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Inhomogeneous initial conditions for inflation

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We address the question whether the inflationary paradigm frees cosmology from the worry about initial conditions. That is, we study the influence of initial inhomogeneities on the inflationary epoch. We derive a simple set of approximate equations and use it to illustrate some features of deviations from homogeneity. However, the approximation methods are limited and in order to explore the full effect of initial inhomogeneities we turn to numerical calculations. The picture that emerges from these calculations is that inhomogeneities can prohibit "new" inflation from taking place. The crucial feature necessary for "chaotic" inflation is a sufficient high average field over a region of several horizon sizes.

I. INTRODUCTION

Cosmological inflation is a phase of exponential expansion that took place at the very early Universe—once the Universe emerged from the quantum era. Inflation was introduced[1] as a solution to the cosmological problems which appear in the standard big-bang model.[2] It is supposed to free cosmology from the need for very special initial conditions, in particular, from the need of having initially very homogeneous and flat conditions. Currently, one of the main unresolved questions concerning inflation is whether inflation is generic or requires specific initial conditions. If inflation is not generic, it will lose its appealing power to free cosmology from the need of specific initial conditions.

During the inflationary phase, the energy density of the matter in the Universe is dominated by a potential of a massive scalar field, $V(\phi)$. The energy density of a scalar field ρ_ϕ contains, in addition to the potential term, also a kinetic term $\rho_{\dot\phi}$ and a gradient term ρ_∇:

$$\rho_\phi=\rho_{\dot\phi}+\rho_\nabla+V=\frac{\dot\phi^2}{2}+\frac{(\nabla\phi)^2}{2R^2}+V \ . \tag{1}$$

We study whether general initial conditions, in which the potential term is not the dominant one, can evolve into a configuration in which the potential term will dominate. We use the framework of two models for inflation: "chaotic" inflation[3] and "new" inflation.[4,5] (A third model, "extended" inflation, was suggested recently.[6] The influence of initial conditions on this model is drastically different from the influence on the two other models, and we do not discuss it here.) We assume that the reader is familiar with the basic concept of inflation (for recent reviews, see, e.g., Ref. 7). Two important results are required for the following discussion: In chaotic inflation models the scalar field must have initially a large value, i.e., $\phi_i >$ a few $m_{\rm Pl}$. In new inflation the scalar field must be initially very near the origin, i.e., $\phi_i \approx 0$, and any significant displacement from the origin reduces the duration of new inflation by a very large factor.

Belinsky *et al.*[8] and Piran and Williams[9] have shown that a large initial kinetic energy hardly affects chaotic inflation. The initial kinetic energy is damped rapidly, while the scalar field hardly changes its value. Unlike chaotic inflation, which is generic, only very specific initial conditions lead to new inflation.[10] In particular, an initial kinetic term can easily prevent the onset of new inflation.

While the effects of the kinetic term are relatively easy to study and are well understood, it is much more difficult to deal with gradients in the scalar field, which imply inhomogeneity. In this paper we review the recent attempts to study this issue and to explore the phase space of initial inhomogeneous conditions which lead to inflation.

One can estimate qualitatively how large the initial inhomogeneity can be without preventing inflation. During inflation, the potential-energy density $V(\phi)$ must be larger than the gradient-energy density $\rho_\nabla \approx (\delta\phi/R\Delta)^2$, where Δ is a "comoving coordinate" measure of the inhomogeneity (i.e., Δ is the "comoving wavelength" of the initial perturbation), $\delta\phi$ is a typical change in ϕ, and R is the cosmological scale factor. The expansion rate $H=(\dot R/R)$ is determined during inflation by the potential of the scalar field: $H^2 \approx 8\pi V/3m_{\rm Pl}^2$. The condition that the gradient-energy density ρ_∇ be smaller than the potential yields

$$R\Delta > \left[\frac{8\pi}{3}\right]^{1/2}\frac{\delta\phi}{m_{\rm Pl}}H^{-1} \ . \tag{2}$$

If $\delta\phi \approx \phi < m_{\rm Pl}$ (as can be expected in chaotic inflation), the physical size of the homogeneous region must be larger than a few horizons. This suggests that the preinflationary Universe must be quite homogeneous and that a large initial inhomogeneity might prevent inflation.

The realization that inhomogeneity at the early Universe influences the onset of inflation and might even prevent it motivated many researchers to look more quantitatively at this issue. We describe here briefly various attempts to deal with this question based on different approximation schemes and the numerical solution to the problem.

II. COSMOLOGICAL "NO HAIR" THEOREMS

The first attempts to deal with deviations from perfect homogeneous models were within the context of the cosmological "no hair" theorems.[11–14] These theorems state that in the presence of a positive cosmological constant, de Sitter space-time is a stable asymptotic solution of the Einstein equations. In other words, perturbations of a de Sitter solution decay leaving an asymptotic de Sitter space-time. The no hair theorems imply that perturbations of an inflating solution where the scalar field potential is an effective positive cosmological constant decay and disappear.

There are three independent reasons why these theorems do not bear much light on the issue of initial inhomogeneity.

(a) All the no hair theorems[11–14] assume that the strong-energy condition[15] ($\rho+3p\geq 0$) is satisfied. Barrow has shown[16] that this assumption is essential and when the strong-energy condition is relaxed the no hair theorems fail. It is not clear,[16,17] however, that this assumption is valid in the preinflationary era, specifically, since it is violated by the inflationary scalar field itself.

(b) The inhomogeneity in the scalar field also influences the potential. Hence the "effective cosmological constant" is not a constant any more. (This was neglected by several authors[13] who argued that since the gradient terms increase H, they will also increase the duration of inflation.)

(c) Garfinkle and Vuille[18] have pointed out that the cosmological no hair theorems do not take into account cases in which only a part of the Universe inflates.

Overall, the no hair theorems have an important role in determining the decay of perturbation and the asymptotic structure of an inflating solution. However, they can be applied only after inflation has begun and all the caveats mentioned above have disappeared. The no hair theorems do not answer the question of how inflation begins under inhomogeneous initial conditions.

III. SMALL-PERTURBATION ANALYSIS

A small-perturbation analysis was applied to both the matter fields and the gravitational field. The perturbations are around a homogeneous scalar field and around the Friedmann-Robertson-Walker (FRW) metric. In view of the cosmological no hair theorems, it is not surprising that this analysis has shown that perturbations on an inflating background decay.

Bardeen[19] has shown that in an expanding universe all perturbations that are not coupled to matter decay, and in particular they decay exponentially in a de Sitter space-time.

Brandenberger and Feldman[20] found, as expected, that new inflation is stable under gravitational perturbations that are not coupled to the matter. Brandenberger, Feldman, and MacGibbon[21] have shown that thermal perturbations in the scalar field do not prevent new inflation. Finally, Feldman and Brandenberger[22] have shown that chaotic inflation is stable to both gravitational and scalar field perturbations.

IV. INHOMOGENEOUS SCALAR FIELDS ON FRW BACKGROUND

Several attempts[17,23–25] were made to perform a different analysis which allows large deviations from homogeneity but without resorting to a full numerical solution. In order to simplify the problem, the metric was assumed to be a homogeneous FRW background even though the scalar field is inhomogeneous. The back reaction of the inhomogeneity of the scalar field on the geometry was included, in some of those works,[17,25] by an average effective gradient-energy density that was added to the energy density.

We outline here one scheme of this kind[23] which contains all the essential features. The metric is FRW type and it is characterized by the scale factor R and the expansion rate H. The scalar field is divided to a homogeneous component ϕ and perturbation $\delta\phi$, which has a single Fourier component of comoving wave number k. We take the physical wavelength $\lambda R\equiv 2\pi R/k$ to be smaller than the horizon size H^{-1}. The influence of $\delta\phi$ on the potential is ignored. We obtain a set of three coupled equations for H, ϕ, and $\delta\phi$:

$$H^2=\frac{8\pi}{3m_{\rm Pl}^2}\left[\frac{\dot\phi^2}{2}+V(\phi)+\frac{\delta\dot\phi^2}{2}+\frac{k^2\delta\phi^2}{2R^2}\right]-\frac{\kappa}{R^2}\ , \tag{3}$$

$$\ddot\phi+3H\dot\phi+\frac{dV}{d\phi}=0\ , \tag{4}$$

and

$$\delta\ddot\phi+3H\delta\dot\phi+\frac{k^2\delta\phi}{R^2}=0\ . \tag{5}$$

Since the gradient-energy density is proportional to R^{-2}, it is expected to decay exponentially once the Universe enters a de Sitter phase. The question is whether the gradients can prevent the Universe from ever entering such a phase. There are two important regimes of the initial phase space of this solution.

(a) If $\delta\phi<m_{\rm Pl}$ (and $2\pi R/k<H^{-1}$), the perturbation oscillates, $\delta\dot\phi$ becomes of the same order as $k\delta\phi/R$, and the perturbation $\delta\phi$ behaves like a (decoupled) radiation field. Like any radiation field the amplitude $\delta\phi$ decays $\propto R$, which in turn expands like $t^{1/2}$. Such perturbations decay, and they do not disturb the onset of inflation. One caveat to this conclusion is the situation in a closed universe. If initially $\rho_{\delta\phi}=\delta\dot\phi^2/2+k^2\delta\phi^2/2R^2>\rho_\phi$, the initial phase will be a radiation-dominated one. During this period, $\rho_{\delta\phi}\propto R^{-4}$, while $V(\phi)\approx$const. The Universe will begin to inflate only when $\rho_{\delta\phi}\approx V(\phi)$, i.e., when

$$\rho_{\delta\phi,i}(R_i/R)^4\approx V(\phi_i) \tag{6}$$

(the index i stands for the initial values). The Universe might collapse before Eq. (6) is satisfied and never inflate.

(b) If $\delta\phi>m_{\rm Pl}$ (and $2\pi R/k<H^{-1}$) and if the gradient term, on the right-hand side of Eq. (3), is large compared to all the other terms,[23] Eq. (5) reduces to

$$\delta\dot\phi=\left[\frac{1}{12\pi}\right]^{1/2}\frac{km_{\rm Pl}}{R}\ . \tag{7}$$

We express $\delta\dot{\phi}$ as $RH(d\delta\phi/dR)$, and we integrate Eq. (7) using $H=\sqrt{(4\pi/3)}(k\delta\phi/m_{\rm Pl}R)$, to obtain

$$\frac{R}{R_0}=\exp\left[2\pi\frac{\delta\phi_0^2-\delta\phi^2}{m_{\rm Pl}^2}\right] . \tag{8}$$

An exponentially large change in R corresponds to a very small change in $\delta\phi$. The increase in R causes, in turn, a decrease of the gradient contribution to the energy density since $\rho_\nabla \propto R^{-2}$. During this phase, the effective pressure (due to the gradients) is $\approx -\rho/3$ and $R \propto t$. This phase ends when $\rho_\nabla = V(\phi) \approx V(\phi_i)$ (the change in ϕ during this phase is of the same order as the change in $\delta\phi$ and, hence, at least for chaotic inflation, we expect that if ϕ was appropriate for inflation, initially, it will remain so) or when $\delta\phi = m_{\rm Pl}$, whichever happens first. In the former case inflation starts (assuming that ϕ_i is "suitable for inflation"); in the latter case the perturbations begin to oscillate, as described earlier.

Albrecht, Brandenberger, and Matzner[24] used a slightly different approach to study the effects of gradients on new inflation. They use $\delta\phi$ also as the effective scalar field in the scalar field potential. They find that the scalar field oscillations decay (since the potential-energy density is negligible initially for the specific parameters used) and $\delta\phi$ is driven toward zero (due to the decay of the scalar field oscillations). They conclude that for certain values of the amplitudes and wavelengths the initial gradients drive the scalar field toward suitable initial conditions for new inflation. Kurki-Suonio *et al.*[26] later confirmed these conclusions in a full numerical study.

The effective average density approximation is quite reasonable in two extreme cases: when the wavelength of the scalar field excitation is much longer or much shorter than the horizon size H^{-1}. In the first case the change in the scalar field takes place on such a long scale that the variations can be ignored and $\delta\phi$ can be simply added to ϕ. In the second case the oscillations are on a very short scale and one can expect their average ρ to be close to the real ρ.

In the first extreme case, it was found[17,25] that an initial large-wavelength excitation with a sufficient amplitude (that by itself leads to chaotic inflation) leads to inflation. This is not surprising since in this case the effective field is simply $\phi+\delta\phi$. An addition of short-wavelength excitation[17] (i.e., the second limit) over a large-wavelength excitation rapidly decays and does not disrupt the evolution.

This approximation breaks down when the gradients in ϕ are of the same order as H^{-1}. In this case the variations in ρ influence the metric. Once the metric is not FRW type, the gradients in the metric might induce a feedback causing the gradients in ϕ to decay much slower. An average ρ cannot reveal the influence of the gradients in the geometry on the matter fields. Only explicit calculations that take into account the matter back reaction on the geometry can reveal the outcome of introducing large gradients.

V. NUMERICAL SOLUTIONS

In view of the above-mentioned shortcoming of the various approximation schemes, one has to resort to a full numerical solution of the problem. In the following we describe the evolution of the Universe starting from various inhomogeneous initial conditions. We derive these results by solving the spherically symmetric Einstein equations coupled to a massive scalar field (for more details, see Goldwirth and Piran[27]). We work in the context of a closed universe (which for simplicity has a reflection symmetry around $\pi/2$). This choice should not affect the results since inflationary scenarios[28,29] assume the existence of large enough regions in which conditions are appropriate for entering an inflationary phase. These regions undergo an exponential expansion, and their evolution does not depend on the global topology. Furthermore, one of the goals of inflation is to explain the flatness problem; such a topology enables us to verify that an initially curved region becomes flat after inflation.

We write the metric as a generalized (we use the term generalized to denote the difference from the common spherical isotropic coordinate system which does not include the Friedmann factor $\sin^2\chi$; it reduces to the usual isotropic coordinates[30] when $\sin\chi=r$) spherical isotropic form

$$ds^2=-(N^2-R^2\beta^2)dt^2+2R\beta\, d\chi\, dt + R^2(d\chi^2+\sin^2\chi\, d\Omega^2) , \tag{9}$$

where $0\le\chi\le\pi$, and R, N, and β are functions of χ and t (in the calculations that we present later, we use $N=1$). R is an inhomogeneous scale factor, a simple generalization of the scale factor R of the FRW metric.

The matter sources are two scalar fields: a massive scalar field ϕ, which acts as the source that drives inflation; a massless scalar field Ψ, which plays the role of a radiation field. This representation enables us to couple the fields both gravitationally and thermally easily and self-consistently. The potential for the scalar field ϕ depends on the inflationary model. The scalar radiation field Ψ does not have a potential of its own, but it can be coupled to the massive scalar field via a potential of the form $g\phi^2\Psi^2$ (g being the coupling constant and usually $g \ll$ the self-coupling of ϕ).

Technical considerations[27] led us to impose the following restrictions on the initial data which do not reduce from the generality of the results.

(i) $\Pi_\phi^{\rm init}=0$. $\Pi_\phi \equiv (R^3/N)(\beta\phi_{,\chi}-\phi_{,t})$ is the conjugate momentum to ϕ. Initially, there is no kinetic energy. Recall that initial kinetic energy (at least in chaotic inflation) decays rapidly and does not influence inflation.[8,9]

(ii) The initial data are isocurvature. The total energy density is constant, but the fields are inhomogeneous; i.e., the total energy at different places has different compositions of a scalar field and a radiation field.

(iii) The self-coupling constants for ϕ are larger than those required by observation.[7]

We introduce several types of initial conditions in order to understand how different gradients influence the inflationary epoch. Each of the cases we present here has been chosen from an explicit group of results with common features to illustrate the typical behavior of a certain type of initial condition, which we describe below.

VI. GAUSSIAN DISTRIBUTION

We start by studying the influence of a gradually changing gradient. We characterize these by a Gaussian centered around the origin:

$$\phi_{\rm init}(\chi)=\phi_0+\delta\phi\left[1-\exp\left[-\frac{\sin^2\chi}{\Delta^2}\right]\right]. \quad (10)$$

The distribution of the ϕ field depends on ϕ_0, the value of the field at the origin [$\phi_{\rm init}(\chi=0)=\phi_0$], $\delta\phi$ the value of ϕ at the "other end of the Universe" (i.e., $\phi_{\rm init}(\chi=\pi/2)\equiv\phi_{\pi/2}=\phi_0+\delta\phi[1-\exp(-1/\Delta^2)]$), and Δ the comoving width of the initial inhomogeneity. Keeping the initial $\rho_{\rm total}$ unchanged, we vary Δ, the size of the initial inhomogeneity, and we calculate how this influences the inflationary epoch.[31]

One can divide the initial conditions into two types.

(i) ϕ_0 and $\delta\phi$ are such that the values $V(\phi_0)$ and $V(\phi_{\pi/2})$ are both "suitable for inflation" (by "suitable" we mean that a homogeneous Universe with such initial ϕ, R, etc., will inflate). The solid curve in Fig. 1 describes the expansion factor at the origin as function of the initial $R\Delta/H^{-1}$ for such a case [see case (a) in Table I for details on the initial parameters]. We can see that large inhomogeneities reduce the duration of inflation, but they do not suppress it completely. Universes of this type enter an inflationary phase everywhere.

(ii) ϕ_0 and $\delta\phi$ are such that $V(\phi_0)$ has a value suitable for inflation, while $V(\phi_{\pi/2})$ does not. In this case the question whether any region will inflate depends on Δ.

Note that in the second type of initial data for chaotic inflation with very small coupling constants the energy density in the gradients is essentially larger than the energy density of the potential. When the potential-energy (with a small coupling constant) density is comparable to the gradient-energy density ($\delta\phi$ is essentially $\ll\phi$), the initial data are of the first type; i.e., the conditions everywhere are suitable for inflation. For example, if $V=10^{-12}\phi^4$ and $V\approx1$, then $\phi\approx10^3$. If the gradient-energy density is comparable to V, it follows that $\delta\phi\approx1$, i.e., $\delta\phi\ll\phi$.

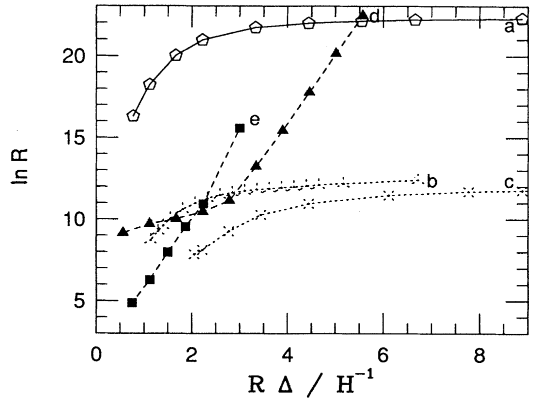

FIG. 1. Scale factor at the origin at the end of the computation as a function of the proper width of the initial Gaussian relative to the horizon size for initial conditions for which only the value of ϕ around the origin is suitable for getting inflation (second type). The various curves correspond to the columns of Table I.

We consider chaotic inflation with $V(\phi)=m^2\phi^2/2$ [case (b)] and with $V(\phi)=\lambda\phi^4/4$ [case (c)] and new inflation with a quartic potential: $V(\phi)=\lambda(\phi^2-\sigma^2)^2/4$ [case (d)] and with a Coleman-Weinberg- (CW-)type potential $V(\phi)=\lambda\phi^4[\ln(\phi^2/\sigma^2)-\frac{1}{2}]+\lambda\sigma^4$ [case (e)]. We follow the evolution of the Universe, with different parameters for the initial fields (see Table I) until it either exits the inflationary phase everywhere or until part of it starts to collapse. $\phi_{\rm end}\equiv\phi(\chi=0)$ at the end of the computation (see Table I) serves as an indication to the conditions at that stage.

The solutions are best described as trajectories in the $(\phi,\overline{\Pi})$ phase plane, where $\overline{\Pi}\equiv\Pi/R^3$. Chaotic inflation is manifested by a $\overline{\Pi}\simeq$const curve. This presentation gives a qualitative picture of the deviation of the solution from the corresponding homogeneous solution and of the dura-

TABLE I. Initial parameters and results for five cases described in Fig. 1. $R^{\rm hom}_{\rm end}$ is the scale factor when $\phi=\phi_{\rm end}$ of an equivalent homogeneous expansion which started with $\phi=\phi_0$.

Case $V(\phi)$	(a) ϕ^4 $\lambda=10^{-4}$	(b) Massive $m=0.1$	(c) ϕ^4 $\lambda=10^{-4}$	(d) ϕ^4 $\lambda=10^{-3}$ $\sigma=4$	(e) CW $\lambda=10^{-1}$ $\sigma=0.2$
$\rho_{\rm total}$	3.75	2.3	1.11	0.449	2×10^{-5}
$R_{\rm init}$	10	6	10	10	2000
$H^{-1}_{\rm init}$	0.9	1.16	1.38	2.68	395
ϕ_0	13	6	7	0.001	5×10^{-4}
$\delta\phi$	-2	-2	-2	1.5	0.01
$\phi_{\rm end}$	~3.8	~0.3	~2.2	~2.3	0.2
$R^{\rm hom}_{\rm end}$	4.98×10^9	2.5×10^5	4656	2.5×10^{15}	7×10^{20}
$\left[\frac{R\Delta}{H^{-1}}\right]_{\rm est}$	1.15	1.15	1.15	0.86	0.0058

tion of the inflationary period. In this picture it is easy to see which part of the expansion is due to inflation (which is after the trajectory reaches the inflationary solution curve) and which part of the expansion is due to the preinflationary epoch. Figures 2(a), 2(b), and 2(c) describe, for cases (a), (c), and (d), the trajectory of the solution at the origin in the $(\phi,\overline{\Pi})$ plane. The overall behavior of the corresponding curves for case (b) resembles the curves of Fig. 2(b), and those of case (e) resemble the curves of Fig. 2(c).

The behavior of the solution with large gradients is clear from Figs. 1, 2(b), and 2(c). In chaotic inflation large gradients suppress inflation and large enough gradients will diminish inflation altogether. When we compare Fig. 2(b) with Fig. 2(a), we see that unlike case (a), in which the Universe inflated for all values of Δ, in case (b) the Universe inflates only when Δ is large enough, i.e., when $R\Delta > 2H^{-1}$. Figures 3(a) and 3(b) describe the trajectories in the $(\phi,\overline{\Pi})$ phase plane at five different spatial points for $\Delta=1$ and for $\Delta=0.25$ for case (b). We see that in this case, when the Universe inflates, it inflates only near the origin.

In new inflation models the influence of large gradients is much more drastic than in chaotic inflation models. One can see in Fig. 1 that new inflation is highly suppressed even when the gradients are reasonably small. If $R\Delta < 4H^{-1}$, the Universe does not inflate at all. This behavior can be seen also in Fig. 2(c) where for $R\Delta < 4H^{-1}$ the trajectories do not reach the inflationary solution. Since new inflation is very sensitive[10] to the initial value of ϕ, relatively small amplitudes $(\delta\phi)$ prevent the regions away from the origin from inflating. Reducing Δ prevents the whole Universe from entering the inflationary phase.

Table I gives an estimate [using Eq. (2)] for the value of $R\Delta$ above which inflation takes place. Comparisons with Fig. 1 show that this estimate is quite good for chaotic inflation. However, for new inflation Eq. (2) underestimates the critical value by a large factor. For comparison we have calculated $R_{\mathrm{end}}^{\mathrm{hom}}$, which is the scale factor at the end of our calculation (when $\phi=\phi_{\mathrm{end}}$) of a homogeneous universe with the same initial conditions as the ones that are given at the origin of our inhomogeneous solutions. For new inflation $R_{\mathrm{end}}^{\mathrm{hom}}$ is much larger than the inhomogeneous R_{end}. In this case the Universe spends most of the inflationary period with $\phi\approx 0$. The deviations from homogeneity induce variation of ϕ_{init}, which speeds up the roll down of ϕ away from the origin.

VII. SHORT-WAVELENGTH EXCITATIONS

In a second series of initial data, we decompose the initial scalar field configuration into Fourier modes:

$$\phi_{\mathrm{init}}(\chi)=\bar{\phi}+\delta\phi\frac{\sin k\chi}{\sin\chi} . \quad (11)$$

We introduced a damping factor $\sin\chi$ in the denominator since "natural" spherical perturbations die away from the center of symmetry (at $\chi=0$). The distribution of ϕ depends on $\bar{\phi}$, the average value of ϕ, $\delta\phi$ the amplitude of the excitations around $\bar{\phi}$, and k the comoving wave num-

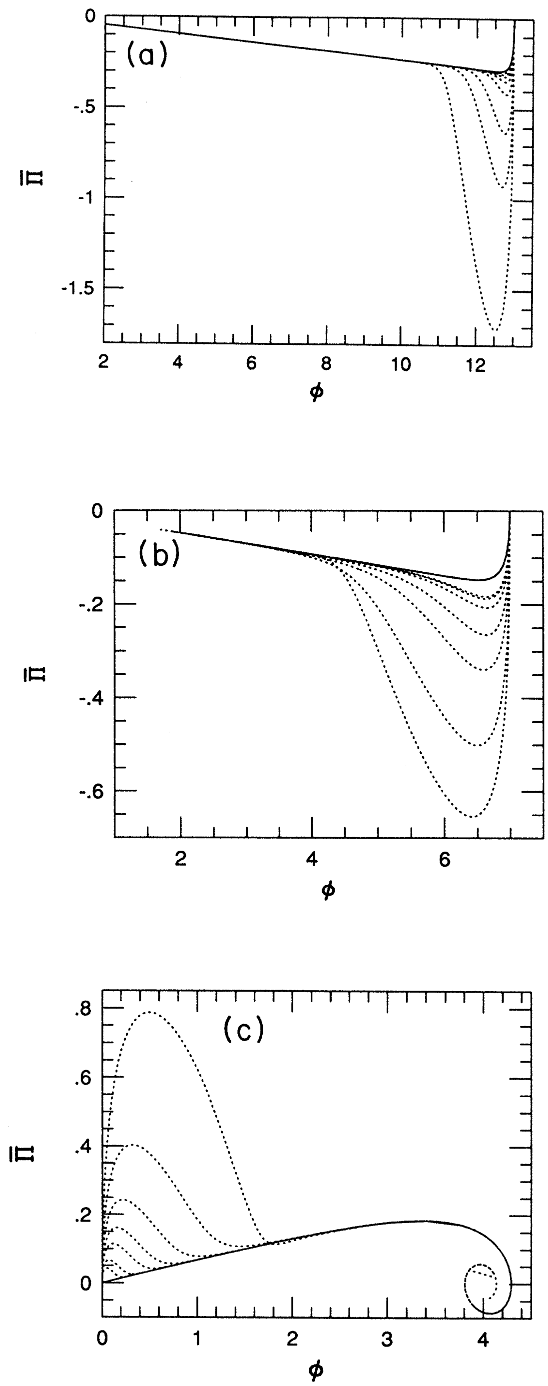

FIG. 2. (a) $\overline{\Pi}$-vs-ϕ trajectories at the origin for case (a) in Table I. The solid line corresponds to the homogeneous solution with $\overline{\Pi}_{\mathrm{init}}=0$ and $\phi_{\mathrm{init}}=13$; the dotted lines correspond to the points in Fig. 1 (large gradients reach further in $\overline{\Pi}$). (b) $\overline{\Pi}$-vs-ϕ trajectories of a chaotic inflation [case (c)]. The solid line corresponds to the homogeneous solution with $\overline{\Pi}_{\mathrm{init}}=0$ and $\phi_{\mathrm{init}}=7$; the dotted lines correspond to the points of curve (c) in Fig. 1. (c) The same as (b), for new inflation case (d), with the homogeneous solution corresponding to $\overline{\Pi}_{\mathrm{init}}=0$ and $\phi_{\mathrm{init}}=0.001$. The straight part of the homogeneous solution is the inflationary phase.

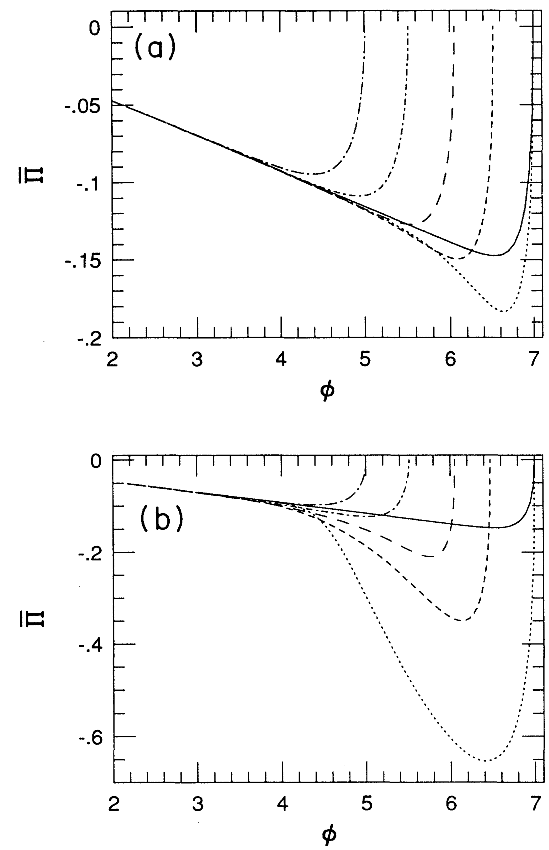

FIG. 3. (a) Trajectories in $(\phi,\overline{\Pi})$ phase plane for five different χ's of case (c) with $\Delta=1$: $\chi=0.5e-4$, dotted line; $\chi=0.42$, short-dashed line; $\chi=0.64$, long-dashed line; $\chi=0.92$, short-dot-dashed line; $\chi=1.5$, long-dot-dashed line. The solid line describes the homogeneous solution for $\overline{\Pi}_{\rm init}=0$ and $\phi_{\rm init}=7$. (b) Trajectories in $(\phi,\overline{\Pi})$ phase plane for five different χ's of case (c) with $\Delta=0.25$: $\chi=0.6e-3$, dotted line; $\chi=0.14$, short-dashed line; $\chi=0.2$, long-dashed line; $\chi=0.3$, short-dot-dashed line; $\chi=0.92$, long-dot-dashed line. The solid line describes the homogeneous solution for $\overline{\Pi}_{\rm init}=0$ and $\phi_{\rm init}=7$.

ber, which is inversely proportional to the comoving wavelength. In all the calculations that we present here, $k=13$, which corresponds to a physical wavelength $2\pi R/k$ of the order of the horizon size.

The key quantity in this initial data is $\overline{\phi}$. If it is not "suitable," the Universe will not inflate, no matter what the gradients are. If $\overline{\phi}$ is in the right range, then the issue depends on the initial gradient-energy density. The main question here is whether the oscillations decay before the Universe starts to collapse.

When the perturbations are around large $\overline{\phi}$, the initial data resemble the first type of Gaussian initial data described earlier. The initial excitation is damped and the whole Universe inflated with $\phi\approx\overline{\phi}$ everywhere. This is demonstrated in Fig. 4 for $\phi_0=40$ and $\delta\phi=0.1$, with $V=10^{-4}\phi^2$ and $R_{\rm init}=10$, where initially 60% of the energy density is in the gradient term. Since we are interested here only in the question of if and how the Universe enters the inflationary phase, we interrupt the calculation once the excitations are frozen; this happens long before the inflationary phase is over.

Figure 5 displays the trajectories in the $(\phi,\overline{\Pi})$ phase plane at the origin for configurations with smaller $\overline{\phi}$ and various $\delta\phi$ (see Table II). As $\overline{\phi}$ decreases, so does $\phi_{\rm inf}$, the value of ϕ at which the Universe starts to inflate. The difference between $\overline{\phi}$ and $\phi_{\rm inf}$ increases when $\overline{\phi}$ decreases, until at a certain stage the Universe does not inflate at all. When $\overline{\phi}$ is small and the gradients are large, there is not enough time for the oscillations to decay before the Universe collapses. In other words, the Universe collapses before the potential-energy density becomes larger than the gradient- and kinetic-energy densities, which is the necessary condition for inflation to occur. Once the collapse starts, the energy densities of the gradient and kinetic terms increase, while the potential term remains a constant. Clearly, it is impossible to reach inflation at this stage.[32]

Variation in the energy density of the gradient term hardly changes the duration of inflation in those cases when inflation commences; i.e., $\phi_{\rm inf}$ is (almost) independent of the initial size of the gradient (see Fig. 5 for

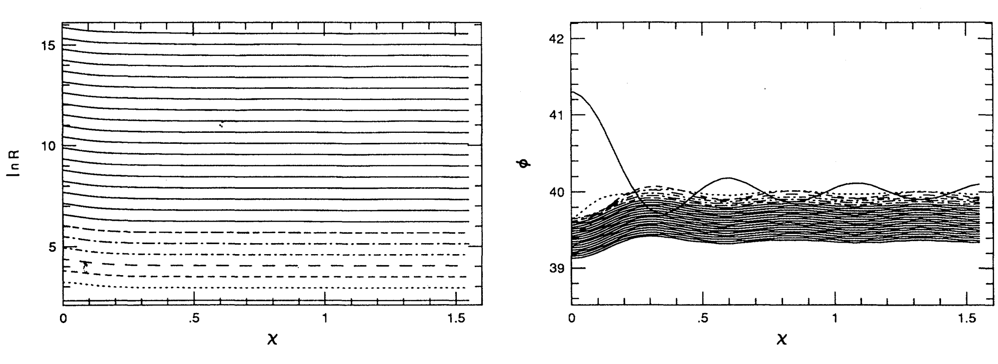

FIG. 4. Evolution of $\ln R$ and ϕ vs χ on different time slices. The solid lines describe the initial conditions, and the other lines represent the evolution in time. The following time slices are described by the dotted lines, the short-dashed lines, the long-dashed lins, dot-dashed lines, long-dot-dashed lines, etc.

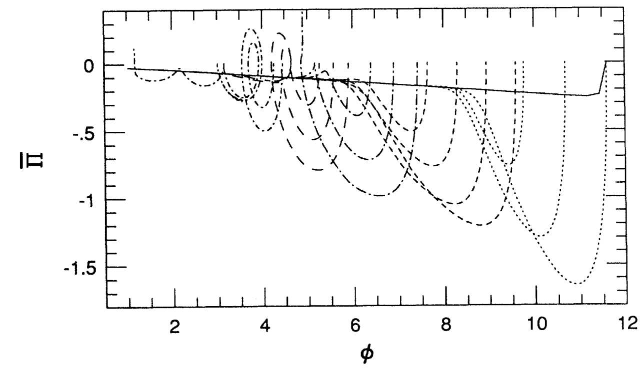

FIG. 5. Trajectories of the cases described in Table II. The solid line describes the homogeneous solution with $\overline{\Pi}_{\rm init}=0$ and $\phi_0=11.6$: case (v), dotted line; case (w), short-dashed line; case (x), long-dot-dashed line; case (y), long-dashed line; case (z), short-dot-dashed line. In all the cases the large gradients reach further in $\overline{\Pi}$.

$\bar{\phi}>6$). This behavior is completely different from the behavior of the Gaussian distribution where an increase of the gradient size reduced the amount of inflation [see Fig. 2(b) for comparison]. Above a certain value (which depends on $\bar{\phi}$), the gradients cause the Universe to collapse before ever reaching the inflationary phase (see Fig. 5 for $\bar{\phi}<7$).

Figure 6(a) describes the trajectories in the $(\phi,\overline{\Pi})$ phase plane for $\bar{\phi}=7$ at five different locations in the Universe. From Fig. 6(a) it is clear that the amount of inflation differs in each of the regions. In particular, the origin, where initially ϕ had the largest value, inflated less than all the other regions. This is the main difference between this type of distribution and the Gaussian type, in which the region where ϕ was initially the largest inflated more than all the other regions. For comparison between these two types of initial conditions, see Figs. 3(a) and 6(a). However, when we compare Fig. 6(a) with Fig. 3(b) we see that the behavior of a very narrow Gaussian is very

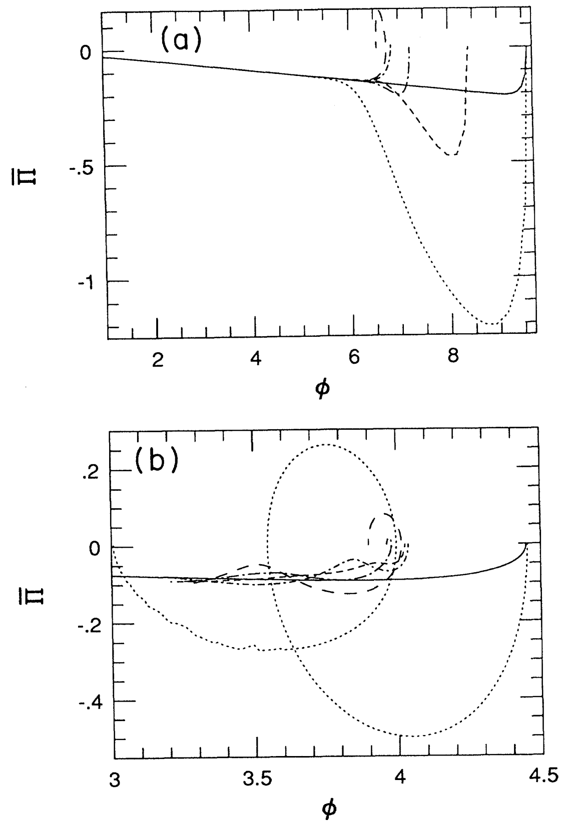

FIG. 6. (a) Trajectories in $(\phi,\overline{\Pi})$ phase plane for five different χ's of case (w) in Table II: $\chi=0.63e-3$, dotted line; $\chi=0.14$, short-dashed line; $\chi=0.3$, long-dashed line; $\chi=0.77$, short-dot-dashed line; $\chi=1.1$, long-dot-dashed line. The solid line describes the homogeneous solution with $\overline{\Pi}_{\rm init}=0$ and $\phi_0=9.6$. (b) Trajectories in $(\phi,\overline{\Pi})$ phase plane for five different χ's of case (z) in Table II: $\chi=0.39e-3$, dotted line; $\chi=0.23$, short-dashed line; $\chi=0.35$, long-dashed line; $\chi=1.1$, short-dot-dashed line; $\chi=1.3$, long-dot-dashed line. The solid line describes the homogeneous solution with $\overline{\Pi}_{\rm init}=0$ and $\phi_0=4.46$.

TABLE II. Initial parameters and results for five cases described in Fig. 5. In all the cases $k=13$ (the wavelength $\lambda=0.24$) and $V=10^{-4}\phi^4$. In case (v), $R_{\rm init}=7$ (the comoving wavelength is $R\lambda=1.68$); in all the other cases $R_{\rm init}=10$ (the comoving wavelength is $R\lambda=2.4$).

Case	$\bar{\phi}$	$\delta\phi$	Gradient term ρ	V	H^{-1}	Inflate y/n
(v)	9	0.2	66%	1.8	0.96	y
		0.13	50%	1.3	1.3	y
		0.06	10%	0.92	1.8	y
(w)	7	0.2	70%	0.85	1.52	y
		0.1	44%	0.475	1.86	y
		0.05	20%	0.34	3	y
(x)	6	0.11	62%	0.305	2.45	n
		0.07	44%	0.23	3.3	y
		0.031	15%	0.1618	4.53	y
(y)	5	0.07	61%	0.122	3.98	n
		0.045	42%	0.1	5.4	y/n
		0.021	15%	0.077	7.1	y
(z)	4	0.035	51%	0.04	9.0	n
		0.02	27%	0.044	17.5	n

similar to that of a short-wave excitation, which is not surprising.

Figure 6(b) for $\bar{\phi}=4$ demonstrates a case in which the excitations oscillate and the Universe does not inflate anywhere. Figure 6(b) shows clearly that none of the trajectories follow the inflationary solution.

The last example that we discuss is for $\bar{\phi}=6$. Here we see two different qualitative types of behavior, depending on the size of the initial gradients (see Fig. 5). For large gradients (more than 60% of ρ), the Universe collapses, while with smaller gradients the ϕ field starts to oscillate, until eventually all the excitations die and the Universe enters inflation. Obviously, the number of oscillations depends on the size of the initial gradient-energy density; the larger it is, the more oscillations are necessary for it to be damped. If the gradient-energy density is damped rapidly and the potential becomes the dominant one, the Universe will start to inflate; otherwise, the Universe will collapse.

We did not address the issue of superposition of several modes with different wavelengths. Configurations with a very long wavelength (which spread over several horizon sizes) resemble the wide Gaussian distribution which was addressed in the previous section. A superposition of such a configuration with short-wavelength excitations changes the average value $\bar{\phi}$, but otherwise does not change the properties of the solution. A superposition of several short wavelengths does not change our basic result that the average value of the field is the main factor that determines the evolution. Furthermore, a Gaussian configuration which we studied in the previous section is indeed a superposition of several wavelengths.

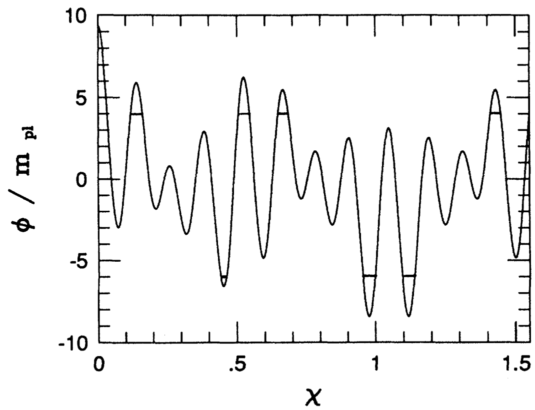

FIG. 7. Schematic picture of high peaks in the ϕ field.

VIII. CONCLUSIONS AND SUMMARY

The picture that emerges from these results is clear. Chaotic inflation can take place in the presence of initial inhomogeneities, but these do not "help" it. The crucial feature necessary for inflation is a sufficiently high average value of the scalar field over a region of several horizon sizes. If the amplitude of the scalar field is large enough, gradients can exist on top of this average field. These gradients will decay and will not disturb the onset of inflation. If the average value of the scalar field is marginal, the gradients may not have sufficient time to decay and they can prevent the onset of inflation. The initial homogeneity requirement is moderate. All that is needed is homogeneity over a few initial horizons. However, this condition cannot be ignored, From this point of view, it seems that inflation is an amplifier of homogeneity, but it is not the sole answer to the homogeneity question. It is essential to realize that an extremely chaotic field which oscillated rapidly will not inflate even if the amplitude of these oscillations is large. The intuitive picture in which (see Fig. 7) sharp peaks of such a random field inflate and lead to separate inflationary regimes is misleading. High peaks can exist, but inflation will start only if by the time that they decay there is a homogeneous and large average field in the background. The inflating region must be moderately large and relatively smooth.

The influence of inhomogeneity on new inflation is much more drastic. The requirement that we find smooth initial data for new inflation joins the other limitations of this model[7,10] as an additional argument which makes it difficult to implement.

Clearly, inflation solves the horizon problem by many orders of magnitude and the initial conditions for inflation are much more general than those required for a Friedmann universe. But there still remains a problem with initial conditions: Is it reasonable to expect that regions of several horizons over which the average scalar field must have a large value, appropriate for inflation, will exist in the preinflationary era?

At the end of the quantum-gravity era, we have $\rho \le \rho_{\rm Pl}$. For a classical scalar field ϕ, this implies [see Eq. (1)] that

$$V(\phi) \le m_{\rm Pl}^4, \quad \dot{\phi}^2 \le m_{\rm Pl}^4, \quad \frac{\mathrm{grad}^2\phi}{R^2} \le m_{\rm Pl}^4 . \tag{12}$$

Linde[3,28] argues that because of the uncertainty principle, the value of the potential energy $V(\phi)$ at $t \sim t_{\rm Pl} \sim m_{\rm Pl}^{-1}$ can be determined only with an accuracy $O(m_{\rm Pl}^4)$. Therefore, ϕ initially can have any value satisfying conditions (12); i.e., the initial distribution of the field ϕ is more or less *chaotic*. One can expect, therefore, that at the end of the quantum era (when $R \approx m_{\rm Pl}^{-1}$ and $\rho \approx m_{\rm Pl}^4$) the energy of the scalar field is distributed equally between the kinetic, gradient, and potential terms:

$$\frac{\dot{\phi}^2}{2} \approx \frac{\delta\phi^2}{2R^2\Delta^2} \approx \frac{m^2\phi^2}{2} \approx m_{\rm Pl}^4 . \tag{13}$$

We use here and in the following the potential $V=m^2\phi^2/2$ as an example. This specific choice is for illustrative purpose and does not affect at all our conclusions. The scalar field varies with a typical wavelength $R\Delta \approx m_{\rm Pl}^{-1}$ and amplitude $\delta\phi \approx m_{\rm Pl}$. $\delta\phi$ is much smaller than the average value of the scalar field $\bar{\phi}$. To see this, recall that the quantum fluctuation constraint[33] $\delta\phi/\phi \approx H/2\pi \ll 10^{-4}$ limits the coupling constant of the scalar field: $m \ll m_{\rm Pl}$. The scalar field must have a very large average value $\bar{\phi} \approx m_{\rm Pl}^2/m \gg m_{\rm Pl}$ in order that the potential term will be in equipartition with the kinetic and gradient terms in spite of its small coupling constant. Since $\phi \gg \delta\phi$, large regions with $\bar{\phi} \gg$ a few $m_{\rm Pl}$ will exist.

The preinflationary scalar field is drastically different if we assume that the scalar field ϕ emerges from the quantum era in thermal equilibrium (with $T \approx m_{\rm Pl}$). (It has been argued[28,34] that a weakly coupled scalar field does not have enough time to thermalize during the quantum era, but Brout, Horowitz, and Weil[35] assert that the scalar field is in a thermal equilibrium during the whole quantum phase.) In this case,

$$\frac{\dot{\phi}^2}{2} \approx \frac{\delta\phi^2}{2R^2\Delta^2} \approx m_{\rm Pl}^4 , \tag{14}$$

and $\delta\phi \approx m_{\rm Pl}$. The potential energy is, in this case, much lower than the kinetic energy:

$$\frac{m^2\bar{\phi}^2}{2} \approx \frac{m^2}{T^2} m_{\rm Pl}^4 , \tag{15}$$

and $\bar{\phi} \approx m_{\rm Pl}$. $\bar{\phi} \approx \delta\phi$ for a thermal field at $T \gg m$, and we do not expect to find the required large regions with $\bar{\phi} \gg$ a few $m_{\rm Pl}$.

We see that the conditions at the preinflationary epoch can be completely different depending on which assumption we make.[36] Only a theory of quantum gravity will be able to reveal what were the conditions at the end of the preinflationary era. The hope is that quantum processes will favor configurations in which the average scalar field will have a large value over domains larger than the horizon, which will lead to inflation. Lacking a clear theory, we are left with an uncertainty about the initial conditions even within the context of the inflationary scenario.

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