Boltzmann gas is the only system for which the dissipation and second-order coefficients are known over the complete range of conditions encountered in the expanding universe.

A great deal of confusion concerning the magnitude and even existence of bulk-viscosity effects arose historically as a result of the fact that the bulk viscosity of a Boltzmann gas vanishes in both the Newtonian and ultrarelativistic limits ("vanishes" is used here in the sense that the ratio of the bulk-viscosity to the shear-viscosity coefficient approaches zero in both limits; at any temperature other than zero or infinity, the bulk viscosity is actually nonzero). Israel[16] seems to have been the first to realize that the bulk viscosity of a Boltzmann gas was in fact nonzero, and that it was largest at mildly relativistic temperatures ($kT \simeq mc^2$). While the general form of the bulk viscosity is known for a variety of systems (e.g., a radiative gas[18]), a precise functional form valid at all temperatures is known only for the Boltzmann gas.[37,38,40]

The thermodynamic properties of the Boltzmann gas may be described by the dimensionless inverse temperature $\beta \equiv mc^2/kT$, the relativistic chemical potential α and a constant $A_0 = m^4 g/(2\pi^2\hbar^3)$, where g is the spin weight of the fluid particles.[11] The ideal-gas law then has the familiar form

$$p = nm\beta^{-1} . \tag{22}$$

The number density is given by

$$nm = A_0 e^{-\alpha} K_2(\beta)/\beta , \tag{23}$$

while the energy density may be written as

$$\rho = A_0[\beta^{-1}K_1(\beta) + 3\beta^{-2}K_2(\beta)] . \tag{24}$$

It is interesting that the second-order coefficients (α_i, β_i, and γ_i) are, like the equilibrium quantities (and unlike the dissipation coefficients), independent of the scattering cross section of the fluid particles. The only second-order coefficient we require, β_0, when evaluated for the Boltzmann gas, may be written as[11]

$$\beta_0 = 3\Omega^*/(\eta^2\Omega^2 p) , \tag{25}$$

where p is the equilibrium pressure defined above, and

$$\eta = K_3(\beta)/K_2(\beta) , \tag{26}$$

$$\Omega = 3\gamma[1 + 1/(\eta\beta)] - 5 , \tag{27}$$

$$\Omega^* = 5 - 3\gamma + 3(10 - 7\gamma)\eta/\beta , \tag{28}$$

where γ is defined as the solution to

$$\gamma/(\gamma - 1) = \beta^2(1 + 5\eta/\beta - \eta^2) . \tag{29}$$

The evaluation of the dissipation coefficients (bulk- and shear-viscosity coefficients, thermal conductivity) for the relativistic Boltzmann gas is a more complex undertaking owing to the need to evaluate collision integrals. The bulk-viscosity coefficient ζ is found to be of the form[37,40]

$$\zeta = mc\eta^2\Omega^2/I , \tag{30}$$

where I is a collision integral:

$$I = \frac{2^7\pi\beta^5}{[K_2(\beta)]^2}\int_0^\pi d\theta \sin^3(\theta)\int_0^\infty d\psi\, \sigma(v,\theta) x^{-3} K_3(x) \sinh^7(\psi)\cosh^3(\psi) , \tag{31}$$

$x = 2\beta\cosh(\psi)$, $2c\tanh(\psi) = v$ is the relative three-velocity of the colliding particles, θ is the angle of deflection in the center-of-mass frame, and $\sigma(v,\theta)$ is the scattering cross section of the fluid particles.

While we have very limited knowledge as to the functional form of σ for realistic particles, it is clear that even if we had complete knowledge, the integral in Eq. (31) could not be evaluated in closed form. In order to obtain a tractable problem, we will make the simplest possible assumption concerning σ: namely, that σ is independent of v. Such a cross section was originally used (in the nonrelativistic context) by Maxwell;[41] it is accordingly often referred to as a "Maxwellian cross section" (or, more physically, as a constant cross section). Specifically we shall define

$$\sigma_0 = \frac{1}{2\pi}\int_0^\pi d\theta\, \sigma(\theta)\sin^3(\theta) = \text{const} . \tag{32}$$

With this choice of cross section, the integration over ψ in Eq. (31) may be carried out explicitly;[37] the result is

$$I = \frac{96\pi^2\sigma_0[\beta K_3(2\beta) + 4K_2(2\beta)]}{\beta^3[K_2(\beta)]^2} . \tag{33}$$

The value we shall use for ζ is obtained by substituting this expression for I into the general expression for ζ for a Boltzmann gas given by Eq. (30). The kinetic theory values we shall need to describe a Boltzmann gas in an expanding Robertson-Walker universe are given by Eqs. (22)–(25), (30), and (33).

V. BOLTZMANN-ROBERTSON-WALKER COSMOLOGICAL MODELS

In order to study the effect of different fluid models of bulk stress on the expanding Universe, the differential equations derived in the previous sections were integrated numerically to yield $k=0$ dissipative Boltzmann-Robertson-Walker cosmological models. For the Eckart fluid theory, the Einstein equation [Eq. (18)] and the conservation of stress-energy equation [Eq. (20)] were integrated, using the kinetic theory values of Sec. IV to relate the thermodynamic variables. The thermodynamics are most easily dealt with if $a(t)$ and $\beta(t)$ are chosen to be the independent variables; Eq. (20) can then be reworked into a differential equation for the inverse temperature β. After a fair amount of algebra, the following differential equation is obtained:

$$\dot\beta = 6a^2\dot a \frac{p+\tau}{N_0 m}[K_2(\beta)]^2 f(\beta) , \tag{34}$$

where N_0 is the number of particles per unit volume at the initial time, an overdot indicates differentiation with respect to time, and

$$f(\beta) = \{K_0(\beta)K_2(\beta) + (1+6\beta^{-2})[K_2(\beta)]^2 - [K_1(\beta)]^2 - K_1(\beta)K_3(\beta)\}^{-1} . \tag{35}$$

In the Israel-Stewart case, the bulk stress is an independent dynamical variable, and so Eqs. (15), (18), and (34) were integrated. The independent dynamical variables were chosen to be $a(t)$, $\beta(t)$, and $\tau(t)$. Equation (15) may be rewritten as

$$\dot\tau = -\frac{1}{\beta_0}\left[\frac{3\dot a}{a} + \left(\frac{3\dot a}{2a}\beta_0 + \frac{1}{\zeta} + \frac{\dot\beta_0}{2} + \frac{\beta_0\dot\beta}{2\beta}\right)\tau\right] . \tag{36}$$

Here $\dot a$ and $\dot\beta$ are defined through Eqs. (18) and (34), respectively, while $\dot\beta_0$ is defined through an algebraic tangle of thermodynamic quantities which ultimately depend only on β, $\dot\beta$, and $\dot a$:

$$\dot\beta_0 = 3\left[\frac{\dot\Omega^*}{\eta^2\Omega^2 p} - \frac{2\Omega^*\dot\eta}{\eta^3\Omega^2 p} - \frac{2\Omega^*\dot\Omega}{\eta^2\Omega^3 p} - \frac{\Omega^*\dot p}{\eta^2\Omega^2 p^2}\right] , \tag{37}$$

where

$$\dot p = -\left(3\frac{\dot a}{a} + \frac{\dot\beta}{\beta}\right)p , \tag{38}$$

$$\dot\eta = \frac{\dot\beta}{2[K_2(\beta)]^2}\{K_1(\beta)K_3(\beta) + [K_3(\beta)]^2 - [K_2(\beta)]^2 - K_2(\beta)K_4(\beta)\} , \tag{39}$$

$$\dot\gamma = -(\gamma-1)^2\left[\beta^2\left(\frac{5\dot\eta}{\beta} - \frac{5\eta}{\beta}\dot\beta + 2\eta\dot\eta\right) + 2\beta\dot\beta + 2\beta\dot\beta\left(1+\frac{5\eta}{\beta} - \eta^2\right)\right] , \tag{40}$$

$$\dot\Omega = 3\gamma\left(-\frac{\dot\eta}{\eta^2\beta} - \frac{\dot\beta}{\eta\beta^2}\right) + 3\left(1+\frac{1}{\eta\beta}\right)\dot\gamma , \tag{41}$$

$$\dot\Omega^* = \left(-3+21\frac{\eta}{\beta}\right)\dot\gamma + 3(10-7\gamma)\left(\frac{\dot\eta}{\beta} - \frac{\eta\dot\beta}{\beta^2}\right) . \tag{42}$$

The equations were numerically integrated over a range of cosmological parameters running from an initially ultrarelativistic universe, with $\beta=0.001$ and scale factor $a=1$, to a final state which would be highly nonrelativistic in the absence of dissipation (scale factor $a=10^6$). The initial energy density was chosen to be equal to that of a radiation-dominated universe with $k=0$; the initial number density ($N_0 a^{-3}$) was then determined from the fluid equation of state [Eq. (22)]. This choice of domain covers the mildly relativistic regime ($\beta \simeq 1$) where bulk-viscous effects are expected to be largest. The relative size of the bulk-viscous effects was varied by constructing models with differing bulk-viscosity coefficients; the bulk-viscosity coefficient was adjusted by varying the Boltzmann gas particle rest mass (ζ is proportional to the cube of the particle mass). Since the Boltzmann gas is not a realistic model of the physical cosmological fluid, the particular values of mass used should not be considered to have any particular cosmological significance. The initial value of the bulk stress τ, which must be specified independently in the Israel-Stewart case, was set equal to its Eckart value.

A primary indicator of the importance of bulk-viscous effects is the ratio $|\tau/p|$, which measures the relative size of the nonequilibrium pressure compared to the equilibrium pressure. For small values of the particle mass ($m < 10^4$ GeV), the behavior of $|\tau/p|$ as a function of the expansion factor $a(t)$ is qualitatively the same for both the Eckart and Israel-Stewart theories. A plot of $|\tau/p|$ vs $a(t)$ for $m=10^4$ GeV is shown in Fig. 1 for both theories. The behavior of the bulk stress is what one

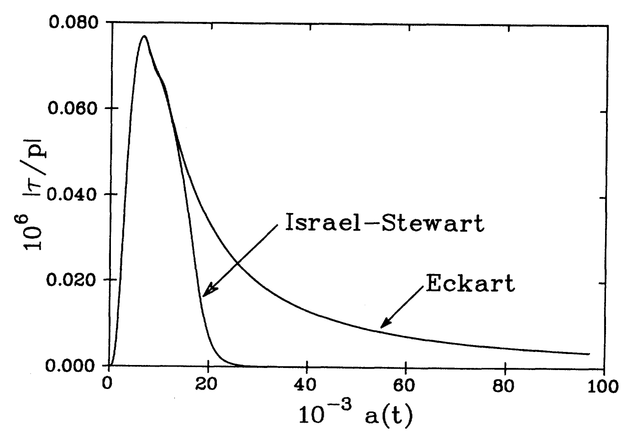

FIG. 1. Magnitude of the ratio of the bulk stress to the equilibrium pressure, $|\tau/p|$, is plotted against the cosmological scale factor $a(t)$ for both the Eckart and Israel-Stewart fluid models. The particle mass is taken to be 10^4 GeV. As would be naively expected, the bulk stress is small in both the ultrarelativistic and nonrelativistic regimes, and reaches a peak when the temperature is comparable to the particle rest mass.

would naively expect: It is small in both the ultrarelativistic and nonrelativistic limits, and is most prominent in the "mildly" relativistic region (the peak here actually occurs at $\beta \simeq 12$). These cosmological models are qualitatively similar and are also similar to nondissipative models. The pathological Eckart theory predicts a larger bulk stress at late times than the Israel-Stewart model, but still the bulk stress decays away as the nonrelativistic limit is approached.

It is perhaps worthwhile to give some further details of these models, since they illustrate nicely the transition between the two standard cosmological models described analytically in every textbook: the radiation-dominated universe ($p = \rho/3$) and the pressureless dust ($p = 0$) universe. Log-log plots are a convenient way to illustrate the transition between the asymptotic power-law behaviors of the various cosmological variables. For all cosmological variables except the bulk stress, the predictions of the two theories are indistinguishable.

The dependence of β on the scale factor $a(t)$ is shown in Fig. 2. For $\beta < 1$, β is linearly proportional to $a(t)$, as would be expected for a radiation gas. There is a smooth transition in the region $1 < \beta < 10$, and for $\beta > 10$, the behavior appropriate to a nonrelativistic gas ($\beta \approx a^2$) is recovered. Figure 3 illustrates the dependence of ρ on $a(t)$, showing the smooth transition from a radiation-gas-like behavior in the early universe ($\rho \approx a^{-4}$) to a pressureless gas in the later universe ($\rho \approx a^{-3}$), when the temperature has become nonrelativistic. Finally, Fig. 4 shows how the expansion of the universe varies with time; again, in the early universe, the Boltzmann gas is well approximated by a radiation gas, and $a \approx t^{1/2}$; in the later universe, when the temperature has dropped to nonrelativistic values, $a \approx t^{2/3}$; there is a smooth transition between these two asymptotes.

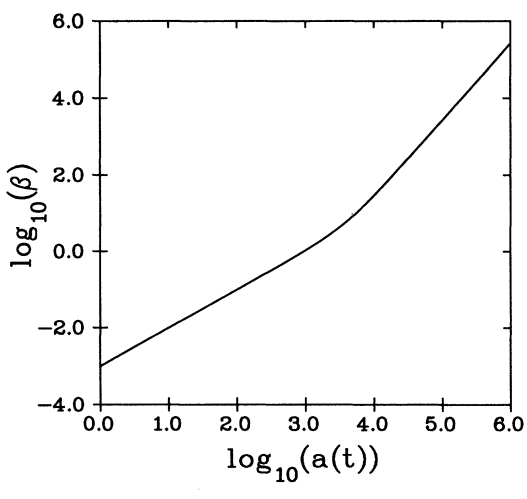

FIG. 2. Inverse dimensionless temperature β is plotted against the scale factor $a(t)$. In the early universe (small scale factor), the temperature evolves as in a radiation gas: $\beta \approx a$. There is then a smooth transition to the nonrelativistic domain, where the temperature evolves as in a nonrelativistic gas, $\beta \approx a^2$. The particle mass is chosen to be 10^4 GeV. The behavior of the Eckart and Israel-Stewart fluids are indistinguishable at this scale.

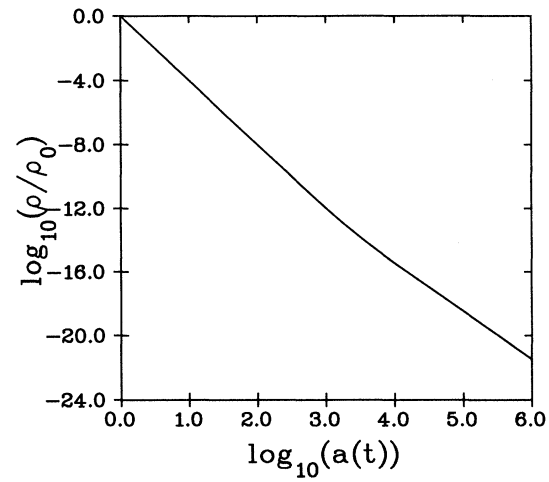

FIG. 3. Energy density ρ (divided by the initial energy density ρ_0) is plotted against the scale factor $a(t)$. A smooth transition is seen between the behavior of a radiation gas in the early universe, $\rho \approx a^{-4}$, and the behavior of a pressureless gas in the late universe, $\rho \approx a^{-3}$. The particle mass is chosen to be 10^4 GeV.

As the particle mass (and bulk-viscosity coefficient) are increased, a striking qualitative change in the cosmological models occurs. Above a critical mass (which is different for the different fluid theories), the bulk stress, instead of rising to a peak and then falling, rises to approach asymptotically a value comparable to the equilibrium pressure.

The critical mass, above which bulk-stress effects dom-

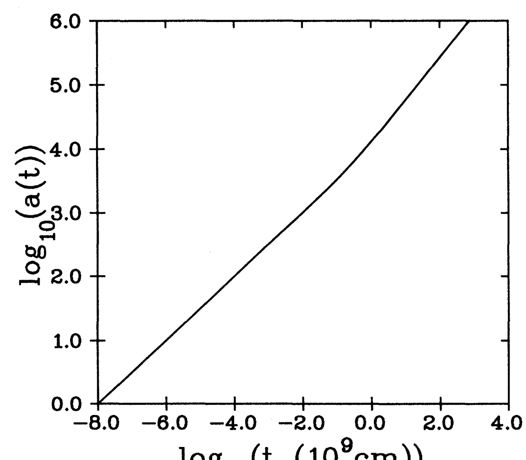

FIG. 4. Scale factor $a(t)$ is plotted against time. There is a smooth transition between the radiation-dominated universe behavior at early times ($a \approx t^{1/2}$) and the matter-dominated universe at late times ($a \approx t^{2/3}$). The particle mass is chosen to be 10^4 GeV.

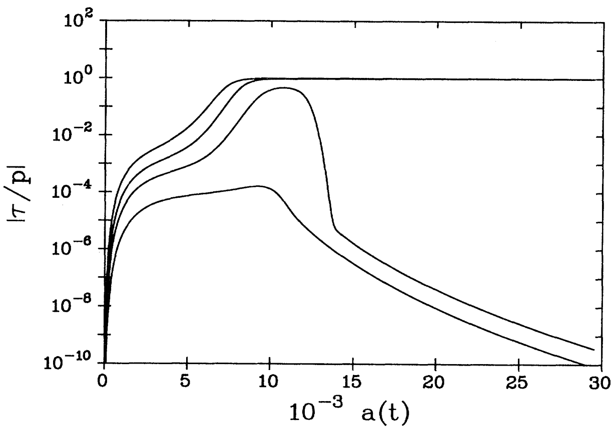

FIG. 5. Behavior of $|\tau/p|$ vs $a(t)$ is shown for the Israel-Stewart model for several choices of particle mass near the critical point. As the particle mass is increased, the maximum value of the magnitude of the bulk stress increases, until, above a certain mass, $|\tau/p|$ asymptotically approaches 1. For all masses above the critical mass, the asymptotic value of $|\tau/p|$ is unity. The illustrated curves represent cosmological models with particle masses equal to (from bottom to top at far right) 7.5×10^4, 1.5×10^5, 2.25×10^5, and 3.75×10^5 GeV.

inate over equilibrium pressure, occurs at a mass of about 10^5 GeV in the Israel-Stewart model: Figure 5 illustrates the approach to this asymptotically nonequilibrium solution. For values of the particle mass above the critical value, the bulk stress asymptotically approaches the equilibrium pressure in magnitude. The reason for the change in behavior above a critical mass can be understood in terms of the differential equation governing the evolution of the cosmic temperature [Eq. (34)]. Increasing the particle mass increases the value of the peak bulk stress; eventually, the peak bulk stress becomes comparable to the equilibrium pressure. At that point, by Eq. (35), $\dot{\beta}$ becomes zero (or can even go negative, if the bulk stress is larger in magnitude than the equilibrium pressure). With $\dot{\beta}$ zero or negative, the universe does not cool off as it expands, and thus the bulk-viscosity coefficient does not return to small values. The universe becomes trapped in a state in which it is expanding, but is not cooling off; the bulk stress and the equilibrium pressure decrease into the future, but only as rapidly as the energy density. Figure 6 illustrates the behavior of β as a function of $a(t)$ in these models. Larger values of bulk viscosity result in an earlier onset of the growth of the bulk stress and, hence, a smaller future asymptotic value of β (higher temperature). In the Israel-Stewart models above the critical point, the asymptotic value of the bulk stress is always equal in magnitude to the equilibrium pressure; thus the total pressure is asymptotically zero. As a result, the universe asymptotically expands at the same rate as a pressureless dust universe: $a \approx t^{2/3}$. The final state of the cosmological fluid is not, however, pressureless dust, since the asymptotic temperature is in the mildly relativistic range ($\beta \approx 10$); the pressure and bulk stress, separately, are thus always within an order of magnitude of the energy density, unlike a nonrelativistic perfect fluid.

The Eckart theory also displays a runaway nonequilibrium behavior, for all particle masses above critical value, which in this case is roughly 10^6 GeV. The approach to this state is illustrated in Fig. 7, where $|\tau/p|$ is

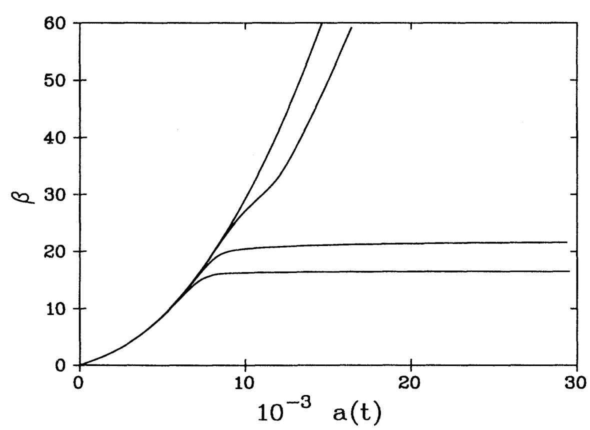

FIG. 6. Evolution of the inverse dimensionless temperature β as a function of scale factor $a(t)$ for Israel-Stewart cosmologies near the critical particle mass. Above the critical particle mass, the total pressure rapidly evolves toward zero, and hence the cosmic temperature evolves toward a constant value. The curves illustrated represent particle masses, from bottom to top, of 3.75×10^5, 2.25×10^5, 1.5×10^5, and 7.5×10^4 GeV.

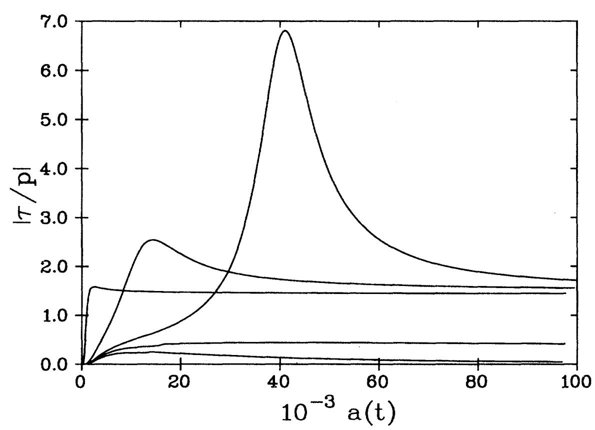

FIG. 7. Behavior of the magnitude of the ratio of the bulk stress to the equilibrium pressure, $|\tau/p|$, vs the scale factor $a(t)$ is shown for the Eckart model for several choices of particle mass near the critical point. As the particle mass is increased, the maximum value of the magnitude of the bulk stress increases, until, above a certain mass, $|\tau/p|$ asymptotically approaches $\frac{10}{7}$. For all masses above the critical mass, the asymptotic value of $|\tau/p|$ is also $\frac{10}{7}$. The illustrated curves represent cosmological models with particle masses equal to (from bottom to top at far right) 10^6, 1.1×10^6, 7.5×10^6, 1.5×10^6, and 1.25×10^6 GeV.

plotted against $a(t)$ for several different values of the particle mass. Again, for all masses greater than the critical mass, the bulk stress approaches the same asymptotic value, in this case $-\frac{10}{7}$ times the equilibrium pressure. There are several interesting consequences of having the asymptotic bulk stress greater in magnitude than the equilibrium pressure. First, since the total pressure $\tau+p$ is now negative, $\dot{\beta}$ is now negative in the late universe. The universe then expands and heats up rather than cooling off; this behavior is shown for several values of the particle mass in Fig. 8. Second, a negative total pressure increases the expansion rate of the universe: The asymptotic states of all Eckart models above the critical mass have $a(t) \approx t^{7/9}$.

It is interesting to compare both of these models with one based on the extended theory of dissipative fluids which has been used in several previous cosmological studies.[29–31] In these treatments, the bulk stress was treated as an independent dynamical variable, as in the Israel-Stewart theory. However, the defining equation for the bulk stress in these treatments all lack the final term on the right-hand side of Eq. (15). We will refer to this theory as a "truncated-Israel-Stewart" model. Such a theory would be a member of the Israel-Stewart class of theories only if β_0 is constant; however, kinetic theory tells us that β_0 is definitely not constant in an expanding universe. In order to determine whether this term plays an important role in the dynamics of a cosmological model, we have numerically integrated the equations with this term left out. For large particle masses, the resulting cosmological model is strikingly different from either the Eckart or Israel-Stewart models, as is shown in the behavior of $a(t)$, plotted in Fig. 9 for all three fluid theories for a particle mass of 10^8 GeV. As discussed above, the Israel-Stewart and Eckart models expand asymptotically

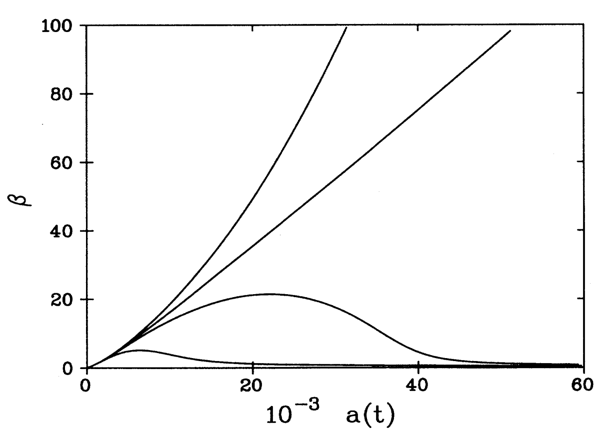

FIG. 8. Evolution of the inverse dimensionless temperature β as a function of scale factor $a(t)$ for Eckart cosmologies near the critical particle mass. Above the critical particle mass, the total pressure rapidly evolves toward a negative final value, and hence the cosmic temperature goes through a minimum and then increases: The late universe expands and heats up. The curves illustrated represent particle masses, from bottom to top, of 1.5×10^6, 1.25×10^6, 10^6, and 10^4 GeV.

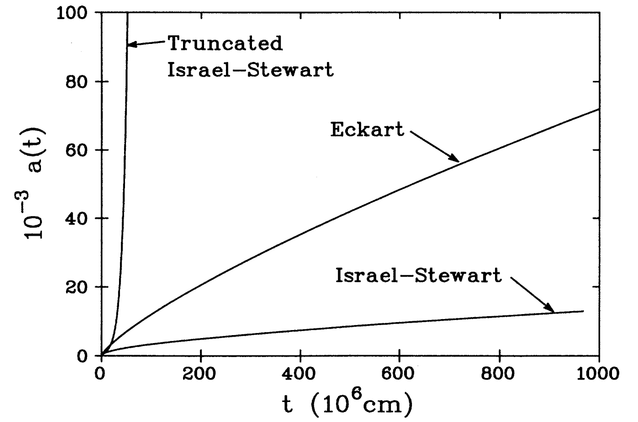

FIG. 9. Evolution of the scale factor $a(t)$ vs time for cosmological models integrated using the Eckart, Israel-Stewart, and truncated-Israel-Stewart fluid theories. The particle mass is the same for all three theories, 10^8 GeV. At late times, the Israel-Stewart cosmology has $a(t)\approx t^{2/3}$, the Eckart cosmology has $a(t)\approx t^{7/9}$, and the truncated-Israel-Stewart cosmology has $a(t)\approx \exp(\kappa t)$.

as $t^{2/3}$ and $t^{7/9}$, respectively. The truncated-Israel-Stewart model, however, expands exponentially asymptotically. In this model, the asymptotic value of the bulk stress is $-4\rho/3$; as in the Eckart case, the universe heats as it expands. The asymptotic equilibrium pressure is thus the same as that of a high-temperature Boltzmann gas, $p=\rho/3$. The total pressure is then $\tau+p=-4\rho/3+\rho/3=-\rho$, the required value for exponential expansion. The energy density of the universe remains constant as it expands: While the fluid is diluted by the expansion, the heating caused by the negative total pressure exactly makes up for the dilution to yield a constant energy density. The asymptotic state is thus exactly de Sitter space, albeit with a rather odd Boltzmann gas as its source.

While it is amusing that the truncated-Israel-Stewart theory yields an asymptotically de Sitter state for a Boltzmann gas cosmology, we do not believe this is an attractive feature. Of the three theories, the Israel-Stewart theory is the most strongly grounded in relativistic kinetic theory (via the Grad method of moments), and we feel it gives the most trustworthy models. We have seen that the Eckart and Israel-Stewart models are quantitatively different over a large range of bulk viscosities; we thus conclude that the simpler (but pathological) Eckart theory is not an adequate approximation to the behavior of the full Israel-Stewart theory in any cosmological model. While for small bulk viscosities many of the cosmological variables are indistinguishable between the two theories, the evolution of the bulk stress is quite different; hence the entropy generated in the expansion will be significantly different. For larger bulk viscosities, we particularly note that only the Israel-Stewart model has zero as a consistent lower bound on the total pressure of a Boltzmann gas: Both the Eckart and truncated-Israel-Stewart theories allow the total pressure to become negative in some cosmological models. We feel that this prediction of a negative total pressure for a Boltzmann gas is an indication of pathology in these fluid theories.

A number of studies in recent years have claimed that an episode of inflation in the early universe might be caused by the fluid phenomenon of bulk viscosity.[23–27,29] While our models considered here are quite different from those which attempt to model a radiative fluid's bulk viscosity, we feel that our comparison of the different fluid theories may be relevant to the discussion of whether bulk-viscous inflation can exist. An accelerated expansion, required for inflation, can exist only when the total pressure is negative. In our models, negative total pressures only occur for the pathological Eckart theory, and the truncated-Israel-Stewart theory. It is perhaps more than a coincidence that these are the same fluid theories in which bulk-viscous inflationary effects have been found. We thus are led to conjecture that "bulk-viscous inflation" occurs as a result of using an inadequate theory of relativistic dissipative fluids. When a truly relativistic, causal, and hyperbolic theory, grounded in relativistic kinetic theory, such as the Israel-Stewart theory, is used, no negative total pressures and, hence, no bulk-viscous-inflation models are found.

ACKNOWLEDGMENTS

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