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Relativistic bound-state effects in heavy-meson physics

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By using a QCD relativistic potential model we compute several physical quantities for heavy-light-quark $Q\bar{q}$ mesons: current-particle matrix elements, leptonic decay constants in the limit $m_Q \to \infty$, and the Kobayashi-Maskawa matrix elements V_{bu} and V_{bc} from recent CLEO and ARGUS data on semileptonic inclusive B decays. A comparison with other theoretical approaches is also presented.

I. INTRODUCTION

In the last few years there has been increasing experimental and theoretical interest for heavy-light-quark $Q\bar{q}$ mesons. This interest arises from the discovery of new phenomena, such as the large B^0-$\overline{B^0}$ mixing,[1] from the hope that CP-violating processes may be observed in these systems[2] and from the possibility that effects of new physics are disclosed by anomalous enhancements of the rates of some rare decay channels.[3]

In the standard model the analysis of $Q\bar{q}$ mesonic systems allows light to be shed on fundamental parameters such as the Cabibbo-Kobayashi-Maskawa (CKM) matrix elements. In this respect, an important experimental result has been recently achieved by the CLEO[4] and ARGUS[5] Collaborations, which observed a signal of direct $b \to u$ transitions in B-meson semileptonic decays, thus finding evidence for $V_{bu} \neq 0$. However, any determination of mesonic weak decay amplitudes (and consequently any quantitative evaluation of weak parameters from experimental data) involves the computation of hadronic matrix elements. The evaluation of these matrix elements is the main subject of this paper.

An approximation scheme has been recently put forward,[6,7] based on the observation that, if the heavy-quark mass is sufficiently large, the heavy quark can be treated as a static color source. In this case, the dependence of hadronic amplitudes (e.g., decay constants and form factors) on the mass m_Q can be obtained by scaling arguments. The problematic aspect of this approach is that there is some evidence[8] that the onset of such a scaling behavior is for masses larger than the charm mass; therefore, one cannot meaningfully extrapolate to b mesons the results obtained for charmed mesons by nonperturbative methods, e.g., lattice QCD. We shall come back to this point below. In any case, it is desirable to investigate the region of moderate m_Q (up to the b quark) by different methods, and to compare their predictions: hopefully this should shed some light on the reliability of the approximations involved in different approaches.

The aim of this paper is to propose in greater detail a model[9] which reproduces some peculiar features of the heavy-light meson systems. It is based on the following main assumptions.

(1) The $Q\bar{q}$ system is described by a wave equation with a potential following the prescriptions dictated by QCD, i.e., a linearly confining behavior at large distances and a Coulombic behavior at short distances. As described below, a particular role in determining the complete form of the potential is played by the duality (suggested in the early 1970s by the authors in Ref. 10) between the free quark behavior observed in $\sigma(e^+e^- \to \text{hadrons})$ at high energies and the production of an infinite number of resonances in the process $e^+e^- \to \gamma^* \to V_n \to \text{hadrons}$.

(2) Since the $Q\bar{q}$ system contains a relativistic light quark, the meson is assumed to obey a wave equation with relativistic kinematics, whose solution provides information on both the spectrum and the wave function of heavy-light-quark mesons.

The plan of the paper is as follows. Section II contains the description of the relativistic QCD potential model with details about the computational technique adopted to solve the mesonic wave equation. We evaluate a number of current-particle matrix elements and compare our findings with the results obtained by different approaches.

In Sec. III the behavior of the pseudoscalar decay constants for increasing heavy quark mass m_Q is studied. A comparison is also carried out with the results of the static quark effective theory proposed in Ref. 6.

In Sec. IV the B-meson wave function and the CLEO results on the semileptonic B decay spectrum are employed in order to evaluate the CKM matrix element V_{bc} and the ratio V_{bu}/V_{bc}. Finally in Sec. V we draw our conclusions.

II. REVIEW OF THE MODEL

In this section we will describe the main aspects of our potential model. To study relativistic effects in $Q\bar{q}$ systems we start by writing the pseudoscalar $P(J^P=0^-)$ and vector mesons $V(J^P=1^-)$ in their rest frame:

$$|P\rangle=\sum Q_{ij}\frac{\delta_{\alpha\beta}}{\sqrt{3}}\int d^3\mathbf{k}\frac{\delta_{rs}}{\sqrt{2}}\tilde{\Phi}_0(|\mathbf{k}|)b_i^\dagger(\mathbf{k},r,\alpha)d_j^\dagger(-\mathbf{k},s,\beta)|0\rangle\ , \tag{2.1}$$

$$|V\rangle=\sum Q_{ij}\frac{\delta_{\alpha\beta}}{\sqrt{3}}\int d^3\mathbf{k}\frac{(\boldsymbol{\sigma}\cdot\boldsymbol{\epsilon})_{rs}}{\sqrt{2}}\tilde{\Phi}_0(|\mathbf{k}|)b_i^\dagger(\mathbf{k},r,\alpha)d_j^\dagger(-\mathbf{k},s,\beta)|0\rangle\ , \tag{2.2}$$

where the sums are over the indices α and β (color indices), r and s (spin indices), i and j (flavor indices). $b^\dagger$ and $d^\dagger$ are quark and antiquark creation operators and ϵ is the V-meson polarization. The momentum-space $s\,(l=0)$ wave function $\tilde{\Phi}_0(k)$ is defined as

$$\tilde{\Phi}_0(k)=\pi\sqrt{2}\frac{\tilde{u}_0(k)}{k}=\frac{\pi\sqrt{2}}{k}\int_0^\infty dr\,\sin(kr)u_0(r)\ ; \tag{2.3}$$

$\tilde{u}_0(k)$ is normalized according to

$$\int_0^{+\infty}dk|\tilde{u}_0(k)|^2=2M\ , \tag{2.4}$$

where M is the meson mass.

As for the r-space wave function $u_0(r)$, we assume that it is the s-wave solution of the eigenvalue equation

$$[\sqrt{-\hbar^2\nabla^2+m_i^2}+\sqrt{-\hbar^2\nabla^2+m_j^2}+V(\mathbf{r})]\Psi(\mathbf{r})=M\Psi(\mathbf{r})\ . \tag{2.5}$$

For the central potential, $V(\mathbf{r})=V(r)$ the angular dependence in $\Psi(\mathbf{r})$ can be factorized:

$$\Psi(\mathbf{r})=Y_{lm}(\hat{\mathbf{r}})\phi_l(r)=Y_{lm}(\hat{\mathbf{r}})\frac{u_l(r)}{r}\ . \tag{2.6}$$

By using the spectral representation of the square-root operator appearing in Eq. (2.5) we obtain the equation ($\hbar=1$)

$$[V(r)-M]u_l(r)+\frac{2}{\pi}\int_0^\infty dr'\int_0^\infty dk[(k^2+m_i^2)^{1/2}+(k^2+m_j^2)^{1/2}]k^2rr'j_l(kr)j_l(kr')u_l(r')=0 \tag{2.7}$$

where $j_l(x)$ are spherical Bessel functions; for $l=0[j_0(x)=(\sin x)/x]$ we thus obtain the s-wave equation satisfied by $u_0(r)$.

Equation (2.5) is the natural relativistic extension of the Schrödinger equation employed in quarkonium phenomenology;[11] it should be useful for systems containing one or two light quarks, when the nonrelativistic approximation is doubtful; as such it has been considered by a number of authors.[12] It should be also stressed that Eq. (2.5) arises from the Bethe-Salpeter equation in QCD replacing the full interaction by the instantaneous local potential $V(r)$ and considering a limited Fock space containing $q\bar{q}$ states only.[13,14]

Let us now discuss $V(r)$. We shall approximate it by the Richardson potential[15]

$$V(r)=\frac{8\pi}{33-2n_f}\Lambda\left[\Lambda r-\frac{f(\Lambda r)}{\Lambda r}\right] \tag{2.8}$$

with Λ a parameter, n_f is the number of flavors[16] and

$$f(t)=\frac{4}{\pi}\int_0^\infty dq\,\frac{\sin(qt)}{q}\left[\frac{1}{\ln(1+q^2)}-\frac{1}{q^2}\right]\ . \tag{2.9}$$

This potential behaves, at small distances, as expected from perturbative QCD:

$$V(r)\to-\frac{8\pi}{33-2n_f}\frac{1}{r\,\ln(1/\Lambda r)} \tag{2.10}$$

whereas, at large distances, it grows linearly, thus providing confinement. In the potential (2.8) and (2.9), already applied to quarkonium phenomenology, spin-spin and spin-orbit terms are not included. The reason is that in heavy-light-quark systems the experimental mass splitting between different spin states is typically of the order of 100 MeV, to be compared to meson masses larger than 1.8 GeV; therefore spin-dependent terms in the potential can be considered as higher-order corrections.

When one introduces the relativistic kinematics by Eq. (2.5) an important difference arises with respect to the usual approach based on the Schrödinger equation. The Coulombic divergence for $r\to 0$ in Eq. (2.10) is harmless in the nonrelativistic case; on the other hand, when Eqs. (2.5)–(2.7) are considered, it reflects in a logarithmic divergence of the wave function $\phi_0(r)$ at the origin.[17] Such a divergence is an unphysical artifact of the approximation employed in the Bethe-Salpeter equation. A possible way to deal with this problem is to smooth the potential for very small r ($r\le k/M$) to a constant value:

$$V(r)=V(r_M)\ ,\quad r\le r_M=\frac{k}{M}\ . \tag{2.11}$$

The value of the constant k can be fixed by QCD duality, i.e., by assuming that summing over infinitely many resonances, bounded by the potential (2.8)–(2.11), is equivalent to the QCD $O(\alpha_s)$ calculation. In this way one finds[18]

$$k=\frac{4\pi}{3}\ . \tag{2.12}$$

We assume Eq. (2.12) valid for the equal-mass case. For $m_i\neq m_j$ we may expect deviations from Eq. (2.12) and write

$$r_M=\frac{\lambda k}{M}\ , \tag{2.13}$$

with λ a parameter to be fitted.

Having fixed the potential we can now solve the eigenvalue equation (2.7) for the $l=0$ case. This is done numerically by the Multhopp method,[19] which we shall now briefly discuss.

Let us consider Eq. (2.7) for $l=0$. The term

$$H_j=\frac{2}{\pi}\int_0^\infty dr'\int_0^\infty dk\,\sqrt{k^2+m_j^2}\sin(kr)\sin(kr')u_0(r') \tag{2.14}$$

under the substitution

$$k=m_j\sinh(x) \tag{2.15}$$

can be rewritten as

$$H_j=\frac{m_j^2}{2\pi}\int_0^\infty dr'\,u_0(r')J_j(r') \tag{2.16}$$

with

$$J_j(r')=\int_0^\infty dx\,[1+\cosh 2x][\cos(m_j|r-r'|\sinh x)-\cos(m_j|r+r'|\sinh x)]\ . \tag{2.17}$$

Introducing the modified Bessel function[20]

$$K_\nu=\frac{1}{\cos(\nu\pi/2)}\int_0^\infty dt\,\cosh(\nu t)\cos(x\,\sinh t) \tag{2.18}$$

that satisfies the recurrency condition

$$zK_{\nu-1}(z)-zK_{\nu+1}(z)=-2\nu K_\nu(z) \tag{2.19}$$

we get

$$J_j=-2\left[\frac{K_1(m_j|r-r'|)}{m_j|r-r'|}-\frac{K_1(m_j|r+r'|)}{m_j|r+r'|}\right] \tag{2.20}$$

so that, by defining $u_0(-r)=-u_0(r)$, we obtain

$$H_j=-\frac{1}{\pi}P\int_{-\infty}^{+\infty}dr'\frac{u_0(r')}{(r-r')^2}+\frac{m_j}{\pi}\int_{-\infty}^{+\infty}dr'\frac{u_0(r')}{|r-r'|}\left[\frac{1}{m_j|r-r'|}-K_1(m_j|r-r'|)\right]\ , \tag{2.21}$$

where P denotes the principal value. It follows that the s-wave radial equation takes the form

$$-\frac{2}{\pi}P\int_{-\infty}^{+\infty}dr'\frac{u_0(r')}{(r-r')^2}+\left[\frac{m_j}{\pi}\int_{-\infty}^{+\infty}dr'\frac{u_0(r')}{|r-r'|}\left[\frac{1}{m_j|r-r'|}-K_1(m_j|r-r'|)\right]+(m_j\to m_i)\right]+[V(r)-M]u_0(r)=0\ . \tag{2.22}$$

The Multhopp method is particularly useful for dealing with singular integral equations of this type. One puts

$$r=-\cot\vartheta \tag{2.23}$$

so that Eq. (2.22) assumes the form

$$\int_0^\pi d\vartheta'\,K(\vartheta,\vartheta')\psi(\vartheta')=M\psi(\vartheta) \tag{2.24}$$

with the conditions $\psi(0)=\psi(\pi)=0$. Let us consider the truncated Fourier series

$$\psi(\vartheta)\simeq\sum_{j=1}^{N}c_j\sin(j\vartheta)\ ; \tag{2.25}$$

by choosing the (Multhopp) angles

$$\vartheta_l=\frac{l}{N+1}\pi\quad(l=1,\dots,N)\ , \tag{2.26}$$

the coefficients c_j are given by

$$c_j=\frac{2}{N+1}\sum_{l=1}^{N}\sin(j\vartheta_l)\psi(\vartheta_l)\ . \tag{2.27}$$

By substituting Eq. (2.25) in Eq. (2.24) and using Eq. (2.27) we get

$$\sum_{m=1}^{N}\frac{2}{N+1}\sum_{j=1}^{N}\left[\int_0^\pi d\vartheta'\,K(\vartheta_l,\vartheta')\sin(j\vartheta')\right]\sin(j\vartheta_m)\psi(\vartheta_m)=M\psi(\vartheta_l)\ . \tag{2.28}$$

In other terms the problem of solving the original equation (2.22) has been recast in the problem of finding the solution of the system

$$\sum_{m=1}^{N}B_{lm}\psi(\vartheta_m)=M\psi(\vartheta_l)\quad(l=1,\dots,N) \tag{2.29}$$

that, when solved, gives both M and the wave function ψ in N points.

As discussed in Ref. 9, from a fit to $Q\bar{q}$ and $Q\bar{Q}$ meson spectra, we fix the values of the parameters as follows: $m_u=m_d=38$ MeV; $m_s=115$ MeV; $m_c=1452$ MeV; $m_b=4890$ MeV; $\Lambda=397$ MeV; $\lambda=0.6$. The results for the heavy-meson masses can be found in Ref. 9; here we only quote the value for $m(B_c)=6279$ MeV in relative agreement with the values obtained from different potential models.[21] We stress that, since we do not include spin-dependent terms, we are not able to distinguish between pseudoscalar and vector states, and therefore our results are for the center of mass of the $0^- - 1^-$ system. The agreement of our results for the meson spectrum with the experimental data is within 100 MeV, the size of the spin splittings. The light-quark masses are not very well determined from the fit of heavy-meson spectrum, so that we assume the quoted value for $m_u=m_d$ only as an indication of the light-quark-mass value. For our purposes such approximation is justified since the heavy-meson wave functions are not sensitive to m_q within a factor 2 or 3.

Using the $Q\bar{q}$ wave functions a number of current-particle matrix elements can be evaluated. Let us define

$$\langle 0|A^{\mu}_{ij}|P(p)\rangle = ip^{\mu}Q_{ij}\sqrt{2}f_P \ , \tag{2.30a}$$

$$\langle 0|V^{\mu}_{ij}|V(p,\epsilon)\rangle = \epsilon^{\mu}Q_{ij}f_V \ , \tag{2.30b}$$

$$\langle 0|T^{\mu\nu}_{ij}|V(p,\epsilon)\rangle = (\epsilon^{\mu}p^{\nu}-\epsilon^{\nu}p^{\mu})Q_{ij}t_V \ . \tag{2.30c}$$

In Eqs. (2.30) Q_{ij} is the meson flavor matrix; the definition of the currents is $A^{\mu}_{ij}(x)=\bar{q}_i(x)\gamma^{\mu}\gamma^5 q_j(x)$, $V^{\mu}_{ij}(x)=\bar{q}_i(x)\gamma^{\mu}q_j(x)$, $T^{\mu\nu}_{ij}(x)=\bar{q}_i(x)\sigma^{\mu\nu}q_j(x)$, with $\sigma^{\mu\nu}=(i/2)[\gamma^{\mu},\gamma^{\nu}]$. As is well known, the pseudoscalar constant f_P, analogous to the pion constant $f_{\pi}=93.2$ MeV, can be measured in leptonic decays. For $Q\bar{Q}$ systems f_V can be related to the electromagnetic decay constant and can be measured, e.g., in $\Upsilon\to e^+e^-$. The tensor coupling $t(B_s^*)$ has been used by the authors in Ref. 22 in an analysis of the rare decay $B\to K^*\gamma$. By writing the currents in terms of creation and annihilation operators, using canonical commutation relations and evaluating Eqs. (2.30) in the meson rest frame, we get

$$f_P=\left[\frac{3}{2}\right]^{1/2}\frac{1}{2\pi M}\int_0^{+\infty}dk\ \tilde{u}_0(k)kN^{1/2}\left[1-\frac{k^2}{(E_i+m_i)(E_j+m_j)}\right] , \tag{2.31a}$$

$$f_V=\sqrt{3}\frac{1}{2\pi}\int_0^{+\infty}dk\ \tilde{u}_0(k)kN^{1/2}\left[1+\frac{1}{3}\frac{k^2}{(E_i+m_i)(E_j+m_j)}\right] , \tag{2.31b}$$

$$t_V=\sqrt{3}\frac{1}{2\pi M}\int_0^{+\infty}dk\ \tilde{u}_0(k)kN^{1/2}\left[1-\frac{1}{3}\frac{k^2}{(E_i+m_i)(E_j+m_j)}\right] , \tag{2.31c}$$

with

$$N=\frac{(E_j+m_j)(E_i+m_i)}{E_jE_i} \ . \tag{2.32}$$

The results of our model can be found in Tables I and II. We estimate a theoretical uncertainty of $\pm 15\%$. This is obtained by varying the parameters of the model (quark masses, potential parameters) around the quoted values and from the numerical error in the Multhopp method. Let us first discuss the pseudoscalar decay constants. There is a substantial agreement (within 25 MeV) on the value of f_D and f_{D_s} between our model, the lattice QCD approach (with the exception of Ref. 27), QCD sum rules and the nonrelativistic potential model in Ref. 37. All these results are compatible with the present experimental bound $f_D\le 2.2f_{\pi}\simeq 205$ MeV.[23] A similar agreement is found for the value of f_{B_c} (with the exception of Ref. 40); we point out that the B_c system should be well described also by nonrelativistic models.

The situation is different for f_B and f_{B_s}. As a matter of fact, QCD sum rules and nonrelativistic models predict $f_B\le f_D$ ($f_{B_s}\le f_{D_s}$). This is at odds with the result of our model, which predicts $f_B/f_D\simeq 1.25$. We shall give an analytic explanation of this behavior in the next section. Let us only observe that a recent[8] evaluation of f_B in the lattice QCD approach using $1/m_Q$ expansion gives the rather large result $f_B=219\pm 18\pm 35$ MeV. Including an estimate of $1/m_Q$ corrections one finds the result $f_B=164\pm 14\pm 26$ MeV which is in agreement with our findings.

Let us conclude this section with some considerations on the vector meson decay constants. As shown by Table II, there is a substantial agreement between our results and the findings obtained by other methods. The only estimate of $t(B_s^*)$ available up to now[22] is substantially lower than our estimate. Finally we have not included the $Q\bar{Q}$ leptonic decay constants whose calculated values can be found in Ref. 9. As discussed in Ref. 9, the present model gives results in good agreement with the experiment for the leptonic decay constants of vector mesons such as J/ψ, Υ, Υ', etc.

III. THE $m_Q\to\infty$ LIMIT

In this section we study the leptonic decay constants f_P and f_V defined in Eqs. (2.31) in the limit $m_Q\to\infty$. In this limit an approximation scheme[6,7] for $Q\bar{q}$ systems has been proposed by considering the heavy quark as a static color source; in this way one derives a logarithmic

TABLE I. Pseudoscalar decay constants evaluated by different theoretical methods. Units are MeV. Here $f_\pi = 93.2$ MeV. The only experimental value is $f_D \le 2.2 f_\pi$.[23] The results of the present model (first line) have a theoretical uncertainty of $\pm 15\%$.

f_D	f_{D_s}	f_B	f_{B_s}	f_{B_c}	f_{η_c}	f_{η_b}	Ref.
129	141	163	173	306	222	405	Present
		165±13±27					8[a,b]
137±11	152±12						24[a]
123±18±33	165±33±39						25[a]
134±23	157±11						26[a]
210							27[a]
		286±117					28[a]
158±19	196±9	102–149					29[c]
				265	246	414	30[c]
122±11	154±14	129±13	140±14				31[c]
120		92					32[c]
156		99					33[c]
117±11	141±11	81±11					34[c]
141		120±14					35[c]
<136	<164	<170	<204				36[c]
106	148	88	124	301			37[d]
202	252	162	196	303			38[d]
83	91	53	61				39[d]
171	237	123		403			40[d]
170±14	205±14	110±11	148±14	290±14			41[d]
159		138					42[d]
78–104		50–69					43[e]

[a]Lattice QCD.
[b]This value is obtained by including their estimation of nonscaling contributions.
[c]QCD sum rules.
[d]Potential models.
[e]Bag models.

correction to the $1/\sqrt{m_Q}$ behavior of f_P which is expected in a nonrelativistic approach.[33] It seems worthwhile to compare such a result with the behavior predicted by the QCD relativistic potential model described in the previous section.

To begin with, we show the presence of a $(m_q)^{-1/2}$ dependence in f_P. In the limit $m_Q \to \infty, m_q \to 0$, one can put $E_Q = \sqrt{m_Q^2 + k^2} \simeq m_Q$ and $E_q \simeq k$; then Eq. (2.31) can be approximated by

$$f_P \simeq \frac{\sqrt{3}}{2\pi M} \int_0^{+\infty} dk\, k \tilde{u}_0(k) = \frac{\sqrt{3}}{2M} \phi_0(0) \tag{3.1}$$

where $\phi_0(r)$ is defined in Eq. (2.6). In the same approximation the wave equation (2.7) becomes $[u_0(r) = r\phi_0(r)]$

TABLE II. (a) Vector-meson constants. Units are GeV. The results of the present model (first line) have a theoretical uncertainty of $\pm 15\%$. (b) Vector meson couplings t_V [see Eq. (2.30c)]. Units are MeV. The results of the present model (first line) have a theoretical uncertainty of $\pm 15\%$.

(a)

f_{D^*}	$f_{D_s^*}$	f_{B^*}	$f_{B_s^*}$	$f_{B_c^*}$	Ref.
0.53	0.56	1.51	1.58	3.03	Present
0.44	0.47				44
		1.23	1.76		35

(b)

t_{D^*}	$t_{D_s^*}$	t_{B^*}	$t_{B_s^*}$	$t_{B_c^*}$	Ref.
222	237	257	271	458	Present
			161±38		22

$$[V(r)-M+m_Q]u_0(r)-\frac{\hbar^2}{2m_Q}u_0''(r)+\frac{1}{\pi\hbar}\int_0^\infty dk\,k\int_0^\infty dr'\sin\left[\frac{kr}{\hbar}\right]\sin\left[\frac{kr'}{\hbar}\right]u_0(r')=0\ . \tag{3.2}$$

We wish to point out that Eq. (3.2) is considerably simpler than Eq. (2.7) (with $l=0$) as far as the Multhopp method is concerned. This is a consequence of the absence in the right-hand side (RHS) of Eq. (2.21) of the mass terms; from a practical point of view we obtain the same accuracy in the wave function with a gain of an order of magnitude in CPU time.

Let us consider a potential $V(r)$ consisting only of a linear confining part (the Coulombic correction will be discussed later on). In this case, one can prove that $\phi_0(r)\to c$, where the constant c is determined by the normalization condition Eq. (2.4). The value of c can be obtained by the WKB approximation[45] of the solution of Eq. (3.2). We search a solution in the form

$$u_0(r)=\exp\left[\frac{i}{\hbar}\sigma(r)\right]=\exp\left[\frac{i}{\hbar}\sigma_0(r)+\sigma_1(r)\right] \tag{3.3}$$

with the boundary condition $u_0(r)\to\text{const}\times r$. The functions $\sigma(r)$ and $\sigma_1(r)$ are determined by the saddle-point equations

$$\begin{aligned} &r'=r\ ,\\ &k=\sigma'(r)\ , \\ &V(r)-M+g(\sigma')+\frac{\hbar}{2i}g''(\sigma')\sigma''(r)=0\end{aligned} \tag{3.4}$$

with

$$g(\sigma')=m_Q+\frac{\sigma'^2}{2m_Q}+\sigma'\ . \tag{3.5}$$

Developing in $\hbar$ we get $[\delta=M-m_Q, V(r)=\mu^2 r]$

$$\sigma_0'(r)=m_Q\left[-1+\left[1+\frac{2(\delta-\mu^2 r)}{m_Q}\right]^{1/2}\right] \tag{3.6}$$

whereas

$$\sigma_1(r)=\ln\frac{1}{\sqrt{\sigma_0'(r)+m_Q}}+\text{const}\ . \tag{3.7}$$

It follows that the WKB solution, for $r\le r_0=\delta/\mu^2$ (r_0 is the classical turning point) is given by

$$u_0(r)=\frac{D}{\sqrt{m_Q}}\frac{1}{\left[1+\frac{2(\delta-\mu^2 r)}{m_Q}\right]^{1/4}}\sin\Sigma(r) \tag{3.8}$$

with

$$\Sigma(r)=\int_r^{r_0}\sigma'(r)dr+\frac{\pi}{4}\ , \tag{3.9}$$

where we have now put $\hbar=1$. For $r>r_0$, $u_0(r)$ falls down exponentially, as it can be shown by general arguments. From the normalization integral Eq. (2.4), evaluated by the saddle-point methods, we obtain

$$D=\eta m_Q \tag{3.10}$$

where η is a constant independent of m_Q. Therefore

$$\phi_0(r)=u'(0)=\text{const}\times\sqrt{m_Q} \tag{3.11}$$

and

$$f_P=\frac{\text{const}}{\sqrt{m_Q}}\ . \tag{3.12}$$

This behavior has been checked for the numerical solution of Eq. (3.2) without the WKB approximation; it is expected also in nonrelativistic potential models.[33]

By including now in the potential $V(r)$ the short-distance Coulombic term, modified as described in the previous section, one gets a logarithmic correction to Eq. (3.12). Indeed Eq. (3.2) can be written as ($\hbar=1$)

$$[V(r)-M+m_Q]r\phi_0(r)-\frac{1}{2m_Q}\frac{d^2}{dr^2}(r\phi_0(r)) -\frac{1}{\pi}P\int_0^\infty dz\,Q_0\left[\frac{1+z^2}{2z}\right]\frac{\partial^2}{\partial z^2}[z\phi_0(rz)]=0 \tag{3.13}$$

where $Q_l(x)$ is the Legendre function of the second kind. Let us search, for r small and $r>r_M=4\pi/3M$ a solution of Eq. (3.13) in the form

$$\phi_0(r)=\text{const}\times\sqrt{m_Q}\left[\ln\left[\frac{1}{\Lambda r}\right]\right]^{\xi}\ . \tag{3.14}$$

By evaluating Eq. (3.13) for $r=r_M=4\pi/3m_Q$, neglecting terms such as $\ln[\ln(m_Q/\Lambda)]$, the following value for the exponent ξ is obtained:

$$\xi=\frac{16}{33-2n_f}\frac{1}{1+\frac{3}{4\pi^2}}\ . \tag{3.15}$$

We now approximate the value of the wave function in the origin as

$$\phi(0)\simeq\phi_0(r_M)=\phi_0\left[\frac{4\pi}{3m_Q}\right]\ , \tag{3.16}$$

which is justified since the potential $V(r)$ is constant for $r\in(0,r_M)$.

From Eqs. (3.1), (3.14)–(3.16) the behavior of the axial-vector-current decay constant f_P for a $Q\bar{q}$ pseudoscalar meson, in the limit $m_Q\to\infty$, is given by

$$f_P\simeq\frac{\text{const}}{\sqrt{m_Q}}\left[\frac{1}{\ln\left[\frac{m_Q}{\Lambda'}\right]}\right]^{-\xi} \tag{3.17}$$

in the leading-log approximation. The mass scale appearing in Eq. (3.17) is

$$\Lambda' = \frac{4\pi}{3}\Lambda \simeq 1.7 \text{ GeV} . \tag{3.18}$$

Equation (3.17) looks similar to the result that can be obtained[6] by taking the renormalization of the axial-vector current into account:

$$f_P \simeq \frac{\text{const}}{\sqrt{m_Q}}[\alpha_s(m_Q)]^{-d} \simeq \frac{\text{const}}{\sqrt{m_Q}} \left[\frac{1}{\ln \left[\frac{m_Q}{\Lambda_{\rm QCD}} \right]} \right]^{-d} . \tag{3.19}$$

However, there are two important differences. First, the exponent d is given by

$$d = \frac{6}{33 - 2n_f} \tag{3.20}$$

to be compared with Eq. (3.15). The second difference is in the mass scales appearing in the logarithmic terms in Eqs. (3.17) and (3.19). In Eq. (3.19) the mass scale is the QCD parameter $\Lambda_{\rm QCD}$. On the other hand, in the relativistic potential model one gets $\Lambda' \simeq 1.7$ GeV. This rather large value arises because of the interplay between the short- and the long-distance dynamics expressed by the duality between asymptotically free quarks and bound states;[9,18] in other terms, it is a phenomenological consequence of the hypothesis that infinitely many resonances, bounded by the potential Eqs. (2.8)–(2.11) can mimic the short-distance QCD behavior.

Both differences, the exponent of the logarithmic corrections and the mass scale, arise from the different approximations used in the two approaches. We cannot assess here the quality of these approximations; nevertheless it is useful to discuss their phenomenological implications. First of all we observe that Eq. (3.17) holds for very large m_Q, since the relevant mass scale is Λ'. For intermediate regions we numerically obtain a small deviation in the exponent ξ. The second observation is that Eq. (3.17) predicts a maximum for f_P at

$$m_Q = \Lambda' \exp(2\xi) , \tag{3.21}$$

i.e., for $m_Q \simeq m_b$. This is at variance with the prediction of the static quark effective theory, where the logarithmic corrections play a minor role; on the other hand our result is in agreement with the outcome of Ref. 8 obtained by an interpolating formula which shows that $f_P(m_Q)$ has a maximum for $m_Q \simeq 3.6$ GeV, with a decreasing behavior for larger masses.

We conclude this section by observing that a similar analysis can be performed for the vector-meson leptonic decay constant f_V; for $m_Q \to \infty$ one gets

$$f_V \simeq \text{const} \times \sqrt{m_Q} \left[\frac{1}{\ln \left[\frac{m_Q}{\Lambda'} \right]} \right]^{-\xi} \tag{3.22}$$

with ξ and Λ' given in Eqs. (3.15) and (3.18).

IV. INCLUSIVE SEMILEPTONIC B-MESON DECAYS AND KOBAYASHI-MASKAWA MATRIX ELEMENTS

As an application of the model developed in the previous sections we wish to study the determination of the Kobayashi-Maskawa matrix elements V_{bu} and V_{bc} by the inclusive $B \to Xe\nu$ spectrum. Recent experimental data from both CLEO[4] and ARGUS[5] Collaborations have shown evidence for a nonvanishing signal for values of the electron momentum such that, for kinematical reasons, only the $B \to X_u e\nu$ decay is possible. This clearly shows that $V_{bu} \neq 0$; however the actual value of V_{bu} depends on the model used for the fragmentation of the b quark; moreover bound-state effects should be taken into account.

To describe fragmentation, we follow the approach developed by Altarelli *et al.*[46] which does not suffer from the large uncertainties in the form factors that are present in the alternative approaches of Refs. 47–49.

In the approach of Ref. 46 one attempts to compute the differential width $d\Gamma(B \to Xe\nu)/dE$ (E is the electron energy) by using mainly information from QCD. One writes $d\Gamma/dE$ as a sum of two pieces:

$$\frac{d\Gamma}{dE} = |V_{bc}|^2 \frac{d\Gamma_c}{dE} + |V_{bu}|^2 \frac{d\Gamma_u}{dE} , \tag{4.1}$$

where $d\Gamma_q/dE$ arises from the decay $b \to qe\nu (q = u, c)$. The B-meson semileptonic decay is described in terms of the β decay of the b quark, treated as a virtual particle of mass m_b with the condition

$$m_b^2 = M_B^2 + m_{\rm sp}^2 - 2M_B \sqrt{m_{\rm sp}^2 + |\mathbf{p}|^2} , \tag{4.2}$$

where M_B is the B meson mass and $m_{\rm sp}$ and $\mathbf{p}$ are the mass and momentum of the spectator quark. $d\Gamma_q/dE$ is obtained by

$$\frac{d\Gamma_q}{dE} = \int_0^{p_{\rm max}} dp\; \tilde{u}_0^2(p) \frac{d\Gamma_q^{\rm part}}{dE} , \tag{4.3}$$

where $\tilde{u}_0(p)$ is the p-space B wave function defined in Eq. (2.3), $d\Gamma_q^{\rm part}/dE$ is the partonic spectrum for the decay $b \to qe\nu$ and $p_{\rm max}$ is obtained from (4.2) when the RHS of this equation becomes equal to m_q^2. Whereas for $d\Gamma_q^{\rm part}/dE$ we use the results of Ref. 46 [containing also $O(\alpha_s)$ corrections], we differ from these authors in the use of quark momentum distribution $\tilde{u}_0^2(p)$. We employ for $\tilde{u}_0(p)$ our computed wave function, whereas in Ref. 46 a phenomenological Gaussian distribution is assumed. It is worth stressing that the shapes of the two distributions are different, which implies that $d\Gamma_q/dE$ significantly depends on $\tilde{u}_0^2(p)$ (see, e.g., Ref. 50 where a comparison between the present model and the results based on the Gaussian distribution is performed).

To compare the results of this analysis with the experimental data we have also included radiative corrections in $d\Gamma_q/dE$ as follows:[51]

TABLE III. Values of $10^2|V_{bu}/V_{bc}|^2$ in the momentum ranges 2.2–2.4 GeV/c, 2.4–2.6 GeV/c and in average, for two values of the light spectator quark mass.

$m_{\rm sp}$ (MeV)	2.2–2.4 GeV/c $10^2|V_{bu}/V_{bc}|^2$	2.4–2.6 GeV/c $10^2|V_{bu}/V_{bc}|^2$	Average $10^2|V_{bu}/V_{bc}|^2$
38	0.7±0.5	1.7±0.6	1.1±0.4
100	0.7±0.5	1.8±0.6	1.2±0.4

$$\frac{d\Gamma_q^{\rm corr}}{dE}=\frac{d\Gamma_q}{dE}\left[1+\frac{2\alpha}{\pi}\ln\frac{M_Z}{M_B}\right][1+\pi\alpha 0.46] \times\left[\frac{3(1-x)}{2x}\right]^{(2\alpha/\pi)[\ln(xM_B/m_e)-1]} \tag{4.4}$$

where α is the fine-structure constant, M_Z and m_e are the Z^0 and the electron masses respectively and $x=E/E_{\rm max}$ (with $E_{\rm max}=2.31$ GeV for $q=c$ and 2.64 GeV for $q=u$). We have extracted the value of V_{bu} from a comparison with the CLEO data[4] in the charged-lepton momentum range 2.2–2.6 GeV/c. We have also included the experimental momentum resolution which for the CLEO apparatus is given by $[\sigma(p)/p]^2=(0.0023p)^2+(0.007)^2$ where p is in GeV. Our results are displayed in Table III where we consider the results for the regions [2.2,2.4] GeV/c, [2.4,2.6] GeV/c and the average result, for two different values of the spectator quark mass which, as discussed in Sec. II, is not very well determined by the fit of the heavy-meson spectra. These results are based on the formula

$$\left|\frac{V_{bu}}{V_{bc}}\right|^2=\frac{\Delta B^{bu}(a,b)}{B_{\rm sl}^{\rm tot}}\frac{\Gamma_c}{\Delta\Gamma_u(a,b)}, \tag{4.5}$$

where

$$\Delta\Gamma_u(a,b)=\int_a^b dE\frac{d\Gamma_u}{dE} \tag{4.6}$$

and

$$\Gamma_c=\int_0^\infty dE\frac{d\Gamma_c}{dE}. \tag{4.7}$$

$B_{\rm sl}^{\rm tot}=(10.2\pm0.2\pm0.7)\times10^{-2}$ (Ref. 4) is the total semileptonic branching ratio and $\Delta B^{bu}(a,b)$ is the semileptonic branching ratio in the interval (a,b). We observe that the results on Table III do not depend strongly on $m_{\rm sp}$; we have checked that the dependence on other fitted parameters of our model is also negligible.

Together with V_{bu}/V_{bc} we can also determine the value of V_{bc} by the formula

$$|V_{bc}|^2\simeq\frac{B_{\rm sl}^{\rm tot}}{\tau_B}\frac{1}{\Gamma_c}. \tag{4.8}$$

By Eqs. (4.5) and (4.8) we obtain the result

$$V_{bc}=0.047\pm0.004 \tag{4.9}$$

and

$$\left|\frac{V_{bu}}{V_{bc}}\right|=0.11\pm0.02. \tag{4.10}$$

These results are in general agreement with the predictions of other models of semileptonic B decays.[47–49]

V. CONCLUSIONS

We have discussed a model for $Q\bar{q}$ mesons based on a relativistic wave equation and a QCD-inspired potential. We have presented some calculations of current-particle matrix elements, the limit $m_Q\to\infty$ and the determination of V_{bu} and V_{bc} Kobayashi-Maskawa matrix elements from recent data obtained by the CLEO and ARGUS Collaborations. The model is in agreement with the experimental data whenever they are available; we have also compared our results with the predictions of other theoretical approaches. We aim to test the present model in other fields of heavy meson physics, e.g., in the calculation of form factors; work is in progress and will be presented elsewhere.

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