Transverse conductivity of a relativistic plasma in oblique electric and magnetic fields

Fulvio Melia and Marco Fatuzzo
Department of Physics and Astronomy, Northwestern University, Evanston, Illinois 60208
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Resistive tearing is a primary candidate for flares occurring in stressed magnetic fields. Its possible application to the strongly magnetized environments ($H_z \sim 10^{12}$ G) near the surface of neutron stars motivates a quantum treatment of this process, which requires knowledge of the electrical conductivity σ of a relativistic gas in a new domain, i.e., that of a low-density (n_e) plasma in oblique electric [$\mathbf{E}=(0,E_y,E_z)$] and magnetic fields. We derive the mathematical formalism for calculating σ and present numerical results for the range of parameter values $10^9 \le H_z \le 10^{12}$ G, $E_z/H_z \lesssim 10^{-4}$, $E_y \lesssim 10^{-4} H_z^2/E_z$, and $10^{20} \le n_e \le 10^{25}$ cm^{-3}. We find that $\sigma \sim E_y^2 E_z n_e^2/H_z^2$ over this range.

I. INTRODUCTION

The electrical conductivity of a relativistic electron gas has received considerable attention, particularly in the cases of high density ($\rho \gtrsim 10^6$ g cm^{-3}) and no external magnetic field,[1] high density with a magnetic field parallel to the electric field,[2] and high density with perpendicular magnetic and electric fields.[3] Subsequent extensions have been made by, e.g., Ventura,[4] Yakovlev,[5] and Hernquist.[6] Our development of a theory of resistive magnetic tearing in a quantizing field[7] has led us to consider the electrical conductivity of a relativistic gas in yet another domain, i.e., that of a low-density plasma ($n_e \lesssim 10^{25}$ cm^{-3}) in oblique electric (**E**) and magnetic (**H**) fields with $H \sim 10^{12}$ G and $E/H \lesssim 10^{-4}$.

It is known from the classical treatment of magnetic-field reconnection that resistive tearing is the only mechanism that develops in time, making it a primary candidate for "flare" processes occurring in stressed magnetic fields. These processes are known to occur on size scales ranging from galaxies[8] down to the Sun.[9] Our quantum treatment of this process is motivated by its possible application to the strong magnetic fields usually encountered in neutron-star environments, particularly as a mechanism for generating the plasma heating and particle acceleration leading to γ-ray bursts. The presence of a strong magnetic field changes the conductivity in the magnetosphere from a scalar to a tensor σ. The free motion of the electron is not altered in the direction of the magnetic field, but can be strongly constrained in a direction perpendicular to it, which results in a larger transverse resistivity with respect to the longitudinal case. This difference is generally greater at lower densities.[3] Our concern here will be to determine the "transverse" conductivity $\sigma_\perp \equiv (\sigma_{yy}^2 + \sigma_{xy}^2)/\sigma_{yy}$ of an electron-ion plasma, where σ_{ij} are the elements of σ, when **H** is along z and $\mathbf{E}=(0,E_y,E_z)$. The component σ_{yy} depends solely on the scattering process, whereas σ_{xy} is due entirely to the drift velocity $v_D = c(\mathbf{E}\times\mathbf{H})/H^2$. If, as we assume in this paper, the ions are free to move, there is no net current in the x direction, so that $\sigma_{xy}=0$.

As is usually done in problems of Ohmic conductivity, the electric field **E** is treated as a perturbation. When **E** is parallel to the magnetic field **H**, the particle transport may be handled adequately with the Boltzmann equation. However, when **E** and **H** are not parallel, this approach cannot be used because the velocity operators perpendicular to **H** vanish. Instead, the density-matrix approach must be used. We discuss the relevant wave function and matrix elements of the velocity operator in Sec. II. A brief summary of the density-matrix formalism is given in Sec. III. In Sec. IV we evaluate the density matrix itself, and we complete the calculation of $\sigma_\perp$ in Sec. V. Numerical values of $\sigma_\perp$ are given for the ranges $10^9 \le H_z \le 10^{12}$ G, $E_z/H_z \lesssim 10^{-4}$, $E_y \lesssim 10^{-4} H_z^2/E_z$, and $10^{20} \le n_e \le 10^{25}$ cm^{-3}, and a comparison with the zero-field conductivity is made in Sec. VI. The relevance of this computation is discussed in Sec. VII.

II. DIRAC WAVE FUNCTIONS FOR OBLIQUE ELECTRIC AND MAGNETIC FIELDS

The general normalized solution to the Dirac equation with oblique electric $\mathbf{E}=(0,E_y,E_z)$ and magnetic $\mathbf{H}=(0,0,H_z)$ fields is[10]

$$\bar{\psi}_{n,p_x,\epsilon,C,E,H} \equiv \frac{1}{N_n(C_1,C_2)} \psi_{n,p_x,\epsilon,C,E,H} = \frac{1}{N_n(C_1,C_2)} \begin{bmatrix} u_{1n}^{+} \\ u_{2n}^{+} \\ u_{1n}^{-} \\ u_{2n}^{-} \end{bmatrix} \exp\left[i \left(\frac{p_x x}{\hbar} - \frac{\epsilon t}{\hbar} + \frac{1}{2}\sqrt{A'}\xi^2 \right) - \frac{1}{2}\sqrt{B'}\eta^2 \right] , \tag{1}$$

where $u^{\pm}_{1n}$, $u^{\pm}_{2n}$, A', B', ξ, η, and the normalization constant $N_n(C_1,C_2)$ are defined by Eqs. (7)–(49) of Ref. 10 and are reproduced in the Appendix. The two independent solutions, corresponding to the positive- and negative-helicity states in the absence of an electric field, are given by the choice of parameters ($C_1=1$, $C_2=0$) (collectively labeled $C=1$) and ($C_1=0$, $C_2=1$) (collectively labeled $C=0$) in the expressions for $u^{\pm}_{1n}$ and $u^{\pm}_{2n}$.

In this scheme the velocity operators α_x and α_y are given as

$$\alpha_k = i\gamma_4\gamma_k = \begin{bmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{bmatrix} . \tag{2}$$

Thus, putting

$$\alpha_\pm = \alpha_x \pm i\alpha_y , \tag{3}$$

the appropriate velocity matrix elements (needed in the evaluation of the transverse currents) are[11]

$$\begin{aligned}\langle np_x\epsilon_n 1;EH|\alpha_+|mp_x\epsilon_m 1;EH\rangle &= \frac{8L_x}{N_n(1,0)N_m(1,0)(B')^{1/4}}(T^{1,1}_{n,m}\delta_{n,m}+T^{1,1}_{n+2,m}\delta_{n+2,m}+T^{1,1}_{n+1,m}\delta_{n+1,m}+T^{1,1}_{n,m-1}\delta_{n,m-1}+T^{1,1}_{n-1,m}\delta_{n-1,m}\\ &\quad +T^{1,1}_{n-2,m}\delta_{n-2,m}+T^{1,1}_{n+1,m-1}\delta_{n+1,m-1}+T^{1,1}_{n-1,m-1}\delta_{n-1,m-1}) ,\end{aligned} \tag{4}$$

$$\begin{aligned}\langle np_x\epsilon_n 0;EH|\alpha_+|mp_x\epsilon_m 0;EH\rangle &= \frac{8L_x}{N_n(0,1)N_m(0,1)(B')^{1/4}}(T^{0,0}_{n,m}\delta_{n,m}+T^{0,0}_{n+2,m}\delta_{n+2,m}+T^{0,0}_{n+1,m}\delta_{n+1,m}+T^{0,0}_{n,m-1}\delta_{n,m-1}+T^{0,0}_{n-1,m}\delta_{n-1,m}\\ &\quad +T^{0,0}_{n-2,m}\delta_{n-2,m}+T^{0,0}_{n+1,m-1}\delta_{n+1,m-1}+T^{0,0}_{n-1,m-1}\delta_{n-1,m-1}) ,\end{aligned} \tag{5}$$

$$\begin{aligned}\langle np_x\epsilon_n 1;EH|\alpha_+|mp_x\epsilon_m 0;EH\rangle &= \frac{8L_x}{N_m(1,0)N_m(0,1)(B')^{1/4}}(T^{1,0}_{n,m}\delta_{n,m}+T^{1,0}_{n+2,m}\delta_{n+2,m}+T^{1,0}_{n+1,m}\delta_{n+1,m}+T^{1,0}_{n,m-1}\delta_{n,m-1}+T^{1,0}_{n-1,m}\delta_{n-1,m}\\ &\quad +T^{1,0}_{n-2,m}\delta_{n-2,m}+T^{1,0}_{n+1,m-1}\delta_{n+1,m-1}+T^{1,0}_{n-1,m-1}\delta_{n-1,m-1}) ,\end{aligned} \tag{6}$$

$$\begin{aligned}\langle np_x\epsilon_n 0;EH|\alpha_+|mp_x\epsilon_m 1;EH\rangle &= \frac{8L_x}{N_n(0,1)N_m(1,0)(B')^{1/4}}(T^{0,1}_{n,m}\delta_{n,m}+T^{0,1}_{n+2,m}\delta_{n+2,m}+T^{0,1}_{n+1,m}\delta_{n+1,m}+T^{0,1}_{n,m-1}\delta_{n,m-1}+T^{0,1}_{n-1,m}\delta_{n-1,m}\\ &\quad +T^{0,1}_{n-2,m}\delta_{n-2,m}+T^{0,1}_{n+1,m-1}\delta_{n+1,m-1}+T^{0,1}_{n-1,m-1}\delta_{n-1,m-1}) ,\end{aligned} \tag{7}$$

$$\langle np_x\epsilon_n 1;EH|\alpha_-|mp_x\epsilon_m 1;EH\rangle = \langle mp_x\epsilon_m 1;EH|\alpha_+|np_x\epsilon_n 1;EH\rangle^* , \tag{8}$$

$$\langle np_x\epsilon_n 0;EH|\alpha_-|mp_x\epsilon_m 0;EH\rangle = \langle mp_x\epsilon_m 0;EH|\alpha_+|np_x\epsilon_n 0;EH\rangle^* , \tag{9}$$

$$\langle np_x\epsilon_n 1;EH|\alpha_-|mp_x\epsilon_m 0;EH\rangle = \langle mp_x\epsilon_m 0;EH|\alpha_+|np_x\epsilon_n 1;EH\rangle^* , \tag{10}$$

and

$$\langle np_x\epsilon_n 0;EH|\alpha_-|mp_x\epsilon_m 1;EH\rangle = \langle mp_x\epsilon_m 1;EH|\alpha_+|np_x\epsilon_n 0;EH\rangle^* , \tag{11}$$

where the constants $T^{i,j}_{n,m}$ are defined in the Appendix, and

$$|np_x\epsilon C;EH\rangle \equiv \bar{\psi}_{n,p_x,\epsilon,C,E,H}\, e^{i\epsilon t/\hbar} . \tag{12}$$

III. DENSITY-MATRIX FORMALISM

The average electron current may be calculated using a density-matrix approach,[12] with a one-electron approximation for the description of the electron gas, in which each electron is described by a normalized wave function

$$\Psi(\mathbf{r},t) = \sum_n a_n(t)\chi_n(\mathbf{r}) . \tag{13}$$

The χ_n form a complete set of orthonormal functions, and the a_n are time-dependent coefficients. The density matrix ρ is defined as[13]

$$\rho_{mn}=\langle a_m(t)a_n^*(t)\rangle \ , \tag{14}$$

where the angular brackets indicate a statistical averaging. The macroscopic average value of any physical variable Q is then calculated by taking the trace of its product with the density matrix:

$$\langle Q\rangle=\mathrm{Tr}(\rho Q) \ . \tag{15}$$

IV. EVALUATION OF THE DENSITY MATRIX

In the presence of electric and magnetic fields and the impurity potential V, the density matrix satisfies the equation of motion

$$-i\hbar\frac{\partial\rho}{\partial t}=[\rho,\mathcal{H}_0+\mathcal{H}_e+V] \ , \tag{16}$$

where $\mathcal{H}_0$ is the Hamiltonian in the absence of an electric field, and $\mathcal{H}_e$ is the potential energy associated with the electric field.

The diagonal elements of the density matrix ρ are interpreted as giving the probability of occupation of the various states; i.e., they play the role of a distribution function.[14] In the absence of an electric field (and hence the effects of scatterings), the system is assumed to be in thermal equilibrium, in which case ρ will have its thermal equilibrium value ρ_0, such that

$$\langle np_xp_z|\rho_0|n'p_x'p_z'\rangle=\frac{f^0(\epsilon^0_{n,p_x,p_z})}{W}\delta_{n,n'}\delta_{p_x,p_x'}\delta_{p_z,p_z'} \ , \tag{17}$$

where W is the volume of the system, and $f^0(\epsilon^0_{n,p_x,p_z})$ is the Fermi-Dirac distribution function, i.e.,

$$f^0(\epsilon^0_{n,p_x,p_z})=\frac{1}{\exp[(\epsilon^0_{n,p_x,p_z}-\mu^0)/kT]+1} \ , \tag{18}$$

where μ^0 is the chemical potential, ϵ^0_{n,p_x,p_z} is the kinetic energy of the given state, and $|np_xp_z\rangle$ is the Dirac wave function for an electron in a magnetic field with zero electric field. Here we have used the postulate of random phases, which implies that the state of the system in equilibrium may be regarded as an incoherent superposition of the independent eigenstates so that the off-diagonal elements vanish. Another way to see this is to realize that in an equilibrium state the density matrix is time independent and must therefore commute with the system Hamiltonian.

The density of particle states can be drastically different in the presence of a magnetic field compared to that in the zero-field case. Not surprisingly, therefore, the chemical potential has the characteristic oscillatory dependence on H, which is, however, only evident in the limit of one Landau level being occupied because of thermal smoothing.[14] In our application the electron number density n_e is sufficiently small ($\lesssim 10^{25}$ cm^{-3}) that the plasma is nondegenerate. The value of μ^0 thus corresponds to the number density normalization in the usual Maxwell-Boltzmann distribution function, i.e., $f^0\rightarrow n_eWe^{-\epsilon^0/kT}/\mathrm{Tr}(e^{-\epsilon^0/kT})$.

To evaluate the density matrix in the presence of an electric field when scattering is present, we employ a two-step process whereby we first calculate the change in ρ_0 due to an electric field E_z parallel to $\mathbf{H}=H_z\hat{z}$ and then find the correction to this new density matrix $\rho_\parallel$ when scattering in the presence of a perpendicular electric field E_y is included.

A. Density matrix for $\mathbf{E}=(0,0,E_z)$

In the first of these steps, we use the Boltzmann technique, generalizing the distribution function to describe populations of electrons over the magnetic quantum states n. Assuming that the first-order effect of the electric field is to shift the equilibrium distribution in the direction of the electric field, the solution of the Boltzmann equation to first order in E_z is given by[14]

$$f(\epsilon_n)=f^0(\epsilon_n)-eE_z\tau_n\frac{\partial f^0(\epsilon_n)}{\partial p_z} \ , \tag{19}$$

where τ_n is the relaxation time, defined by

$$\frac{1}{\tau_n}\equiv\sum_{n',p_x',p_z'}\frac{p_z-p_z'}{p_z}V_{n,p_x,p_z;n',p_x',p_z'} \ , \tag{20}$$

with

$$V_{n,p_x,p_z;n',p_x',p_z'}\equiv\frac{2\pi}{\hbar}|\langle np_xp_z|V|n'p_x'p_z'\rangle|^2 \times\delta(\epsilon^0_{n,p_x,p_z}-\epsilon^0_{n',p_x',p_z'}) \ . \tag{21}$$

When the chief scattering process is due to Coulomb interactions with slowly moving, randomly distributed ions, the interaction potential V between the electron at $\mathbf{r}$ and the αth ion at position $\mathbf{R}_\alpha$ may be written

$$V=\frac{Ze^2}{|\mathbf{R}_\alpha-\mathbf{r}|} =\frac{4\pi Ze^2}{W}\sum_q\frac{1}{q^2}\exp[-i\mathbf{q}\cdot(\mathbf{R}_\alpha-\mathbf{r})] \ , \tag{22}$$

where Z is the ion's atomic number and W is the volume. In this case it can be shown that the relaxation time is given by the expression[15]

$$\frac{1}{\tau_n}=\frac{1}{\tau_0}\sum_{n'=0}^{\infty}\mathcal{E}(\mathcal{E}^2-a_{n'}^2)^{-1/2}\left[1+\left[\frac{\mathcal{E}^2-a_{n'}^2}{\mathcal{E}^2-a_n^2}\right]^{1/2}\right]R_+ +\frac{1}{\tau_0}\sum_{n'=0}^{\infty}\mathcal{E}(\mathcal{E}^2-a_{n'}^2)^{-1/2} \times\left[1-\left[\frac{\mathcal{E}^2-a_{n'}^2}{\mathcal{E}^2-a_n^2}\right]^{1/2}\right]R_- \ , \tag{23}$$

where

$$\frac{1}{\tau_0}=4\pi g\alpha^2Z^2\left[\frac{N_i}{W}\right]\left[\frac{\hbar^2}{m_e^2c}\right]\left[\frac{H_c}{H_z}\right] \ , \tag{24}$$

and $g=(2s+1)=2$, $\alpha=e^2/\hbar c$, $H_c=m_e^2c^3/e\hbar=4.414\times10^{13}$ G, $a_n^2=1+2n(H_z/H_c)$,

$$\mathcal{E}^2=1+\left[\frac{p_z}{m_e c}\right]^2+2n\left[\frac{H_z}{H_c}\right] , \tag{25}$$

$$R_\pm=\int_0^\infty dt\frac{|A_{n,n'}(t,u,\mp u')|^2}{[t+(u\pm u')^2]^2}\Phi(b_\pm) , \tag{26}$$

$$u=\frac{p_z}{m_e c}\left[\frac{H_c}{2H_z}\right]^{1/2} ,$$
$$u'=(\mathcal{E}^2-a_{n'}^2)^{1/2}\left[\frac{H_c}{2H_z}\right]^{1/2} , \tag{27}$$

$$b_\pm(t)=\left[\frac{3}{4\pi}\right]^{1/3}\left[\frac{W}{N_i}\right]^{1/3}\left[\frac{m_e c}{\hbar}\right] \times\left[\frac{2H_z}{H_c}\right]^{1/2}[t+(u\pm u')^2]^{1/2} , \tag{28}$$

$$\Phi(b_\pm)=1+3\int_0^\infty (xb_\pm)^{-1}\sin(xb_\pm)dx , \tag{29}$$

$$|A_{n,n'}^\pm|^2=[\omega_1^\mp\Psi(n|n')-\omega_2^\mp\Psi(n-1|n'-1)]^2 , \tag{30}$$

$$\Psi(n|n')=\Psi(n'|n) =(n!n'!)^{-1/2}e^{-t/2}t^{(n+n')/2} \times {}_2F_0(-n',-n;-1/t) , \tag{31}$$

$${}_2F_0(a,c;x)=1+\sum_{k=1}^{\infty}\frac{(a)_k(c)_k x^k}{k!} , \tag{32}$$

$$(a)_k=\prod_{s=1}^{k}(a+s-1) , \tag{33}$$

$$(\omega_1^\mp)^2=(\omega_2^\pm)^2=\tfrac{1}{2}[1+\mathcal{E}^{-2}\pm\mathcal{E}^{-2}(\mathcal{E}^2-a_n^2)^{1/2} \times(\mathcal{E}^2-a_{n'}^2)^{1/2}] , \tag{34}$$

and

$$\omega_1\omega_2=\mathcal{E}^{-2}(nn')^{1/2}\left[\frac{H_z}{H_c}\right] . \tag{35}$$

We note that these values of ω have been averaged over the initial and final spins, and that N_i/W is the ion number density. In addition, although n' could in principle take any value $0\le n'\le\infty$, the requirement that $\mathcal{E}^2-a_{n'}^2>0$ places a finite upper limit on its range.

As a result of scattering in the presence of E_z, the density matrix ρ evolves away from ρ_0 and, in general, can no longer be diagonalized simultaneously with the Hamiltonian. Its diagonal elements, representing the probability of occupation of the new states in the presence of both H_z and E_z, will now be given approximately by

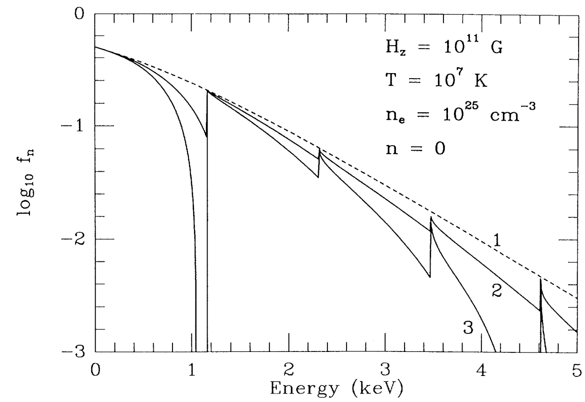

FIG. 1. Distribution function $f_n\equiv f(\epsilon_n^0)$ as a function of energy ϵ_n^0 for $H_z=10^{11}$ G, $T=10^7$, K, $n_e=10^{25}$ cm^{-3}, and $n=0$, and three electric-field intensities: $E_z=0$ (curve 1), 5×10^5 V/cm (curve 2), and 1.1×10^6 V/cm (curve 3).

$$\langle np_x\epsilon C;E_y=0,E_zH|\rho_\parallel|np_x\epsilon C;E_y=0,E_zH\rangle=\frac{f(\epsilon_n^0)}{W} , \tag{36}$$

where $|np_x\epsilon C;E_y=0,E_z,H\rangle$ is the Dirac wave function given in Eq. (12) with $E_y=0$. Although the off-diagonal elements of $\rho_\parallel$ cannot be determined in this way, they do not contribute to the transverse current and may therefore be neglected in our present application.

The distribution function $f_n\equiv f(\epsilon_n^0)$ is shown in Fig. 1 for the illustrative parameter values $H_z=10^{11}$ G, $T=10^7$ K, $n_e=10^{25}$ cm^{-3}, and $n=0$, and three electric field intensities $E_z=0$ (curve 1), 5×10^5 V/cm (curve 2), and 1.1×10^6 V/cm (curve 3). The relaxation time τ_n depends on the value of the particle energy relative to the bottom of the oscillator states. Thus, as p_z varies, ϵ_n^0 crosses the Landau levels, causing an oscillatory behavior in τ_n and hence f_n. The least variation in f_n relative to the field-free case arises when the relaxation time is very short, which occurs at multiples of the cyclotron fundamental energy.

Because of the reduced scattering cross section at high energy, the relaxation time τ_n is longer ($\sim10^{-2}$–10^{-14} sec) than the time ($\lesssim10^{-18}$ sec) required to "deplete" the distribution function $f(\epsilon_n)$ [Eq. (19)] when the particle Lorentz factor is large ($\gamma\approx1.2$–10^3). Thus, in this regime, it is more appropriate to solve the relativistic Boltzmann equation without the collisional term $\partial f/\partial t+\mathbf{F}\cdot\nabla_p f=0$ (where $\mathbf{F}=e\mathbf{E}$ is the driving force), whose solution is

$$f(\epsilon)=(n_e W)e^{-\epsilon/kT}/\mathrm{Tr}(e^{-\epsilon/kT})=\{n_e(2\pi\hbar)^2\hbar\exp[(m_e c^2-\epsilon)/kT]\}/(2m_e\sqrt{2m_e kT}\pi kT) ,$$

as indicated above. Here,

$$\epsilon\equiv[m_e^2c^4+(p_z-eE_zt)^2c^2+2n|e|c\hbar H_z]^{1/2}-m_ec^2 , \tag{37}$$

where e carries the sign of the charge [cf. Eq. (46) below]. Figure 2 shows the region in $\gamma-E_z$ phase space where the collisional term may be neglected. Above a given curve, labeled by the appropriate value of magnetic-field

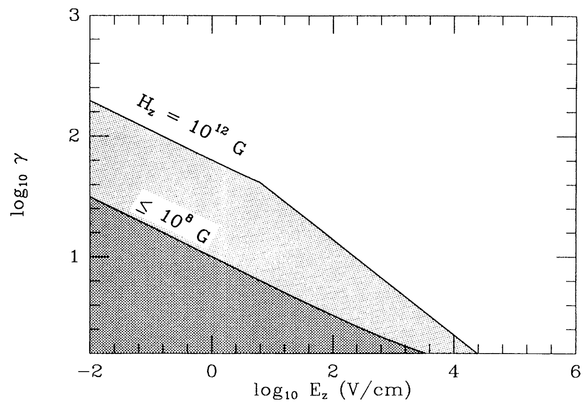

FIG. 2. Regions (shaded) in $\gamma - E_z$ phase space for which the relaxation time τ_n (with $n=5$ in this example) is sufficiently short that no significant depletion in the distribution function f_n takes place. The solid curves are labeled by values of the magnetic-field intensity H_z. For each value of H_z, the zone *above* the corresponding curve represents the values of γ and E_z for which the collisionless Boltzmann equation gives an adequate representation of f_n.

intensity H_z, τ_n is sufficiently long that depletion of f_n takes place. This region thus corresponds to values of γ and E_z for which the collisionless Boltzmann equation gives an adequate representation of f.

B. Density matrix for $\mathbf{E}=(0,E_y,E_z)$

Following a procedure first employed by Kohn and Luttinger,[16] we now let the electric-field component E_y turn on adiabatically; i.e., we replace E_y by $E_y e^{st}$, where s is a small positive number that will eventually be set to zero. This effectively disconnects the y component of the electric field at $t \to -\infty$, so that knowledge of the probability function f_n calculated above (for $\mathbf{E}=[0,0,E_z]$) can be used to determine the initial value of the solution to Eq. (16), giving the density matrix for the full electric field $\mathbf{E}=(0,E_y,E_z)$ at time $t=0$.

With the assumption that $\mathcal{H}_e \ll \mathcal{H}_0$, we can write the density matrix as

$$\rho=\rho_{\parallel}+\rho_1 e^{st} , \tag{38}$$

and then substitute this into the equation of motion [Eq. (16)] to solve for ρ_1, which represents the effect of scattering in the presence of E_y. Solving for ρ_1 at $t=0$, we get

$$-i\hbar s\rho_1=[\rho_{\parallel},\mathcal{H}_0+\mathcal{H}_e]+[\rho_1,\mathcal{H}_0+\mathcal{H}_e] + [\rho_{\parallel},V]+[\rho_1,V] , \tag{39}$$

for which we now take matrix elements in the reference system of Eq. (12). This results in the relation

$$(\epsilon_{n,p_x,C}-\epsilon'_{n',p_x',C'}-i\hbar s)\langle np_x\epsilon C;EH|\rho_1|n'p_x'\epsilon' C';EH\rangle = \frac{[f(\epsilon_n^0)-f(\epsilon_{n'}^{0'})]}{W}\langle np_x\epsilon C;EH|V|n'p_x'\epsilon' C';EH\rangle+\langle np_x\epsilon C;EH|[\rho_1,V]|n'p_x'\epsilon' C';EH\rangle , \tag{40}$$

where $\epsilon_{n,p_x,C}\equiv\epsilon_{n,p_x,C}(E,H)$ is the full energy in the presence of the electric field. As may be seen from Eqs. (4)–(11) and (15), the elements of ρ_1 diagonal in the momentum (i.e., p_x) determine the current. These elements satisfy the equation

$$\langle np_x\epsilon C;EH|\rho_1|n'p_x\epsilon' C';EH\rangle = \frac{1}{(\epsilon_{n,p_x,C}-\epsilon'_{n',p_x,C'}-i\hbar s)} \times\left[\frac{f(\epsilon_n^0)-f(\epsilon_{n'}^{0'})}{W}\langle np_x\epsilon C;EH|V|n'p_x\epsilon' C';EH\rangle+\langle np_x\epsilon C;EH|[\rho_1,V]|n'p_x\epsilon' C';EH\rangle\right] . \tag{41}$$

Since

$$\langle np_x\epsilon C;EH|V|n'p_x'\epsilon' C';EH\rangle=\langle n'p_x'\epsilon' C';EH|V|np_x\epsilon C;EH\rangle ,$$

the first term on the right-hand side does not contribute to the current and may therefore be ignored. Thus[10],

$$\langle np_x\epsilon C;EH|\rho_1|n'p_x\epsilon' C';EH\rangle = \frac{1}{\epsilon_{n,p_x,C}-\epsilon'_{n',p_x,C'}-i\hbar s}\sum_{n''p_x''\epsilon''C''}(\langle np_x\epsilon C;EH|\rho_1|n''p_x''\epsilon'' C'';EH\rangle\langle n''p_x''\epsilon'' C'';EH|V|n'p_x\epsilon' C';EH\rangle - \langle np_x\epsilon C;EH|V|n''p_x''\epsilon'' C'';EH\rangle\langle n''p_x''\epsilon'' C'';EH|\rho_1|n'p_x\epsilon' C';EH\rangle) . \tag{42}$$

We note here that the summation over n'' and p_x'' includes the degeneracy arising from the particle's motion in the xy plane. This degeneracy is finite if the motion is restricted to an area L_xL_y. All values of the momentum p_x'' are permitted for which the Landau orbit center (x_0,y_0) lies inside L_xL_y (neglecting the orbit's radius $r_{n''}$ in comparison to L_y). From the condition that $0<y_0<L_y$ [see Eq. (A7) in the Appendix], we find that the number of states per Landau level is therefore $\approx[eH_zL_xL_y]/[2\pi\hbar c]$. In practical terms, this degeneracy determines the range $\Delta p_x''$ of integration over p_x'' when converting the sum through the relation $\sum_{p_x''}\rightarrow(L_x/2\pi\hbar)\int dp_x''$, such that $\Delta p_x''\approx eH_zL_y/c$, which is independent of n'' as long as $r_{n''}\ll L_y$. To determine $\langle n''p_x''\epsilon''C'';EH|\rho_1|n'p_x\epsilon'C';EH\rangle$, we note that the first term in Eq. (41) is of order lower than the second, so that

$$\langle n''p_x''\epsilon''C'';EH|\rho_1|n'p_x\epsilon'C';EH\rangle \approx \frac{1}{\epsilon''_{n'',p_x'',C''}-\epsilon'_{n',p_x,C'}-i\hbar s}\frac{1}{W}\{[f(\epsilon_{n''}^{0''})-f(\epsilon_{n'}^{0'})]\langle n''p_x''\epsilon''C'';EH|V|n'p_x\epsilon'C';EH\rangle\} . \quad (43)$$

If we now let $s\rightarrow0$ and use the relation

$$\frac{1}{x-is}=P\left[\frac{1}{x}\right]+i\pi\delta(x) , \quad (44)$$

then, since the principal part P does not contribute to the current, we get

$$\langle np_x\epsilon C;EH|\rho_1|n'p_x\epsilon'C';EH\rangle = \frac{i\pi}{(\epsilon_{n,p_x,C}-\epsilon'_{n',p_x,C'})W}\sum_{n''p_x''\epsilon''C''}\langle np_x\epsilon C;EH|V|n''p_x''\epsilon''C'';EH\rangle\langle n''p_x''\epsilon''C'';EH|V|n'p_x\epsilon'C';EH\rangle$$
$$\times\{[f(\epsilon_n^0)-f(\epsilon_{n''}^{0''})]\delta(\epsilon-\epsilon'')+[f(\epsilon_{n'}^{0'})-f(\epsilon_{n''}^{0''})]\delta(\epsilon'-\epsilon'')\} . \quad (45)$$

The electric field enters into this expression only through the argument of the δ functions and the matrix elements of V. If we now choose the origin of our coordinate system to have zero potential energy [i.e., $\mathcal{H}_e(0,0,0)=0$, where $\mathcal{H}_e=-yeE_y-zeE_z$], then the "shift" (y_0,z_0) in the center of the cyclotron orbit corresponds to the electron energy in the presence of the electric field.[10] In our application we have $E_y,E_z\ll H_z\ll H_c$, so that

$$\epsilon_{n,p,C}\approx(m_e^2c^4+p_z^2c^2+2n|e|c\hbar H_z)^{1/2}+(E_y/H_z)cp_x , \quad (46)$$

giving

$$\epsilon_{n,p_x,C}-\epsilon''_{n'',p_x'',C''}\approx(\epsilon^0_{np_xC}-\epsilon^{0''}_{n''p_x''C''})+(E_y/H_z)c(p_x-p_x'') , \quad (47)$$

and similarly for $\epsilon'_{n',p_x,C'}-\epsilon''_{n'',p_x'',C''}$, where $\epsilon^0_{n,p_x,C}\equiv\epsilon_{n,p_x,C}(E=0,H)$ is the full energy in the absence of an electric field. Therefore, we find after some manipulation that the off-diagonal elements to first order in E are

$$\langle np_x\epsilon C;EH|\rho_1|n'p_x\epsilon'C';EH\rangle = \frac{-i\pi cE_y}{(\epsilon_{n,p_x,C}-\epsilon'_{n',p_x,C'})H_zW}$$
$$\times\sum_{n''p_x''\epsilon''C''}(p_x-p_x'')\langle np_x\epsilon C;E_y=0,H|V|n''p_x''\epsilon''C'';E_y=0,H\rangle$$
$$\times\langle n''p_x''\epsilon''C'';E_y=0,H|V|n'p_x\epsilon'C';E_y=0,H\rangle[\delta(\epsilon^{0''}-\epsilon^0)+\delta(\epsilon^{0''}-\epsilon^{0'})]f'(\epsilon_{n''}^{0''}) , \quad (48)$$

where f' is the derivative of the probability function f with respect to energy.

V. TRANSVERSE CONDUCTIVITY

In the single-particle Dirac theory, the quantity $s_\mu=c\psi^\dagger\alpha_\mu\psi$ is interpreted as the four-vector probability current. In this context the single-particle charge-current density is $j_\mu=es_\mu=ec\psi^\dagger\alpha_\mu\psi$. With Eqs. (4)–(11) and (48), we can now determine the expectation value of the *average* current components:

$$J_\pm\equiv J_x\pm iJ_y , \quad (49)$$

given by

$$\langle J_\pm\rangle=ec\,\mathrm{Tr}(\rho\alpha_\pm) , \quad (50)$$

that is,

$$\langle J_{\pm}\rangle = ec \sum_{n,p_x,\epsilon,C}\sum_{n',\epsilon',C'} \langle np_x\epsilon C;EH|\rho|n'p_x\epsilon' C';EH\rangle\langle n'p_x\epsilon' C';EH|\alpha_{\pm}|np_x\epsilon C;EH\rangle \ . \tag{51}$$

Substituting for $\rho=\rho\|+\rho_1$ and keeping only the dominant (i.e., diagonal) elements of $\rho_{\|}$, we get

$$\langle J_{\pm}\rangle = ec \sum_{n,p_x,\epsilon,C}\sum_{n',\epsilon',C'} \frac{f(\epsilon_n^0)}{W}\delta_{n,n'}\delta_{\epsilon,\epsilon'}\delta_{C,C'}\langle n'p_x\epsilon' C';EH|\alpha_{\pm}|np_x\epsilon C;EH\rangle$$
$$+ec\sum_{n,p_x,\epsilon,C}\sum_{n',\epsilon',C'} \langle np_x\epsilon C;EH|\rho_1|n'p_x\epsilon' C';EH\rangle\langle n'p_x\epsilon' C';EH|\alpha_{\pm}|np_x\epsilon C;EH\rangle \ . \tag{52}$$

The first term in this expression describes a current in the x direction of magnitude $n_e|e|cE_y/H_z$ and is related to the well-known classical result that a charged particle in crossed electric and magnetic fields drifts on the average with a velocity $v_D=c(\mathbf{E}\times\mathbf{H})/H^2$. This current is independent of the scattering and is not affected by orbital quantization. Thus, for each species of charge e,

$$\sigma_{xy}=J_x/E_y=n_e|e|c/H_z \ . \tag{53}$$

Evidently,

$$\sigma_{yy}=\frac{J_y}{E_y}=\left[\frac{i}{2E_y}\right](J_- - J_+)$$
$$=\left[\frac{iec}{2E_y}\right]\sum_{n,p_x,\epsilon,C}\sum_{n',\epsilon',C'}\langle np_x\epsilon C;EH|\rho_1|n'p_x\epsilon' C';EH\rangle$$
$$\times(\langle n'p_x\epsilon' C';EH|\alpha_-|np_x\epsilon C;EH\rangle-\langle n'p_x\epsilon' C';EH|\alpha_+|np_x\epsilon C;EH\rangle) \ . \tag{54}$$

Introducing Eq. (48) into Eq. (54), we obtain

$$\sigma_{yy}=\left[\frac{\pi ec^2}{2H_zW}\right]\sum_{n,p_x,\epsilon,C}\sum_{n',\epsilon',C'}(\langle n'p_x\epsilon' C';EH|\alpha_-|np_x\epsilon C;EH\rangle-\langle n'p_x\epsilon' C';EH|\alpha_+|np_x\epsilon C;EH\rangle)$$
$$\times\frac{1}{\epsilon_{n,p_x,C}-\epsilon'_{n',p_x,C'}}\sum_{n'',p_x'',\epsilon'',C''}(p_x-p_x'')\langle np_x\epsilon C;E_y=0,H|V|n''p_x''\epsilon''C'';E_y=0,H\rangle$$
$$\times\langle n''p_x''\epsilon''C'';E_y=0,H|V|n'p_x\epsilon' C';E_y=0,H\rangle$$
$$\times[\delta(\epsilon^{0''}-\epsilon^0)+\delta(\epsilon^{0''}-\epsilon^{0'})]f'(\epsilon_n^{0''}) \ . \tag{55}$$

If we now carry out the sum over p_x, ϵ and ϵ' in the limit where Eq. (46) can be used to express ϵ in terms of p_z, then

$$\sigma_{yy}=\frac{2\pi^2 ec}{(2\pi)^4\hbar^3H_z}\sum_{n,C}\sum_{n',C'}\sum_{n'',p_x'',p_z'',C''}\int dp_x\frac{p_x-p_x''}{\epsilon_{n,p_x,C}-\epsilon'_{n',p_x,C'}}f'(p_z'')\epsilon^{0''}$$
$$\times\Bigg[\int dp_z'(\langle n'p_x\epsilon' C';EH|\alpha_-|np_x\bar{\epsilon}C;EH\rangle$$
$$-\langle n'p_x\epsilon' C';EH|\alpha_+|np_x\bar{\epsilon}C;EH\rangle)$$
$$\times\langle np_x\bar{\epsilon}C;E_y=0,H|V|n''p_x''\epsilon''C'';E_y=0,H\rangle$$
$$\times\langle n''p_x''\epsilon''C'';E_y=0,H|V|n'p_x\epsilon' C';E_y=0,H\rangle$$
$$\times[(p_z'')^2c^2+2|e|c\hbar H_z(n''-n)]^{-1/2}$$
$$+\int dp_z(\langle n'p_x\bar{\epsilon}C';EH|\alpha_-|np_x\epsilon C;EH\rangle$$
$$-\langle n'p_x\bar{\epsilon}C';EH|\alpha_+|np_x\epsilon C;EH\rangle)$$
$$\times\langle np_x\epsilon C;E_y=0,H|V|n''p_x''\epsilon''C'';E_y=0,H\rangle$$
$$\times\langle n''p_x''\epsilon''C'';E_y=0,H|V|n'p_x\bar{\epsilon}C';E_y=0,H\rangle$$
$$\times[(p_z'')^2c^2+2|e|c\hbar H_z(n''-n')]^{-1/2}\Bigg] \ , \tag{56}$$

where

$$\bar{\epsilon} \equiv \epsilon^{0''} + \frac{E_y}{H_z} c p_x \ . \tag{57}$$

For the electron-ion (elastic scattering) interaction potential given by Eq. (22),

$$\langle n p_x \epsilon C; EH | V | n'' p_x'' \epsilon'' C''; EH \rangle = \frac{4\pi Z e^2}{W} \sum_{\mathbf{q}} \frac{1}{q^2} \langle n p_x \epsilon C; EH | e^{i\mathbf{q}\cdot\mathbf{r}} | n'' p_x'' \epsilon'' C''; EH \rangle \sum_{\alpha=1}^{N_i} e^{-i\mathbf{q}\cdot\mathbf{R}_\alpha} \ , \tag{58}$$

where N_i is the total number of ions. If we now take the ensemble average over the scattering centers, we get

$$\left\langle \sum_{\alpha\alpha''}^{N_i} e^{i\mathbf{q}''\cdot\mathbf{R}_{\alpha''} - i\mathbf{q}\cdot\mathbf{R}_\alpha} \right\rangle = N_i \delta_{\mathbf{q},\mathbf{q}''} \Phi(q) \ , \tag{59}$$

where

$$\Phi(q) = 1 + 3\int_0^\infty (xs)^{-1} \sin(xs)\, dx \ , \tag{60}$$

and

$$s \equiv \left[\frac{3}{4\pi}\right]^{1/3} \left[\frac{W}{N_i}\right]^{1/3} |\mathbf{q}| \ . \tag{61}$$

In our application, $\Phi(q) - 1 \ll 1$, so that with a conversion of the sum into an integration through the usual relation

$$\sum_{\mathbf{q}} \rightarrow \frac{W}{(2\pi)^3} \int d^3 q \ , \tag{62}$$

we get

$$\langle n p_x \epsilon C; EH | V | n'' p_x'' \epsilon'' C''; EH \rangle \langle n'' p_x'' C''; EH | V | n' p_x \epsilon' C'; EH \rangle$$
$$= (4\pi Z e^2)^2 \left[\frac{N_i}{W}\right] \left[\frac{1}{2\pi}\right]^3 \int d^3 q \frac{1}{q^4} \langle n p_x \epsilon C; EH | e^{i\mathbf{q}\cdot\mathbf{r}} | n'' p_x'' \epsilon'' C''; EH \rangle \langle n'' p_x'' \epsilon'' C''; EH | e^{-i\mathbf{q}\cdot\mathbf{r}} | n' p_x \epsilon' C'; EH \rangle \ . \tag{63}$$

Note that with the dependence on p_x indicated in Eq. (1), q_x in this integral must satisfy the condition $q_x = (p_x - p_x'')/\hbar$.

VI. RESULTS

The conductivity $\sigma_\perp \equiv (\sigma_{yy}^2 + \sigma_{xy}^2)/\sigma_{yy} = \sigma_{yy}$ calculated from Eq. (56) is shown graphically in Figs. 3–5 for various combinations of the parameters H_z, E_y, E_z, γ, and n_e. (A more complete survey of the results will be published elsewhere.[17]) The horizontal scale shows the "characteristic" (perpendicular) temperature $T_{\rm ch}$, such that the highest accessible Landau level is given as

$$n_{\max} \equiv \text{int} \left[\frac{m_e c k T_{\rm ch}}{|e| \hbar H_z}\right] . \tag{64}$$

These calculations assume a minimum value of the electron-ion momentum transfer corresponding to the Debye wave number

$$k_D \equiv \left[\frac{4\pi n_e e^2}{3kT_{\rm ch}}\right]^{1/2} . \tag{65}$$

In all cases the conductivity is zero when the thermal energy is insufficient to excite particles above the ground state, a necessary condition for the interaction potential to yield a nonzero contribution to the particle transport. For temperatures above the value $kT_{\rm ch} = |e|\hbar H_z / m_e c$, $\sigma_\perp$ displays the familiar oscillatory behavior resulting from sweeps across increasing Landau excitations. The overall

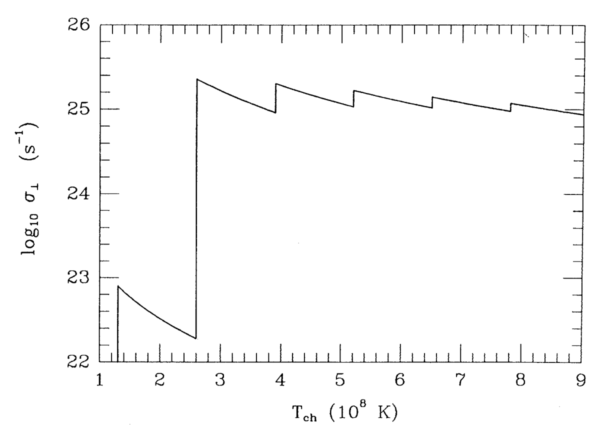

FIG. 3. Perpendicular conductivity $\sigma_\perp$ as a function of characteristic temperature $T_{\rm ch}$, for a magnetic field $H_z = 10^{12}$ G, electric field $\mathbf{E} = (0, 10^8, 10^8)$ V/cm, Lorentz factor $\gamma = 10^3$, and particle number density $n_e = 10^{25}$ cm^{-3}.

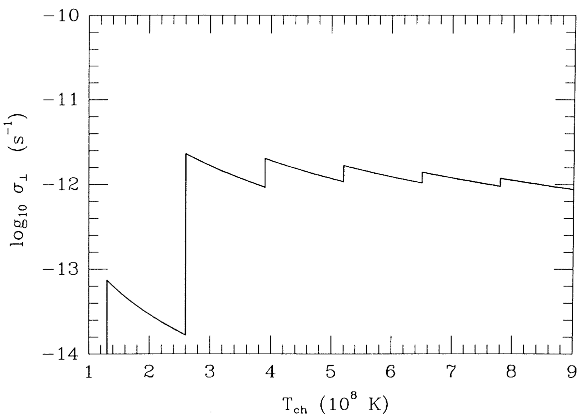

FIG. 4. Perpendicular conductivity $\sigma_\perp$ as a function of characteristic temperature $T_{\rm ch}$, for a magnetic field $H_z=10^{12}$ G, electric field $\mathbf{E}=(0,0.1,0.1)$ V/cm, Lorentz factor $\gamma=10^3$, and particle number density $n_e=10^{20}\ \mathrm{cm}^{-3}$.

dependence of $\sigma_\perp$ on $T_{\rm ch}$ is due to two main competing effects. The first of these results from the temperature dependence of the particle distribution function f_n (see Sec. IV A above), which yields $f_n' \sim T^{-5/2}$. Since f_n' is a measure of the phase space available to electrons accelerated by the applied electric field, this effect results in a decrease of $\sigma_\perp$ as $T_{\rm ch}$ increases. However, the Debye wave number k_D [Eq. (65)], specifying the minimum allowed momentum transfer during electron-ion interactions, decreases with increasing temperature and results in a greater contribution to the particle transport. This effect is particularly important at the higher densities, as illustrated in Fig. 5.

The conductivity is only weakly dependent on γ because the relaxation time τ_n is sufficiently long that the dependence of the scattering cross section on particle energy is not an important factor. However, $\sigma_\perp$ depends very strongly on the electric field, though we are still justified in using a first-order analysis due to the very small value of the "expansion" parameter E/H_z ($\ll 1$). These results indicate that $\sigma_\perp \sim E_y^2$ to better than 0.01% and is roughly linear in E_z (to within 10%) for temperatures $kT_{\rm ch} \gtrsim |e|\hbar H_z/m_e c$. As discussed in Sec. III, the theory we have developed here assumes a one-electron approximation for the description of the electron gas; i.e., it does not take into account fluctuations of the particle motion when the particle mean free path is small compared to the classical orbital radius of the Landau excitations. Thus the strong dependence of $\sigma_\perp$ on E_y is not realistic for densities $n_e \gtrsim (3kT)^2\sqrt{eH_z}/8\pi e^4\sqrt{c}\hbar \ln\Lambda_c$, where $\ln\Lambda_c$ is the usual Coulomb cutoff arising from the Debye screening. In fact, it is expected that for these physical conditions the conductivity tensor should reduce to its classical counterpart[18] $\sigma_\perp^c$, which is independent of E_y. We have thus chosen to "normalize" $\sigma_\perp$ so that it correctly reduces to $\sigma_\perp^c$ in the region where the thermal fluctuations are important, and we have included this scaling in the results presented in Figs. 3–5.

This second-order dependence on the electric field E_y reflects the intrinsic difference between the particle wave function for $E_z \neq 0$ [Eq. (1)] and that for $E_z=0$. In the latter case, the particle motion is unrestricted (i.e., a plane wave) in the z direction,[3] whereas in the former, the z momentum directly affects the quantization of the particle motion in the xy plane through the energy of the state. The fact that motion in the z direction is now coupled to the harmonic oscillator greatly restricts the al-

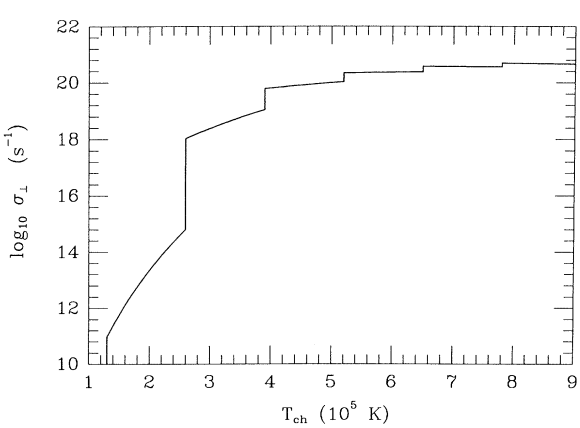

FIG. 5. Perpendicular conductivity $\sigma_\perp$ as a function of characteristic temperature $T_{\rm ch}$, for a magnetic field $H_z=10^9$ G, electric field $\mathbf{E}=(0,10^2,10^2)$ V/cm, Lorentz factor $\gamma=10^3$, and particle number density $n_e=10^{25}\ \mathrm{cm}^{-3}$.

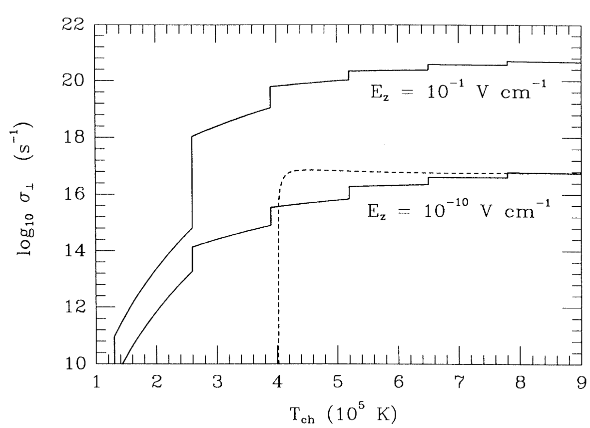

FIG. 6. Comparison between the conductivity calculated here (solid with two values of E_z) with the *classical* expression (dashed) $\sigma_\perp^c \sim \omega_p^2/[4\pi\nu(1+\omega_B^2/\nu^2)]$, where ω_p is the electronic plasma (precession) frequency, $\omega_B=eB/m_e c$ is the cyclotron frequency, and ν is the collision frequency. The parameter values are $H_z=10^9$ G, $n_e=10^{25}\ \mathrm{cm}^{-3}$, and $E_y=8\times10^6\ \mathrm{V\,cm}^{-1}$. The steep dropoff in $\sigma_\perp^c$ at low $T_{\rm ch}$ is due to the temperature dependence of the classical Debye screening length, which results in $\nu \to 0$ for $T \lesssim 4\times10^5$ K.

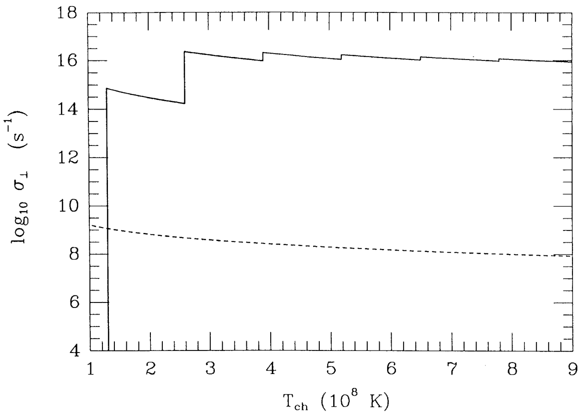

FIG. 7. Comparison between the conductivity calculated here (solid) with the *classical* expression (dashed) $\sigma_\perp^c \sim \omega_p^2/[4\pi\nu(1+\omega_B^2/\nu^2)]$, assuming $H_z=10^{12}$ G, $n_e=10^{25}$ cm^{-3}, $E_y=10^8$ V cm^{-1}, and $E_z=0.1$ V cm^{-1}. The difference resulting from the strong dependence of $\sigma_\perp$ on **E** is clearly evident (see also Fig. 6).

lowed transitions in the α-matrix elements given by Eqs. (4)–(11), in such a way that only second- (and higher-) order terms in E_y contribute to the conductivity $\sigma_\perp$ [Eq. (56)].

A simple relation that incorporates all of these effects is $\sigma_\perp \sim E_y^2 E_z n_e^2/H_z^2$, with $\sigma_\perp \approx 5\times 10^{16}$ s^{-1} when $H_z=10^9$ G, $n_e=10^{25}$ cm^{-3}, $E_y=8\times 10^6$ V cm^{-1}, and $E_z=10^{-10}$ V cm^{-1} (see Figs. 3–7). It is interesting to see how this conductivity compares with the results of earlier work. As mentioned above, the oscillatory behavior of $\sigma_\perp$ is a manifestation of the quantized particle motion in strong magnetic fields and has, of course, been evident whenever these quantizing effects have been taken into account. The previous calculations that most closely resemble those discussed here are those reported by Canuto and Chiuderi,[3] in which the electric and magnetic fields were assumed to be perpendicular. These earlier calculations also assumed a degenerate, high-density gas with $\rho \gtrsim 10^6$ g cm^{-3} (corresponding to a particle number density $n_e \gtrsim 10^{30}$ cm^{-3}), as opposed to the ideal plasma with $\lesssim 10^{25}$ cm^{-3} used here. This difference in the physical conditions is reflected in the functional form of σ_{yy} for the former case (i.e., $\sigma_{yy}^{CC} \sim n_i H_z^{1/2}$, with n_i the ion number density), where the dependence on density enters only in first order because of the electron degeneracy. A direct comparison between our conductivity and the *classical* expression $\sigma_\perp^c \sim \omega_p^2/[4\pi\nu(1+\omega_B^2/\nu^2)]$, where ω_p is the electronic plasma (precession) frequency, $\omega_B = eB/m_e c$ is the cyclotron frequency, and ν is the collision frequency, is shown in Figs. 6 and 7 for representative field intensities. Although $\sigma_\perp$ and $\sigma_\perp^c$ are comparable in regions where the thermal fluctuations are important (see above and Fig. 6), the difference between the two can be significant as a result of the strong dependence of $\sigma_\perp$ on **E**. Thus, for example, $\sigma_\perp \approx \sigma_\perp^c$ when $H_z \lesssim 10^9$ G, $n_e \gtrsim 10^{25}$ cm^{-3}, $E_y = 8\times 10^6$ V cm^{-1}, and $E_z=10^{-10}$ V cm^{-1} (Fig. 6), but $\sigma_\perp \gg \sigma_\perp^c$ when $E_y \gg 10^6$ V cm^{-1} and/or $E_z \gg 10^{-10}$ V cm^{-1} (Figs. 6 and 7).

VII. CONCLUDING REMARKS

We have developed a formalism for calculating the "transverse" conductivity $\sigma_\perp \equiv (\sigma_{yy}^2+\sigma_{xy}^2)/\sigma_{yy}$ of an electron-ion plasma, where σ_{ij} are the elements of the conductivity tensor σ, in the case where **H** is along z and $\mathbf{E}=(0,E_y,E_z)$. The component σ_{yy} depends solely on the scattering process, whereas σ_{xy} is due entirely to the drift velocity $v_D = c(\mathbf{E}\times\mathbf{H})/H^2$.

These results will be applied to a quantum treatment of magnetic-field reconnection, which has been motivated by its relevance to the superstrong magnetic fields usually encountered in neutron-star environments, particularly as a mechanism for generating the plasma heating and particle acceleration leading to the sudden release of magnetospheric energy. The dependence of the perpendicular conductivity on the electric-field intensity implies that strong electric fields generated by tearing mode fluctuations in the magnetic field can significantly alter the classically derived particle transport properties. A description of the theory for resistive magnetic tearing in a quantizing field and the application of the results presented in this paper will be reported elsewhere.[17]

ACKNOWLEDGMENTS

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APPENDIX

In this appendix we reproduce the constants appearing in the normalized Dirac wave function and the velocity matrix elements:

$$A' = \frac{e^2}{2c^2\hbar^2}[-F+(F^2+4G^2)^{1/2}] , \tag{A1}$$

$$B' = \frac{e^2}{2c^2\hbar^2}[F+(F^2+4G^2)^{1/2}] , \tag{A2}$$

$$F = \mathbf{H}^2 - \mathbf{E}^2 , \tag{A3}$$

$$G = \mathbf{E}\cdot\mathbf{H} , \tag{A4}$$

$$\xi = (y-y_0)\cos\varphi_0 + (z-z_0)\sin\varphi_0 , \tag{A5}$$

$$\eta = -(y-y_0)\sin\varphi_0 + (z-z_0)\cos\varphi_0 , \tag{A6}$$

$$y_0 = -\frac{c}{e}\frac{p_x}{H_z} , \tag{A7}$$

$$z_0 = \frac{c}{e}\frac{p_x E_y - \epsilon H_z/c}{G} , \tag{A8}$$

$$\tan\varphi_0 = [2E_z^2 + F + (F^2+4G^2)^{1/2}]/(2E_yE_z) , \tag{A9}$$

$$u_{1n}^{\pm}=\sum_{j=1}^{2}C_j\{K_1[i\hbar\sin\varphi_0\Omega_jH_n+i\hbar\cos\varphi_0(-K_2\eta H_n+K_{3n}H_{n-1})\theta_j\mp mcH_n\theta_j]$$
$$-\hbar\cos\varphi_0\Omega_jH_n+\hbar\sin\varphi_0(-K_4\eta H_n+K_{3n}H_{n-1})\theta_j\}\ , \tag{A10}$$

$$u_{2n}^{\pm}=\sum_{j=1}^{2}C_j\{K_1[-\hbar\cos\varphi_0\Omega_jH_n+\hbar\sin\varphi_0(-K_5\eta H_n+K_{3n}H_{n-1})\theta_j]$$
$$+i\hbar\sin\varphi_0\Omega_jH_n+i\hbar\cos\varphi_0(K_6\eta H_n+K_{3n}H_{n-1})\theta_j\pm mcH_n\theta_j\}\ , \tag{A11}$$

where

$$K_1=\frac{E_y}{H_z+(c\hbar/e)\sqrt{B'}\ \operatorname{sgn} H_z+i[E_z+(c\hbar/e)\sqrt{A'}]}\ , \tag{A12}$$

$$K_2=\sqrt{B'}+\frac{c\hbar}{e}\frac{iB'}{E_z}\ , \tag{A13}$$

$$K_{3n}=2n(B')^{1/4}\ , \tag{A14}$$

$$K_4=\sqrt{B'}+\frac{e}{c}\frac{H_z}{\hbar}\ , \tag{A15}$$

$$K_5=\sqrt{B'}-\frac{e}{c}\frac{H_z}{\hbar}\ , \tag{A16}$$

and

$$K_6=-\sqrt{B'}+\frac{c\hbar}{e}\frac{iB'}{E_z}\ . \tag{A17}$$

Here $H_n=H_n((B')^{1/4}\eta)$ are the Hermite polynomials, and θ_1 and θ_2 are the confluent hypergeometric functions

$$\theta_1(\xi)=F\left[\frac{i\Lambda}{4\sqrt{A'}\hbar^2},\frac{1}{2},-i\sqrt{A'}\xi^2\right]\ , \tag{A18}$$

$$\theta_2(\xi)=\xi F\left[\frac{1}{2}+\frac{i\Lambda}{4\sqrt{A'}\hbar^2},\frac{3}{2},-i\sqrt{A'}\xi^2\right]\ , \tag{A19}$$

and Ω_1 and Ω_2 are their first derivatives:

$$\Omega_1(\xi)=\frac{d\theta_1}{d\xi}=\xi\frac{\Lambda}{\hbar^2}F\left[1+\frac{i\Lambda}{4\sqrt{A'}\hbar^2},\frac{3}{2},-i\sqrt{A'}\xi^2\right] \tag{A20}$$

and

$$\Omega_2(\xi)=\frac{d\theta_2}{d\xi}=F\left[\frac{1}{2}+\frac{i\Lambda}{4\sqrt{A'}\hbar^2},\frac{1}{2},-i\sqrt{A'}\xi^2\right]\ , \tag{A21}$$

where

$$\Lambda=[(mc)^2+(2n+1+\operatorname{sgn} H_z)\hbar^2\sqrt{B'}]\ . \tag{A22}$$

The constants $T_{n,m}^{i,j}\equiv T_{n,m}(C_1=i,\ C_2=j)$ appearing in the matrix elements of the velocity operator $\boldsymbol{\alpha}$ are given in terms of the integrals

$$I_{1,1}^{\theta\theta}(\Delta)=\int_{-L_\xi}^{L_\xi}\theta_1^*(\xi)\theta_1(\xi+\Delta)e^{i(\Delta^2+2\xi\Delta)\sqrt{A'}/2}d\xi\ , \tag{A23}$$

$$I_{2,2}^{\theta\theta}(\Delta)=\int_{-L_\xi}^{L_\xi}\theta_2^*(\xi)\theta_2(\xi+\Delta)e^{i(\Delta^2+2\xi\Delta)\sqrt{A'}/2}d\xi\ , \tag{A24}$$

$$I_{1,1}^{\Omega\Omega}(\Delta)=\int_{-L_\xi}^{L_\xi}\Omega_1^*(\xi)\Omega_1(\xi+\Delta)e^{i(\Delta^2+2\xi\Delta)\sqrt{A'}/2}d\xi\ , \tag{A25}$$

$$I_{2,2}^{\Omega\Omega}(\Delta)=\int_{-L_\xi}^{L_\xi}\Omega_2^*(\xi)\Omega_2(\xi+\Delta)e^{i(\Delta^2+2\xi\Delta)\sqrt{A'}/2}d\xi\ , \tag{A26}$$

$$I_{1,2}^{\theta\Omega}(\Delta)=\int_{-L_\xi}^{L_\xi}\theta_1^*(\xi)\Omega_2(\xi+\Delta)e^{i(\Delta^2+2\xi\Delta)\sqrt{A'}/2}d\xi\ , \tag{A27}$$

$$I^{\Omega\theta}_{2,1}(\Delta)=\int_{-L_\xi}^{L_\xi}\Omega_2^*(\xi)\theta_1(\xi+\Delta)e^{i(\Delta^2+2\xi\Delta)\sqrt{A'}/2}d\xi\ , \tag{A28}$$

$$I^{\Omega\theta}_{1,2}(\Delta)=\int_{-L_\xi}^{L_\xi}\theta_2^*(\xi)\Omega_1(\xi+\Delta)e^{i(\Delta^2+2\xi\Delta)\sqrt{A'}/2}d\xi\ , \tag{A29}$$

$$I^{\theta\Omega}_{2,1}(\Delta)=\int_{-L_\xi}^{L_\xi}\Omega_1^*(\xi)\theta_2(\xi+\Delta)e^{i(\Delta^2+2\xi\Delta)\sqrt{A'}/2}d\xi\ , \tag{A30}$$

$$I^{\theta\Omega}_{1,1}(\Delta)=\int_{-L_\xi}^{L_\xi}\theta_1^*(\xi)\Omega_1(\xi+\Delta)e^{i(\Delta^2+2\xi\Delta)\sqrt{A'}/2}d\xi\ , \tag{A31}$$

$$I^{\Omega\theta}_{1,1}(\Delta)=\int_{-L_\xi}^{L_\xi}\Omega_1^*(\xi)\theta_1(\xi+\Delta)e^{i(\Delta^2+2\xi\Delta)\sqrt{A'}/2}d\xi\ , \tag{A32}$$

$$I^{\theta\Omega}_{2,2}(\Delta)=\int_{-L_\xi}^{L_\xi}\theta_2^*(\xi)\Omega_2(\xi+\Delta)e^{i(\Delta^2+2\xi\Delta)\sqrt{A'}/2}d\xi\ , \tag{A33}$$

and

$$I^{\Omega\theta}_{2,2}(\Delta)=\int_{-L_\xi}^{L_\xi}\Omega_2^*(\xi)\theta_2(\xi+\Delta)e^{i(\Delta^2+2\xi\Delta)\sqrt{A'}/2}d\xi\ , \tag{A34}$$

where $\Delta\equiv z_0-z_0'$. They are

$$T^{j,k}_{n,m}=(2^n\pi^{1/2}n!)K_7I^{\Omega\Omega}_{2-j,2-k}+[2^{n-1}\pi^{1/2}(2n+1)n!](1/\sqrt{B'})K_8I^{\theta\theta}_{2-j;2-k}$$
$$+(2^{n-1}\pi^{1/2}n!)[1/(B')^{1/4}]K_9I^{\theta\theta}_{2-j,2-k}+(2^n\pi^{1/2}n!)K_1^*m^2c^2I^{\theta\theta}_{2-j,2-k}\ , \tag{A35}$$

$$T^{j,k}_{n+2,m}=[2^n\pi^{1/2}(n+2)!](1/\sqrt{B'})K_8I^{\theta\theta}_{2-j,2-k}\ , \tag{A36}$$

$$T^{j,k}_{n-2,m}=(2^{n-2}\pi^{1/2}n!)(1/\sqrt{B'})K_8I^{\theta\theta}_{2-j,2-k}+[2^{n-2}\pi^{1/2}(n-1)!][1/(B')^{1/4}]K_9I^{\theta\theta}_{2-j,2-k}\ , \tag{A37}$$

$$T^{j,k}_{n+1,m-1}=[2^n\pi^{1/2}(n+1)!][1/(B')^{1/4}]K_{10}I^{\theta\theta}_{2-j,2-k}\ , \tag{A38}$$

$$T^{j,k}_{n-1,m-1}=(2^{n-1}\pi^{1/2}n!)[1/(B')^{1/4}]K_{10}I^{\theta\theta}_{2-j,2-k}+[2^{n-1}\pi^{1/2}(n-1)!]K_{11}I^{\theta\theta}_{2-j,2-k}\ , \tag{A39}$$

$$T^{j,k}_{n,m-1}=(2^n\pi^{1/2}n!)K_{12}I^{\Omega\theta}_{2-j,2-k}\ , \tag{A40}$$

$$T^{j,k}_{n+1,m}=[2^n\pi^{1/2}(n+1)\ ][1/(B')^{1/4}]K_{13}I^{\Omega\theta}_{2-j,2-k}+[2^n\pi^{1/2}(n+1)!][1/(B')^{1/4}]K_{14}I^{\theta\Omega}_{2-j,2-k}\ , \tag{A41}$$

and

$$T^{j,k}_{n-1,m}=(2^{n-1}\pi^{1/2}n!)[1/(B')^{1/4}]K_{13}I^{\Omega\theta}_{2-j,2-k}+(2^{n-1}\pi^{1/2}n!)[1/(B')^{1/4}]K_{14}I^{\theta\Omega}_{2-j,2-k}$$
$$+[2^{n-1}\pi^{1/2}(n-1)!]K_{15}I^{\theta\Omega}_{2-j,2-k}\ , \tag{A42}$$

where

$$K_7=|K_1|^2i\hbar^2\sin\phi_0\cos\phi_0+K_1^*\hbar^2\sin^2\phi_0+K_1\hbar^2\cos^2\phi_0-i\hbar^2\sin\phi_0\cos\phi_0\ , \tag{A43}$$

$$K_8=-|K_1|^2i\hbar^2\cos\phi_0\sin\phi_0K_2^*K_5-K_1^*\hbar^2\cos^2\phi_0K_2^*K_6+K_1\hbar^2\sin^2\phi_0K_4K_5-i\hbar^2\sin\phi_0\cos\phi_0K_4K_6\ , \tag{A44}$$

$$K_9=|K_1|^2i\hbar^2\cos\phi_0\sin\phi_0K_3K_5+K_1^*\hbar^2\cos^2\phi_0K_3K_6-K_1\hbar^2\sin^2\phi_0K_3K_5+i\hbar^2\sin\phi_0\cos\phi_0K_3K_6\ , \tag{A45}$$

$$K_{10}=|K_1|^2i\hbar^2\cos\phi_0\sin\phi_0K_2^*K_3-K_1^*\hbar^2\cos^2\phi_0K_2^*K_3-K_1\hbar^2\sin^2\phi_0K_4K_3-i\hbar^2\sin\phi_0\cos\phi_0K_4K_3\ , \tag{A46}$$

$$K_{11}=-|K_1|^2i\hbar^2\cos\phi_0\sin\phi_0|K_3|^2+K_1^*|K_3|^2\hbar^2\cos^2\phi_0+K_1|K_3|^2\hbar^2\sin^2\phi_0+i\hbar^2\sin\phi_0\cos\phi_0|K_3|^2\ , \tag{A47}$$

$$K_{12}=-|K_1|^2i\hbar^2\sin^2\phi_0K_3+K_1^*\hbar^2\sin\phi_0\cos\phi_0K_3-K_1\hbar^2\sin\phi_0\cos\phi_0K_3-i\hbar^2\cos^2\phi_0K_3\ , \tag{A48}$$

$$K_{13}=|K_1|^2i\hbar^2\sin^2\phi_0K_5+K_1^*K_6\hbar^2\sin\phi_0\cos\phi_0+K_1\hbar^2\sin\phi_0\cos\phi_0K_5-i\hbar^2\cos^2\phi_0K_6\ , \tag{A49}$$

$$K_{14}=-|K_1|^2i\hbar^2\cos^2\phi_0K_2^*-K_1^*\hbar^2\cos\phi_0\sin\phi_0K_2^*+K_1\hbar^2\cos\phi_0\sin\phi_0K_4-i\hbar^2\sin^2\phi_0K_4\ , \tag{A50}$$

and

$$K_{15}=|K_1|^2i\hbar^2\cos^2\phi_0K_3+K_1^*\hbar^2\cos\phi_0\sin\phi_0K_3-K_1\hbar^2\cos\phi_0\sin\phi_0K_3+i\hbar^2\sin^2\phi_0K_3\ . \tag{A51}$$

The normalization constant is given as

$$N_n^2(C_1,C_2)\equiv\int|\psi_{n,p_x,\epsilon}|^2dx\,dy\,dz=\int_{-L_x}^{L_x}dx\int_{-\infty}^{+\infty}d\eta\int_{-L_\xi}^{L_\xi}d\xi|\psi_{n,p_x,\epsilon}|^2$$
$$=\frac{4L_x}{(B')^{1/4}}\sum_{j=1}^{2}|C_j|^2\Bigg\{(2^n\pi^{1/2}n!)(|K_1|^2+1)[\hbar^2I_{j,j}^{\Omega\Omega}+(mc)^2I_{j,j}^{\theta\theta}]+[2^{n-1}\pi^{1/2}(n-1)!4n^2]\hbar^2\sqrt{B'}(|K_1|^2+1)I_{j,j}^{\theta\theta}$$
$$+\frac{2^{n-1}}{\sqrt{B'}}\pi^{1/2}(2n+1)n![(|K_1K_5|^2+|K_4|^2)\hbar^2\sin^2\phi_0+(|K_1K_2|^2+|K_2|^2)\hbar^2\cos^2\phi_0$$
$$+i(K_1K_2-K_1^*K_2^*)(K_4-K_5)\hbar^2\sin\phi_0\cos\phi_0]I_{j,j}^{\theta\theta}$$
$$+(2n2^{n-1}\pi^{1/2}n!)[-2(|K_1|^2K_5+K_4)\hbar^2\sin^2\phi_0-2(|K_1|^2+1)\sqrt{B'}\hbar^2\cos^2\phi_0$$
$$+i\hbar^2\sin\phi_0\cos\phi_0(K_1-K_1^*)(K_5-K_4)]I_{j,j}^{\theta\theta}\Bigg\}\,. \quad \text{(A52)}$$

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