Intertwining of the equations of black-hole perturbations

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For both Schwarzschild and Reissner-Nordström black holes, equations have been given in which the dynamics of perturbations are governed by effective potentials. These potentials have different form for odd- and for even-parity perturbations, yet they are equivalent. In particular, they give rise to the same spectrum of quasinormal frequencies. Though all the potentials are rational functions of the radial coordinate, the odd-parity potentials are markedly simpler and are necessary for certain semianalytic approaches to numerical studies. We investigate here whether there can be yet simpler equivalent potentials which could further simplify numerical work. With the "intertwining operator" viewpoint we show that no such further simplification is possible. This viewpoint also gives added insights into the relationship of the equivalent effective potentials.

I. INTRODUCTION

The perturbations of a Schwarzschild black-hole spacetime can be described by a "potential-form" equation, the Zerilli equation[1]

$$\frac{d^2\psi}{dx^2} + (\omega^2 - V_Z)\psi = 0. \tag{1}$$

Here, time dependence $e^{i\omega t}$ (in terms of Schwarzschild time t) is assumed, and the x variable is related to the Schwarzschild radial coordinate r by

$$\frac{d}{dx} = \frac{r-1}{r}\frac{d}{dr}. \tag{2}$$

(The x variable is often written as r_* in equations describing black-hole dynamics.) Units are used in which $G = c = 2M = 1$, so that the horizon is at $r = 1$. In these units the Zerilli potential V_Z is given by

$$V_Z = \frac{2(n+1)r^3 + 3r^2 + 9r/2n + 9/4n^2}{r^4(r+3/2n)^2}(r-1), \tag{3}$$

where the parameter n, in terms of the multipole index $l \geq 2$ of the perturbation, is

$$n \equiv \tfrac{1}{2}(l-1)(l+2). \tag{4}$$

The perturbations can also be described by another potential-form equation, the Regge-Wheeler (RW) equation[2]

$$\frac{d^2\phi}{dx^2} + (\omega^2 - V_{\rm RW})\phi = 0, \tag{5}$$

which differs only in the details of the potential

$$V_{\rm RW} = \frac{2(n+1)r - 3}{r^4}(r-1). \tag{6}$$

The Zerilli equation arose initially in the study of even-parity metric perturbations of the Schwarzschild solution in a particular gauge choice;[3] the RW equation arose in the study of odd-parity perturbations in the same formalism. A very different formalism for perturbations is based on Newman-Penrose projections on null tetrads. This formalism leads to the Bardeen-Press equation[4]—an equation of a somewhat different form from that of Eqs. (1) or (5)—which has the same form for even and odd parity. These equations all describe the same dynamics, and Chandrasekhar[5] showed explicitly the connection between the Zerilli and Bardeen-Press equations, and between the RW and Bardeen-Press equations, and finally,[6] between the RW and Zerilli equations. (Many of the original papers in this area did not make clear the relationships among the various descriptions, relationships that were not understood until later. An excellent presentation which includes an exhaustive treatment of results and relationships is the monograph by Chandrasekhar.[7])

Work on perturbations of the Reissner-Nordström spacetime paralleled that for the Schwarzschild case. For a charged black hole the two degrees of freedom—gravitational and electromagnetic—require two wave functions. Equations were given by Zerilli,[8] and decoupled potential-form equations were given by Moncrief.[9,10] Two different potentials were found for each of the decoupled wave functions, with the simpler potentials, analogous to the RW potential, arising from the equations for odd-parity perturbations, and the more intricate potentials, analogous to the Zerilli potential, arising from the even-parity equations. The Newman-Penrose formalism gave equations, analogous to the Bardeen-Press equation, which simultaneously described both even and odd perturbations.[11] This was used by Chandrasekhar[11] to show the relationship between the potential-form equations of different parity.

The existence of different descriptions of perturbations of black holes led Chandrasekhar[12] (see also Ref. 7, Sec. 28) to consider the general question of the relationship of two potentials which are "equivalent" in the sense of producing the same physical consequences (more specifically, having the same reflection and transmission coefficients). He found that a sufficient condition for two potentials $V^{(+)}$ and $V^{(-)}$ to be equivalent is that they are related by

$$V^{(\pm)} = \pm\beta\frac{df}{dx} + \beta^2 f^2 + \kappa f, \tag{7}$$

where f is some function of x. It turns out that this relationship is mathematically identical to the relationship of two potential-form equations in Anderson's formalism[13] of "intertwining operators." That formalism, however, gives a slightly different viewpoint on the issue and clarifies the fact that for any potential there are equivalent potentials, in fact, an infinite number of equivalent potentials. For Schwarzschild black holes, for example, the Zerilli and RW potentials are only two of an infinite set of possible potentials.

This range of possibilities raises a question that is not only of interest as a matter of principle, but which has important practical consequences. With laser interferometric gravity-wave detection[14] more or less imminent, renewed attention has been given to quasinormal (QN) modes of black holes since quasinormal oscillations may provide the strongest gravitational waves and would provide a signal with a signature characterizing their black-hole origin.[15]

The computation of QN frequencies, and other aspects of QN excitations, can involve numerical problems. Analytic or semi-analytic approaches are often far superior to brute-force numerical computations. One clear example of this is the high-precision computation of the Schwarzschild QN frequencies by Leaver.[16] Leaver's approach is to seek a solution of the RW equation in the form of a power series entirely in terms of the r variable. After certain simplifications [in particular, after the behavior at infinity is factored out; see Eq. (19) below], the equation leads to a three-term recursion relation for the coefficients in the power-series solution. Three-term recursion relations can be treated by continued-fraction methods. With such a method, Leaver shows that the QN frequencies are those complex values of ω for which the continued fractions converge. He then uses the convergence of the continued fraction as the basis for a precise and stable computation scheme for the QN frequencies.

Clearly this method could be applied only to the RW equation. Since the numerator in the Zerilli potential contains four different powers of r, a power-series solution for Eq. (1) would entail at least a four-term recursion relation of coefficients. Had only the Zerilli equation been available, therefore, efficient semianalytic methods could not have been brought to bear. This is practical motivation for understanding the special features of the Zerilli equation which allow it to be transformed to a simpler equation. If we had had a physical derivation only for the Zerilli equation, should we have been able to transform it to the much simpler RW? We might very well ask how much further such simplification can be pushed. That is, how can we be sure that the RW equation is the simplest representation of perturbation dynamics? Perhaps it can be shown to be equivalent to a yet simpler equation, one which yields to the familiar techniques of two-term recursion relations, and for which the computation of QN frequencies and other aspects of perturbation dynamics would be rather straightforward.

An understanding of the simplifiability of equivalent potential equations is, finally, quite relevant to a very recent problem. Leaver[17] has recently approached, in the same spirit as his previous work, the computation of the QN frequencies of the Reissner-Nordström black holes. This has required taming four-term recursion relations for the coefficients occurring in the power-series representations.

The difficulties of dealing with the 3-term and 4-term recursion relations provides ample motivation for searching for an equivalent problem with a simpler potential. As a general question about equations with potentials, this is a crucial question, but for perturbations of spherically symmetric holes, physical intuition suggests that such a search is futile. Both the RW equation for Schwarzschild and the odd-parity Reissner-Nordström equation used by Leaver have as their only singular points, points with physical significance. In the case of RW, these points are $r = 0$, 1, and ∞ (the curvature singularity, the horizon, and spatial infinity). In the case of the odd-parity Reissner-Nordström equation, the points are at $r = 0$, r_-, r_+, and ∞, where $r_\pm$ are the inner and outer horizon locations.

As will be seen in the details that follow, simplification is possible only by removing singularities. Removal of any of the physically meaningful singularities seems unlikely. Nevertheless, as a reinforcement of this intuition, it would be comforting to have a proof that no simplification is possible. If no simplification is possible (and this indeed does turn out to be the case), it is still of interest as a matter of principle, if not computational pragmatism, to understand what the conditions are that allow us, e.g., to make the RW equation worse, i.e., to introduce the equivalent Zerilli equation.

The outline of the remaining sections of this paper is as follows. In Sec. II A the "intertwining operator"[13] is introduced and is related to Chandrasekhar's formalism for equivalent potentials. A criterion for analytic simplicity is introduced in Sec. II B, and it is shown in Secs. II C and II D that there is a straightforward algorithm for testing whether a potential-form problem is simplifiable. In Sec. III this algorithm is applied to the Schwarzschild and the Reissner-Nordström cases, and the inapplicability of the method, in its present form, to the Kerr case is explained. Conclusions are presented in Sec. IV.

II. MATHEMATICAL APPROACH

A. Intertwining of potentials

The intertwining operator D of Anderson[13] is an operator which changes a differential operator L to another operator $\overline{L}$ according to

$$DL = \overline{L}D. \tag{8}$$

Clearly, any eigenfunction ψ of L corresponds to an eigenfunction $\overline{\psi}$ of $\overline{L}$ according to $\overline{\psi} = D\psi$. (By "eigenfunction," here we mean a formal solution of $L\psi = \lambda\psi$ without regard to boundary conditions or other auxiliary conditions.) The two operators L and $\overline{L}$ will therefore be isospectral (aside from issues of boundary conditions).

The converse question, i.e., to what extent isospectral operators are related by an intertwining operator, is more subtle. We will be interested in particular in "first-order" intertwinings in which D involves only first derivatives. Limitation to first-order operators is based on two considerations: (i) a higher-order differential operator can be decomposed into a product of first-order operators; (ii) the relationship of the RW and Zerilli equations, and those for different parity equations for the Reissner-Nordström problem, turn out to be given by first-order intertwinings.

In the context of potential-form equations, the search for an intertwining amounts to the following question: Given an original potential $V_{\rm orig}(r)$, can an intertwining operator $d/dx - g(r)$ be found which relates the potential to a new potential $V_{\rm new}(r)$ via the transformation

$$\left(\frac{d^2}{dx^2} + \omega^2 - V_{\rm new}(r)\right)\left(\frac{d}{dx} - g(r)\right) = \left(\frac{d}{dx} - g(r)\right)\left(\frac{d^2}{dx^2} + \omega^2 - V_{\rm orig}(r)\right), \tag{9}$$

where x and r are related by

$$\frac{d}{dx} = h(r)\frac{d}{dr}, \tag{10}$$

a generalization of Eq. (2)? It is straightforward to verify that $V_{\rm new}$ and $V_{\rm orig}$ are related by

$$V_{\rm new} - V_{\rm orig} = -2\frac{dg}{dx}, \tag{11}$$

and that the condition for an intertwining to exist is

$$g = \frac{d\psi}{dx}\Big/\psi, \tag{12}$$

where ψ is a solution of the original potential-form equation, for any specific eigenvalue ω^2.

These results are equivalent to Chandrasekhar's sufficient conditions Eq. (7) for equivalent potentials. To see this we note that if ψ satisfies $[d^2/dx^2 + (\omega^2 - V_{\rm orig})]\psi = 0$, then g, by Eq. (12), must satisfy the Riccati equation

$$V_{\rm orig} = \frac{dg}{dx} + g^2 + \omega^2. \tag{13}$$

We can then define f by

$$g \equiv -\beta f + i\omega, \tag{14}$$

where β is any constant, and find that f must satisfy

$$V_{\rm orig} = -\beta\frac{df}{dx} + \beta^2 f^2 + \kappa f, \tag{15}$$

where $\kappa \equiv -2i\omega\beta$. In terms of f, Eq. (11) becomes

$$V_{\rm new} = V_{\rm orig} + 2\beta\frac{df}{dx}. \tag{16}$$

With the change in notation $V_{\rm orig} \equiv V^{(-)}$ and $V_{\rm new} \equiv V^{(+)}$, the intertwining results Eqs. (15) and (16), in terms of f, become precisely Chandrasekhar's condition Eq. (7).

The above results, whether in terms of f or g, show that it is possible to use any solution to the original eigenvalue equation as the basis for introducing a new potential. Most new potentials, however, will be highly complicated transcendental functions and of little interest in the present context, where we are concerned only with relating simple potentials. To proceed in this direction we must first specify what is meant by a "simple" potential.

B. Classification of singularities

In general, potential-form equations will allow analytic approaches only if the coefficients of the differential equation are rational functions. We focus, therefore, on the case that $V_{\rm orig}$, $V_{\rm new}$, and $h(r)$ are rational functions of r. The "simpler" these rational functions are, the simpler will be the application of analytic methods. A direct measure of this simplicity might be the number of terms in the recursion relation for coefficients of a power-series expansion. A closely related, but more useful, criterion for simplicity is provided by the number of singular points of the differential equation, and their type, i.e., whether they are regular singular or irregular singular points. (A categorization of irregular singular points is possible by considering them to result from the confluence of "elementary" regular singular points,[18] but such a classification scheme will not be needed here.) We note, for example, that the RW equation has two regular singular points, at $r = 0$ and $r = 1$, and an irregular singular point at $r = \infty$. The Zerilli equation has these same singular points with the addition of a regular singular point at $r = -3/2n$.

Whether a potential-form equation can be simplified will be tantamount to asking whether intertwining can decrease the number or intensity of singularities. This question turns out to be tractable; the requirement that the potentials be rational functions immediately greatly constrains the intertwining function g. It will be shown below that with this constraint the problem can be reduced to that of finding a polynomial solution of the differential equation in a standard form.

C. Standard form

For definiteness, and simplicity of description, we narrow our consideration to potential-form equations which are—in the sense to be defined—similar to the RW, Zerilli, and RN equations. It will be clear that the underlying ideas—and the scheme for finding whether equivalent potentials exist—are rather more generally applicable.

We consider potential-form equations

$$\frac{d^2\psi}{dx^2} + (\omega^2 - V)\psi = 0, \quad \frac{d}{dx} \equiv h(r)\frac{d}{dr}, \tag{17}$$

with the following constraints.

(i) We require that $h(r)$ is a rational function of the form

$$h(r) = \frac{p_A}{p_B}, \tag{18}$$

where p_A and p_B are polynomials. For perturbation type equations, x and r should agree as $r \to \infty$, so we require that p_A, p_B are polynomials of the same order.

(ii) We require that $V(r)$ be a rational function, asymptotically of order r^{-2}.

(iii) We require that the resulting differential equation has no irregular singular points in the finite r plane. This means that V has, at worst, second-order poles, and this constrains the form of p_A, p_B.

Our approach now involves reducing all such equations to a standard form. The first step in this procedure is the simple transformation

$$\psi = e^{i\omega x}\Psi, \tag{19}$$

to put the equation into the form

$$\frac{d^2\Psi}{dx^2} + 2i\omega\frac{d\Psi}{dx} - V\Psi = 0. \tag{20}$$

As examples, the RW equation, in terms of the r variable [see Eq. (2)], is

$$\frac{d^2\Psi}{dr^2} + \frac{2i\omega r + 1/r}{r-1}\frac{d\Psi}{dr} - \frac{2(n+1)r - 3}{r^2(r-1)}\Psi = 0, \tag{21}$$

and the Zerilli equation is

$$\frac{d^2\Psi}{dr^2} + \frac{2i\omega r + 1/r}{r-1}\frac{d\Psi}{dr} - \frac{2(n+1)r^3 + 3r^2 + 9r/2n + 9/4n^2}{r^2(r-1)(r+3/2n)^2}\Psi = 0. \tag{22}$$

The nature of differential equations, and their solutions, near regular singular points may be characterized by the indices at the regular singular points. If near the point $r = r_0$ a solution can behave as $(r - r_0)^s$, then s is an index at r_0. In general, there are two indices at each regular singular point. [It is possible that the two indices are identical or differ by an integer; in this case the solution can have the behavior $(r - r_0)^s \log(r - r_0)$ or $(r - r_0)^s$.] It is straightforward to show that for the RW and Zerilli equations, the indices are as follows.

For RW	at $r = 0$,	$s = 3,\ -1$,
	at $r = 1$,	$s = 0,\ -2i\omega$;
for Zerilli	at $r = 0$,	$s = 1,\ 1$,
	at $r = 1$,	$s = 0,\ -2i\omega$,
	at $r = -3/2n$,	$s = 2,\ -1$.

To facilitate a systematic study of intertwinings we put Eq. (17) into a standard form in which one of the indices is zero at each regular singular point. If the regular singular points are $r = r_1,\ r_2, \ldots,\ r_N$, with indices s_{1a}, s_{1b} at r_1, and s_{2a}, s_{2b} at r_2, and so forth, then the transformation

$$\Psi = (r - r_1)^{s_{1a}}(r - r_2)^{s_{2a}} \cdots (r - r_N)^{s_{Na}}\Phi \tag{23}$$

leads to a differential equation with the same regular singular points—and with the same sort of irregular singularity at ∞—but with indices $s_1' = 0$, $s_{1b} - s_{1a}$ at $r = r_1$, with indices $s_2' = 0$ and $s_{2b} - s_{2a}$ at $r = r_2$, and so forth. Note that this transformation to an equation having a zero index at each regular singular point can be made in 2^N ways, depending on which index at each regular singular point is shifted to zero. The general equation of this type has the appearance

$$\frac{d^2\Phi}{dr^2} + \left(2i\omega + \frac{A_1}{r-r_1} + \frac{A_2}{r-r_2} + \cdots + \frac{A_N}{r-r_N}\right)\frac{d\Phi}{dr} + \left(\frac{B_1}{r-r_1} + \frac{B_2}{r-r_2} + \cdots + \frac{B_N}{r-r_N}\right)\Phi = 0. \tag{24}$$

D. Intertwining solutions

For Eq. (24), every point in the finite r plane is regular except $r = r_1,\ r_2, \ldots,\ r_N$, and the allowed form of the solution at these regular singular points is proscribed, e.g., near r_1 the solution must behave as $\Phi \to$ const, or $\Phi \to (r - r_1)^{s_{1a} - s_{1b}}$, or $\Phi \to (r - r_1)^{s_{1a} - s_{1b}} \log(r - r_1)$. By the appropriate choice of which indices are set to zero, we can always arrange to have the solution of interest be that for which $\Phi \to$ const near $r = r_1,\ r_2, \ldots,\ r_N$. For this choice Φ must be an entire function, that is, a function regular everywhere in the finite r plane.

We now recall that any original and new potentials are related by

$$V_{\rm new} - V_{\rm orig} = -2\frac{dg}{dx} = -2\frac{p_A}{p_B}\frac{dg}{dr}. \tag{25}$$

From Eqs. (12), (18), (19), and (23), we find

$$g = \frac{1}{\psi}\frac{d\psi}{dx} = i\omega + \frac{p_A}{p_B}\left(\frac{s_{1a}}{r-r_1} + \frac{s_{2a}}{r-r_2} + \cdots + \frac{s_{Na}}{r-r_N} + \frac{1}{\Phi}\frac{d\Phi}{dr}\right). \tag{26}$$

The condition that $V_{\rm new} - V_{\rm orig}$ is a rational function, asymptotically of order r^{-2}, requires that, for $r \to \infty$

$$\frac{d}{dr}\left(\frac{1}{\Phi}\frac{d\Phi}{dr}\right) = \text{rational function} \to \frac{1}{r^2}. \tag{27}$$

The general form of Φ that satisfies this requirement is, at $r \to \infty$,

$$\Phi = Ke^{kr}\left[r^M \exp\left(\frac{\alpha}{r} + \frac{\beta}{r^2} + \cdots\right)\right], \quad (28)$$

where M must be an integer, since $r = \infty$ cannot be a branch point. It follows that the function in square brackets does not have an essential singularity and therefore must be a polynomial (the only entire function without an essential singularity at ∞). If the solution form in Eq. (28) is tried in Eq. (24), it is immediately clear from the dominant terms at large r that either $k = 0$ or $k = -2i\omega$. It is straightforward to show that the latter case corresponds to replacing $e^{i\omega x}$ by $e^{-i\omega x}$ in Eq. (19). We shall consider ω to be an adjustable parameter. Thus a solution for Φ with the asymptotic form $e^{-2i\omega r}\times$polynomial exists if and only if a polynomial solution exists. We need therefore only consider the polynomial case.

We conclude that a solution which accomplishes an intertwining of two rational potentials will be a solution of one of the 2^N equations of the form Eq. (24) in which Φ is a polynomial. The question of whether an intertwining is possible is then simply the question of whether a polynomial solution of Eq. (24) exists. It is easy to see that, in general, such a solution does not exist. If we multiply Eq. (24) by $(r - r_1)\cdots(r - r_N)$ and try as a solution Φ an mth-order polynomial, the structure of the equation becomes

$$p_{m+N-2} + (2i\omega p_{m+N-1} + p_{m+N-2}) + p_{m+N-1} = 0, \quad (29)$$

where p_k denotes a polynomial of order $\leq k$. This polynomial equation of order $m + N - 1$ requires that each of the $m + N$ powers of r in the equation vanish. To make these terms vanish we can choose the value of the free parameters ω and make m choices for the coefficients (modulo an overall scaling factor) of the mth-order polynomial Φ.

Thus, we have $m + 1$ choices and $m + N$ conditions. Aside from the trivial case, $N = 1$, the existence of an intertwining between two rational potentials depends on coincidences in the coefficients. In the standard form, checking whether such a coincidence occurs is straightforward.

III. EXAMPLES

A. Schwarzschild perturbations

With the indices $s = -1,\ 3$ at $r = 0$ shifted to $s = 0,\ 4$, the RW equation takes the standard form

$$\frac{d^2\Phi}{dr^2} + \left(2i\omega + \frac{2i\omega + 1}{r-1} - \frac{3}{r}\right)\frac{d\Phi}{dr} + \left(\frac{2n}{r} - \frac{2n + 2i\omega}{r-1}\right)\Phi = 0. \quad (30)$$

If we take Φ to be a polynomial of order m, then in Eq. (30) the dominant term at large r is $2i\omega(m-1)r^{m-1}$. Unless $\omega = 0$ it follows that the only possible polynomial form for Φ is a linear solution. Since this is a case of $N = 2$ a single "numerical coincidence" is required for the linear solution to satisfy the equation. That "numerical coincidence" in fact occurs (it can be viewed as the cancellation of the terms proportional to r in the linear solution) and the choice $i\omega = -\frac{2}{3}n(n+1)$ leads to the only polynomial solution of

$$\Phi = 1 + \frac{2}{3}nr. \quad (31)$$

This form of Φ corresponds, by Eq. (23), to

$$\Psi = \frac{1}{r} + \frac{2n}{3} \quad (32)$$

and, according to Eq. (26), to

$$g = i\omega + \frac{r-1}{r}\frac{1}{\Psi}\frac{d\Psi}{dr} = i\omega - \frac{r-1}{r^2(1+\frac{2}{3}nr)}. \quad (33)$$

This value of g gives us, by Eq. (25),

$$\begin{aligned} V_{\text{new}} &= V_{\text{RW}} - 2\frac{r-1}{r}\frac{dg}{dr} \\ &= V_{\text{RW}} + \frac{2(r-1)}{r^4(r+\frac{3}{2n})^2}\left(-\frac{3r^2}{n} + \frac{9(2n-1)r}{4n^2} + \frac{9}{2n^2}\right) \\ &= V_{\text{Z}}\,. \end{aligned} \quad (34)$$

Thus we see that the polynomial in Eq. (31) is precisely the solution that gives the intertwining from the RW potential to the Zerilli potential. (This transformation, and the reverse—from the Zerilli to the RW equations—are given in Ref. 6 and Ref. 7, Sec. 26.) It is the vanishing of Φ at $r = -3/2n$ that induces the regular singular point at $r = -3/2n$ in the Zerilli equation.

If we choose $\omega = 0$ it turns out that there is an infinite number of polynomial solutions of the form

$$\Phi = \sum_{k=4}^{\ell+2} \alpha_k r^k, \quad (35)$$

where ℓ is the multipole index. [These solutions, with behavior $\Psi = r^{-1}\Phi \approx r^3$, near $r = 0$, correspond to the $s = 3$ index of Eq. (21).] The simplest of the solutions is that for the quadrupole

$$\Phi = r^4, \quad g = 3\left(\frac{r-1}{r^2}\right),$$

$$V_{\text{new}} = 3\frac{r-1}{r^4}(4r-5). \quad (36)$$

The new potential has the same form as the RW potential in Eq. (6) and is neither better nor worse for numerical or semianalytical studies. The $\ell = 3$ solution is

$$\Phi = 5r^4 - 6r^5, \quad g = 3\frac{r-1}{r^2}\frac{5-8r}{5-6r}, \tag{37}$$
$$V_{\rm new} = V_{\rm RW} - \frac{6(r-1)}{r^4(6r-5)^2}(-48r^3 + 156r^2 - 155r + 50).$$

This new potential has an added ("unphysical") singularity at $r = \frac{5}{6}$ and has the same analytic structure as the Zerilli potential. (Note, however, that the new potential does not agree with $V_{\rm RW}$ to order r^{-2}, whereas $V_{\rm Z}$ does.) For higher values of ℓ the $\omega = 0$ polynomials add further poles to the potential. (In general there are $\ell - 2$ unphysical singular points added.) We can thus conclude that the RW potential can be interwined with other rational potentials, but not with any potential that is simpler than $V_{\rm RW}$ itself.

The examples above illustrate how a polynomial for Φ tends to add regular singular points to the potential. Only in the case of a single-term polynomial can the addition of singularities be avoided. Cancellations can then lead to the elimination of singularities and simplification of the potential. The Zerilli equation, put into standard form, shows how this can come about:

$$\frac{d^2\Phi}{dr^2} + \left(2i\omega + \frac{1}{r} - \frac{2}{r+3/2n} + \frac{2i\omega+1}{r-1}\right)\frac{d\Phi}{dr} + \frac{2n(n+1) - 3i\omega}{n+3/2}\left(\frac{1}{r+3/2n} - \frac{1}{r-1}\right)\Phi = 0. \tag{38}$$

With the choice $3i\omega = 2n(n+1)$, clearly $\Phi = $ const is a solution, and indeed the solution intertwines the Zerilli potential and the RW potential.

B. Reissner-Nordström perturbations

Perturbations of the Reissner-Nordström black-hole spacetime have both gravitational and electromagnetic degrees of freedom, and their description therefore requires two wave functions. For the odd-parity case two functions $\psi_1^{(-)}$ and $\psi_2^{(-)}$ can be found—each describing a different mixture of gravitational and electromagnetic perturbations—which satisfy decoupled potential form equations [see Refs. 8 and 9, and Ref. 7, Sec. 42, Eqs. (150), (151)]

$$\frac{d^2\psi_i^{(-)}}{dx^2} + \left(\omega^2 - V_i^{(-)}\right)\psi_i^{(-)} = 0. \tag{39}$$

Here r and x are related by

$$\frac{dr}{dx} = \frac{\Delta}{r^2} = (r-r_-)(r-r_+)/r^2 = 1 - r^{-1} + Q^2/r^2\,, \tag{40}$$

where r_-, r_+ are the radii for the inner and outer horizons, and Q is the electric charge of the hole. As in the Schwarzschild case we use units in which $2M = 1$. The potentials are

$$V_i^{(-)} = \Delta\left[2(n+1)r^2 - q_j r + 4Q^2\right]/r^6\,, \tag{41}$$

where

$$q_1 = (3 + \sqrt{9 + 32nQ^2})/2\,,$$
$$q_2 = (3 - \sqrt{9 + 32nQ^2})/2\,. \tag{42}$$

In Eq. (41) we follow the notation of Ref. 7, Secs. 43 and 44, in using $i, j = 1, 2$ with $i \neq j$ so that, for example, q_2 appears in the equation above defining $V_1^{(-)}$.

With the transformation $\psi_i^{(-)} = e^{i\omega x}\Psi_i^{(-)}$, we find

$$\frac{d^2\Psi_i^{(-)}}{dr^2} + \left[\frac{2i\omega r^2}{(r-r_+)(r-r_-)} + \frac{1}{r}\left(\frac{r_-}{r-r_-} + \frac{r_+}{r-r_+}\right)\right]\frac{d\Psi_i^{(-)}}{dr} - \frac{2(n+1)r^2 - q_j r + 4Q^2}{r^2(r-r_-)(r-r_+)}\Psi_i^{(-)} = 0. \tag{43}$$

The indices for this equation are

at $r = r_+$, $s = 0$, $-2i\omega r_+^2/(r_+ - r_-)$,
at $r = r_-$, $s = 0$, $-2i\omega r_-^2/(r_- - r_+)$,
at $r = 0$, $s = -1$, 4.

Finally, the transformation $\Psi_i^{(-)} = r^{-1}\Phi_i^{(-)}$ brings the equations into standard form

$$\frac{d^2\Phi_i^{(-)}}{dr^2} + \left[\frac{2i\omega r^2}{(r-r_+)(r-r_-)} - \frac{4}{r} + \frac{1}{(r-r_+)} + \frac{1}{(r-r_-)}\right]\frac{d\Phi_i^{(-)}}{dr} + \left[-\frac{2i\omega r}{(r-r_+)(r-r_-)} - \frac{2nr + 3 - q_j}{r(r-r_+)(r-r_-)}\right]\Phi_i^{(-)} = 0. \tag{44}$$

If $\Phi_i^{(-)}$ is a polynomial of order m the dominant term as $r \to \infty$ gives $2i\omega(m-1)r^{m-1} = 0$. It follows that (unless $\omega = 0$) m must be unity, and the only possible polynomial solution must have the form $\Phi_i^{(-)} = 1 + k_i r$. To see whether such a solution exists we multiply Eq. (44) by $r(r-r_+)(r-r_-)$, substitute $1 + k_i r$ for $\Phi_i^{(-)}$, and set all powers of r to zero. The terms of order r^2, r^1, and r^0 give, respectively,

$$i\omega + (n+1)k_i = 0, \tag{45}$$

$$q_j k_i - 2n = 0, \tag{46}$$

$$q_j - 3 - 4k_i Q^2 = 0. \tag{47}$$

The first two of these are satisfied if we take

$$k_i = 2n/q_j\,, \tag{48}$$

$$i\omega = -(n+1)k_i = -2n(n+1)/q_j\,. \tag{49}$$

The third equation is compatible with the first two if and only if

$$q_j^2 - 3q_j - 8nQ^2 = 0, \tag{50}$$

which is, in fact, satisfied by both q_1 and q_2 in Eq. (42).

With the solution $\Phi_i^{(-)} = 1 + k_i r$ we can generate an intertwining operator, of the required type, in which

$$\begin{aligned} g_i &= \left(e^{i\omega x} r^{-1}\Phi_i^{(-)}\right)^{-1} \frac{d}{dx}\left(e^{i\omega x} r^{-1}\Phi_i^{(-)}\right) \\ &= i\omega + \frac{\Delta}{r^2}\left(\frac{1}{r}+k_i\right)^{-1}\frac{d}{dr}\left(\frac{1}{r}+k_i\right) \\ &= i\omega - \frac{\Delta}{r^3(k_i r+1)} = -\frac{2n(n+1)}{q_j} - \frac{\Delta q_j}{r^3(2nr+q_j)}\,. \end{aligned} \tag{51}$$

This form of g_i can be used to find the new potentials

$$V_i^{(+)} = V_i^{(-)} - 2\frac{\Delta}{r^2}\frac{dg_i}{dr}\,. \tag{52}$$

The resulting potentials $V_1^{(+)}$ and $V_2^{(+)}$ turn out to be the potentials that arise in the description of even-parity perturbations. (The even-parity formalism is described in Refs. 8, 10, and Ref. 7, Sec. 42. Transformations between $V_1^{(+)}$ and $V_2^{(+)}$ are given in Ref. 11 and in Ref. 7, Sec. 43.) These even-parity potentials $V_i^{(+)}$ each contain an extra, unphysical, singularity at $r = -q_j/2n$ and are therefore less convenient for analytic approaches than are the equivalent odd-parity potentials $V_i^{(-)}$.

As was the case in the RW analysis, choosing $\omega = 0$ leads to polynomial solutions. Equation (44), with $\omega = 0$, has solutions which are polynomials of order $\ell + 2$. Unlike the RW case, these polynomials contain all terms of lower order. The simplest polynomial, that for $\ell = 2$, will be a quartic with four distinct (unphysical) roots. This polynomial intertwines Eq. (44) with an equation having these roots as additional regular singular points. Similarly the $\ell = 3$ polynomial would lead to an equation with five unphysical regular singular points, and so forth. As in the RW case these solutions are of interest for their existence, but are unlikely to simplify the analysis of perturbation dynamics. As physical intuition would suggest, there are no potentials for the Reissner-Nordström problem simpler than the potentials $V_i^{(-)}$, which contain only singularities with a physical origin.

C. Kerr perturbations

Chandrasekhar and Detweiler have shown[19] that there are four potential-form equations analogous to the equations for even- and odd-parity perturbations of Schwarzschild or Reissner-Nordström. One might hope to show that these four equations are related by the analysis presented above. There are two distinct reasons that this is not possible.

First, the potentials fall outside of the class considered in Sec. II C. The explicit forms of the potentials are rational (and quite complicated) but contain fourth-order poles at $r^2 + a^2 = 0$, where a is the specific angular momentum of the black hole. These represent irregular singular points of the potential form equation. The analysis presented above, which assumed that the equation had no irregular singular points in the finite r plane, is therefore inapplicable.

The second difficulty with the Kerr equations is that the potentials are frequency (eigenvalue) dependent. In the Kerr equations the constant analogous to n, of the Schwarzschild and Reissner-Nordström potentials, is

$$\nu = E - 2a^2\sigma^2, \tag{53}$$

where σ^2 is the eigenvalue and E is a frequency-dependent separation constant. The consequence of this is that there cannot be a frequency-independent intertwining operator which transforms the spectrum in one Kerr potential to that in another. An operator is needed which transforms between particular solutions for each value of σ^2. Such an operator is exhibited by Chandrasekhar and Detweiler,[19] but constitutes an approach different from the intertwining formalism discussed above.

IV. OBSERVATIONS AND CONCLUSIONS

In the metric perturbation formalism for black holes, very different effective potentials appear for even-parity perturbations and for odd-parity perturbations. This was initially puzzling since the two potentials gave the same spectrum of quasinormal modes. Chandrasekhar, in particular, has emphasized that the equivalence of the two potentials is required by the existence of a different description of black-hole perturbations—that based on the Newman-Penrose formalism—in which there is no distinction between even and odd parity. (See Ref. 7, especially Sec. 33.) Chandrasekhar pointed out, furthermore, that the potentials appear to be special, having the form of Eq. (7) which is a sufficient condition for them to be physically equivalent.

One result of the present paper is to suggest a slightly different viewpoint on the relations of these equations. We show that any potential can be "intertwined" with a multitude of other potentials. More specifically: For any potential there exists an infinite number of other potentials which are physically equivalent [and which satisfy Eq. (7) above]. Furthermore, these alternative potentials are easily found; they are generated by solutions to the original potential equation.

This all suggests that the appearance of two different but equivalent potentials for black-hole perturbations was not particularly exotic and did not require a unique set of circumstances. With the clarity of hindsight we might even say that the mathematical generation of two different potentials was to be expected. The relationship of the metric perturbations to the Weyl tensor, and (in the Reissner-Nordström case) the relationship of perturbations of the electromagnetic field tensor and of the stress-energy tensor, lead to different patterns of radial differentiation in the even and the odd parity cases. This is all that is required to produce different appearing, but intertwined, differential equations.

The analysis of the present paper has been concerned almost exclusively with relating potentials which are rational functions. That rational potentials arose in the Schwarzschild and the Reissner-Nordström case is easily understood; it follows from the fact that the background metric coefficients and electromagnetic field components are rational functions. The major concern of the paper has been to find whether there could be other rational potentials, especially rational functions "simpler" than those occurring in the odd parity equations. The discovery of such a simplification would be of great benefit in numerical work. New, equivalent, rational potentials were found, but not simpler potentials. To a large extent the search for a simpler potential was doomed from the start. The only singular points in the odd parity equations are those corresponding to physical radii [the radii for the central singularity, the horizon(s), spatial infinity]. To eliminate the special status of these points from the mathematical description seems unlikely. But one can imagine, in some different physical context, equations in which the physical significance of the singular points is not so transparent. The search in such a case for a simpler description would not be *a priori* futile, and (in view of the potential benefits) might well be worth the effort. The techniques presented in this paper, which served to verify that there is no simpler description than the odd-parity one, are easily modified to a much broader class of problems.

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