Instantons and Borel resummability for the perturbed supersymmetric anharmonic oscillator

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In this paper we give an analytical derivation of the large-order behavior of the perturbation series for both the ground state and the excited states of the supersymmetric anharmonic oscillator and of the anharmonic oscillator obtained from the supersymmetric case by varying the strength of the fermion coupling. The results which are obtained with the help of instanton calculus coincide with those obtained numerically in previous work. The large-order perturbation series of the ground state vanishes in the supersymmetric case, whereas away from the supersymmetric point the perturbation series diverges factorially. The perturbation series of the excited states diverges factorially both at the supersymmetric point and away from this point.

I. INTRODUCTION

The large-order behavior of the perturbation series for the anharmonic oscillator and its connection to the semiclassical limit was investigated in an early paper by Vainshtein.[1] However, this paper did not get much attention, and the subject only got widely known after the work of Bender and Wu[2,3] who, at that moment, were not aware of the work of Vainshtein. It was found both numerically and by a WKB approximation that the coefficients of the perturbation series for the eigenvalues diverges factorially. Later, with the help of instanton calculus it was shown that this type of large-order behavior is generic to a much wider class of theories including field theories in 2, 3, and 4 dimensions.[4–6] The original results of Bender and Wu could be reproduced with this technique.[7] In supersymmetric quantum mechanics, as in any supersymmetric theory,[8,9] the ground state vanishes to all orders of perturbation theory. This raises the question whether the large-order behavior of the excited states of supersymmetric theories is special.

In previous papers[10,11] we investigated this question numerically. It was found that for the excited states of supersymmetric quantum-mechanical models the large-order behavior was generically of a factorial type. However, by varying the couplings in the potential away from their supersymmetric values, it was seen that the supersymmetric point is a point of bifurcation in the sense that the large-order behavior changed in some discontinuous way as one moved the couplings through their supersymmetric values. This demonstrated the existence of a cancellation of contributions from large-order boson and fermion loops.

The aim of this paper is to give an analytical derivation of these numerical results and also to extend them using the instanton techniques of Lipatov.[4,5] The dependence of the ground state of the double well on the fermion coupling was already investigated by Balitsky and Yung.[12] We study the large-order behavior of both the ground state and the excited states for two different potentials: the double well and the triple well. We will be able to show the vanishing of the ground-state energy but also recover the generically factorial results for the excited states. As such we will study the theory for the "Yukawa term" for an arbitrary value of its coupling in order to be able to study the deviations from supersymmetry and trace in a clear way the vanishing of the ground-state energy in the supersymmetric case.

As we will see in Sec. III the fermionic zero modes play an important role. They give rise to a zero fermion determinant in the presence of one instanton or one anti-instanton. Therefore, the leading contribution originates from classical paths that are a superposition of an instanton and an anti-instanton. These paths are not classical solutions. However, they are the closest one can come to classical solutions: they are streamlines[13–15] which flow through the classical configuration space as a river flows through a mountainous landscape. As we will show in Sec. IV, to leading order in the instanton–anti-instanton separation we do not need an explicit streamline solution, and it is sufficient to expand about a configuration given by the superposition of an instanton and an anti-instanton. The fermion determinant for this configuration is treated analogous to the fermion determinant of QCD instantons.[16,17,12] It is factorized in a factor that involves the zero modes and another factor that involves the nonzero modes (this procedure was also used in Ref. 18). As we will see in Sec. V, the latter factor is calculated in the one-instanton approximation, whereas the first factor is calculated exactly (it would be zero in the one-instanton approximation). This procedure amounts to a semiclassical approximation and suffices to obtain the large-order behavior of the perturbation series for the eigenvalues.

II. LARGE-ORDER PERTURBATION THEORY

In order to investigate the Borel resummability of a theory we study the large-order behavior of the perturbation series for its eigenvalues. In this section we outline some of the steps taken when deducing the large-order behavior. By necessity the discussion will be somewhat generic. Reviews on this subject can be found in Refs. 19 and 20.

When we denote the eigenvalues for zero coupling constant by E_l^0, the eigenvalues for coupling constant g can be written as

$$E_l(g)=E_l^0+\delta E_l(g) . \tag{2.1}$$

The corrections to the energy levels $\delta E_l(g)$ are analytic functions of g except in some interval $[a,b]$ of the real axis. For the usual anharmonic oscillator this interval is $\langle -\infty,0]$, while for potentials in this paper both this interval and the interval $[0,\infty\rangle$ occur. A consequence of this analyticity property is that

$$\delta E_l(g)=\frac{g^m}{2\pi i}\oint\frac{\delta E_l(g')dg'}{g'^m(g'-g)} =\frac{g^m}{2\pi i}\int_a^b\frac{\Delta(\delta E_l(g'))dg'}{g'^m(g'-g)}, \tag{2.2}$$

where $\Delta(\delta(E_l(g))$ is the discontinuity of $\delta E_l(g)$ across $[a,b]$. The integer m is adjusted such that when distorting the contours the contribution at infinity vanishes.

The perturbation coefficients are given by

$$a_k^l=\frac{1}{2\pi i}\oint dg\frac{\delta E_l(g)}{g^{k+1}} , \tag{2.3}$$

where the contour encircles the origin counterclockwise. Substituting $\delta E_l(g)$ from Eq. (2.2) into (2.4) we find that

$$a_k^l=\frac{-1}{2\pi i}\int_a^b dg'\frac{\Delta(\delta E_l(g'))}{g'^{k+1}} . \tag{2.4}$$

Consequently, to find the large-order behavior requires the discontinuity $\Delta(\delta E_l(g))$. However, for $k\to\infty$ it turns out that the above integral is dominated by the region $g\to 0$, and in this region the discontinuity also approaches zero.

The energy levels can be evaluated using the relation between the partition function and the path integral

$$P(g)\equiv\mathrm{Tr}\exp(-\beta H) =\int_{q(-\beta/2)=q(\beta/2)}\mathcal{D}q\,\exp(-S/g) , \tag{2.5}$$

where S is the Euclidean action. The path integral is defined for some real coupling, but we may actually extend it to complex g by rotating g into the complex plane, and, at the same time, rotating the $q(\tau)$ integration from the real axis to an appropriate contour in the complex $q(\tau)$ space. By rotating g to be above and below the discontinuity,

$$P(g+i0)-P(g-i0)=\int_{C_+}\mathcal{D}q\,\exp(-S/g) -\int_{C_-}\mathcal{D}q\,\exp(-S/g) , \tag{2.6}$$

where C_+ and C_- are the contours in the $q(\tau)$ space resulting from the above rotations. Further distorting the contours C_+ and C_- to run through the nontrivial saddle points (given by the solutions of the classical equations of motion) and taking into account that $\Delta(\delta E_l(g))\to 0$ for $g\to 0$, we find in this limit

$$\sum_l -\beta\Delta(\delta E_l(g))\exp(-\beta E_l^0)=\sum_{\mathrm{SP}} D\exp(-S/g) , \tag{2.7}$$

where D is a factor resulting from the Gaussian oscillations about the classical solutions of the equation of motion. The sum is over all nontrivial saddle points. In the case of zero modes this sum also includes an integration over the corresponding collective coordinates with the integration contour determined by C_+ and C_-. Combining Eqs. (2.7) and (2.4) we obtain

$$\sum_l a_k^l\exp(-\beta E_l^0)=\frac{1}{2\pi i\beta}\int_a^b\frac{dg'}{g'^{k+1}}\sum_{\mathrm{SP}} D\exp(-S/g) , \tag{2.8}$$

which is the final result of this section.

III. THE PERTURBED SUPERSYMMETRIC ANHARMONIC OSCILLATOR

In this paper we study the effect of fermions on the large-order behavior of the eigenvalues of the anharmonic oscillator. We do this in a quantum-mechanical model in which the fermions are coupled to the potential via a Yukawa-type interaction. The model is defined by the Euclidean action

$$S=\int dt\tfrac{1}{2}(\dot{x}^2+W'^2+\psi_\alpha\dot{\psi}_\alpha+cW''\psi\sigma_2\psi) , \tag{3.1}$$

where σ_2 is one of the Pauli spin matrices and a time derivative is denoted by a dot. The functions W' and W'' are the first and second derivative of the function $W(x)$. For $c=1$ the model is supersymmetric. The supersymmetry transformation is given by

$$\delta x=\zeta\sigma_2\psi , \tag{3.2a}$$

$$\delta\psi_\alpha=\sigma_{2\alpha\beta}\zeta_\beta\dot{x}-W'\zeta_\alpha . \tag{3.2b}$$

where ζ_α is a fixed Grassmann variable. Below we will study the large-order behavior for two different choices of W':

$$W_n'=x+gx^n,\quad n=2,3 . \tag{3.3}$$

[The ground state of the model (3.1) for $n=2$ was also studied in Ref. 12.] To exhibit clearly the dependence on the coupling constant we introduce new variables $g^{1/n-1}x\to x$ and redefine the coupling constant by $g^{2/n-1}\to g$. In terms of these variables the action is given by

$$S=\frac{1}{g}\int dt\tfrac{1}{2}(\dot{x}^2+W'^2+\psi_\alpha\dot{\psi}_\alpha+cgW''\psi\sigma_2\psi) . \tag{3.4}$$

Now, g plays the role of Planck's constant. As usual the fermionic term is subleading in g.

In order to evaluate the large-order behavior we have to evaluate the functional integral corresponding to the action (3.1). This will be performed with the help of a saddle-point approximation. The saddle points are given by the solutions $x_{\mathrm{cl}}(t)$ of the classical equations of motion

$$\lambda(x_{\rm cl}) \equiv \ddot{x}_{\rm cl} - W''(x_{\rm cl}) W'(x_{\rm cl}) = 0 \ . \tag{3.5}$$

The classical solutions with a finite action are called instantons and anti-instantons. As an example we give explicit expressions for the one-instanton solutions. For $n=2$ we find

$$x_{\rm cl} = \frac{-1}{1+e^{-t}} \ , \tag{3.6}$$

and for $n=3$ the instanton solutions are given by

$$x_{\rm cl} = \frac{i}{(1+e^{-2t})^{1/2}} \ , \tag{3.7a}$$

$$x_{\rm cl} = \frac{-i}{(1+e^{-2t})^{1/2}} \ . \tag{3.7b}$$

The anti-instanton solutions are obtained by replacing t by $-t$. Instantons or anti-instantons located at τ_I or τ_A, respectively, are obtained by translating the corresponding solution in time.

We evaluate the path integral by expanding a given path $x(t)$ around a classical solution $x_{\rm cl}$:

$$x(t) = x_{\rm cl}(t) + \xi(t) \ . \tag{3.8}$$

Up to second order in ξ we obtain, for the action (see Ref. 21),

$$S = S(x_{\rm cl}) + \int dt\, \lambda(x_{\rm cl})\xi + \int dt\, \xi \hat{Q}(x_{\rm cl})\xi + \int dt (\psi_\alpha \dot{\psi}_\alpha + c W'' \psi \sigma_2 \psi) \ , \tag{3.9}$$

where $\hat{Q}(x_{\rm cl})$ is defined by

$$\hat{Q}(x_{\rm cl}) = -\frac{d^2}{dt^2} + W''^2(x_{\rm cl}(t)) + W'(x_{\rm cl}(t)) W'''(x_{\rm cl}(t)) \ . \tag{3.10}$$

Most simply $x_{\rm cl}$ is a classical solution for which $\lambda(x_{\rm cl})$ vanishes. To leading order in g we have to evaluate [see Eq. (2.7)]

$$\sum_{\rm SP} D(x_{\rm cl}) \exp\left[-\frac{S(x_{\rm cl})}{g}\right] , \tag{3.11}$$

where the sum is over all saddle points (including an integration over the collective coordinates). The contributions of the Gaussian oscillations around a classical solution, denoted by $D(x_{\rm cl})$, can be factorized as

$$D(x_{\rm cl}) = [D_B^{\rm ZM}(x_{\rm cl}) D_B^{\rm NZM}(x_{\rm cl})]^{-1/2} \times [D_F^{\rm ZM}(x_{\rm cl}) D_F^{\rm NZM}(x_{\rm cl})]^{1/2} \ . \tag{3.12}$$

The nonzero-mode (NZM) parts of the bosonic and the fermionic determinants (a prime indicates that zero eigenvalues are excluded in the calculation of the determinant) are defined by

$$D_B^{\rm NZM}(x_{\rm cl}) \equiv \det{}' \left[-\frac{d^2}{dt^2} + W'^2(x_{\rm cl})\right] , \tag{3.13a}$$

$$D_F^{\rm NZM}(x_{\rm cl}) \equiv \det{}' \left[\frac{d}{dt} + c\sigma_2 W''(x_{\rm cl})\right] . \tag{3.13b}$$

Fluctuations in the directions of the zero modes cannot be evaluated via a saddle-point approximation. They have to be treated separately. Their contribution is denoted by the factors $D_B^{\rm ZM}$ and $D_F^{\rm ZM}$.

We will first discuss the contribution of the bosonic zero modes. They obey the equation

$$\hat{Q} x_0(t) = 0 \ . \tag{3.14}$$

By differentiating the classical equation of motion that solution is given by

$$x_0(t) = \dot{x}_{\rm cl}(t) \ . \tag{3.15}$$

This solution corresponds to the translation of the instanton solution. It can be proved that this is the only bosonic zero mode. Instead of integrating over the coefficient of this mode we will integrate over the position of τ of the instanton. This change of variables gives rise to a Jacobian. It can be shown[22,23] that its value is equal to $(A_0/2\pi)^{1/2}$, where A_0 is the absolute value of the action of a single instanton. The contribution of the bosonic zero mode is thus given by

$$[D_B^{\rm ZM}(x_I)]^{-1/2}(\cdots) = (A_0/2\pi)^{1/2} \int d\tau (\cdots) \ , \tag{3.16}$$

where the integrand is constituted by all other contributions to the partition function.

The fermionic zero modes obey the equation

$$\left[\frac{d}{dt} + c\sigma_2 W''(x_{\rm cl}(t))\right] \chi(t) = 0 \ . \tag{3.17}$$

The solutions of this equation are given by

$$\chi^{\overset{I}{A}}(t) = N \exp\left[\mp \int_{-\infty}^{t} c W''(x_{\rm cl}) dt'\right] \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ \mp i \end{bmatrix} = N W'^{c}(x_{\rm cl}^{\pm}(t)) \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ \mp i \end{bmatrix} , \tag{3.18}$$

where N is a normalization factor (an explicit solution for the potential for $n=2$ [see Eq. (3.3)] can also be found in Ref. 12). In this equation, the upper sign holds for the instanton and the lower sign for the anti-instanton. The existence of a normalizable zero mode generically does not depend on the value of the constant c but follows from the topological properties of the classical solution. For an extended discussion we refer to Secs. 3e and 6 of the review by Gendenshtein and Krive.[24] Only one of the two zero modes corresponding to an instanton is normalizable (an explicit discussion can be found in Ref. 25). Obviously, in the presence of a zero mode the fermion determinant $D_F^{\rm ZM}$ in Eq. (3.12) is zero and the one-instanton contribution to the partition function vanishes. In the next section we will study the leading nonvanishing contribution which is given by a path that is a succession of an instanton and an anti-instanton.

IV. CLASSICAL ACTION OF THE PERIODIC INSTANTON–ANTI-INSTANTON CONFIGURATION

A path given by the succession of an instanton and an anti-instanton does not allow for a normalizable zero mode. The reason is that the topological charge of this path is zero (it converges to the same value for $t \to \pm\infty$). Such a path is not a solution of the classical equations of motion. Therefore, the linear term $\lambda(x(t))$ does not vanish, and after completing the square in Eq. (3.9) the classical action is given by[21]

$$S = S(x_{\rm cl}) - \tfrac{1}{2}\int dt\, dt' \lambda(x(t)) \hat{Q}^{-1}(t,t') \lambda(x(t')) \ . \tag{4.1}$$

The additional term is of $O(\lambda^2)$, and below we will construct a path for which this term can be neglected to leading order in our expansion parameters.

For the instanton–anti-instanton path

$$x_{IA}(t) = x_I(t-\tau_I) + x_A(t-\tau_A) - x_I(\infty) \ , \quad t \in \langle -\infty, \infty \rangle \tag{4.2}$$

the linear term is equal to

$$\lambda(x_{IA}) = [x_I - x_I(\infty)][x_A - x_A(-\infty)] F(x_I, x_A) \ , \tag{4.3}$$

where F is a function free of singularities (note that $\lambda(x_I) = \lambda(x_A) = 0$ [see Eq. (3.5)]). For $n=2$ and $n=3$ we find

$$\lambda(x_{IA}) = 6(x_I+1)(x_A+1)(1+x_I+x_A) \ , \tag{4.4a}$$

$$\lambda(x_{IA}) = 3(x_I - i)(x_A - i)[5x_I^2 + 5x_A^2 + 5x_A^2 + 5x_I x_A - 5i(x_I + x_A) - 1] \ , \tag{4.4b}$$

respectively. From the explicit expressions given in Eqs. (3.6) and (3.7) it can be easily verified[21] that the factor $(x_I+1)(x_A+1)$ in Eq. (4.4a) is of $O(e^{-\theta})$. Analogously, we find that the factor $(x_I - i)(x_A - i)$ in Eq. (4.4b) is of $O(e^{-2\theta})$.

The path integral (2.5) is over all periodic trajectories $q(t)$ with $q(-\beta/2) = q(\beta/2)$. In analogy with the caloron solution of QCD,[26] it is possible to construct from $x_{IA}(t)$ a path that satisfies these boundary conditions. The solution is given by

$$X(t) = \sum_{n=-\infty}^{\infty} x_{IA}(t - n\beta) \ . \tag{4.5}$$

We already have shown that the linear term $\lambda(x(t))$ is of $O(e^{-\theta})$ (for $n=2$) or $O(e^{-2\theta})$ (for $n=3$) for each of the terms contributing to this sum. From the observation that $x_{IA}(t)$ decreases exponentially both for $t \to \infty$ and $t \to -\infty$ we conclude that the only other relevant contribution is from neighboring instantons. The magnitude of these contributions to the linear term $\lambda(x(t))$ is of $O(e^{-(\beta-\theta)})$, both for $n=2$ and $n=3$. Therefore, we have shown that to leading order in the small parameters $e^{-\theta}$ and $e^{-(\beta-\theta)}$ the quadratic term in Eq. (4.1) does not contribute to the instanton interaction.

For a more systematic treatment of these "almost classical solutions" we have to generalize the notion of a classical solution to the so-called streamline configurations.[13,12,14,15] They are defined as paths for which the linear term vanishes in all directions except for one (which is called the streamline). Since we are interested only in the leading-order contributions to the instanton–anti-instanton interactions there is no need to use this concept (see e.g., Ref. 12 for an extensive discussion of streamlines and their relevance in the present context).

The instanton–anti-instanton ansatz Eq. (4.1) has two approximate zero modes. As explained above we may choose to integrate over the collective coordinates τ_I and τ_A of the instanton and the anti-instanton. This yields the factor

$$\int_{-\beta/2}^{\beta/2} d\tau_A \int_{-\beta/2}^{\beta/2} d\tau_I \frac{A_0}{2\pi} \cdots \ , \tag{4.6}$$

in the path integral. Changing the variable to $s = (\tau_I + \tau_A)/2$ and $\theta = \tau_A - \tau_I$ and taking into account that, due to the translation invariance and the periodicity in the τ_I and the τ_A variables, the integrand only depends on θ, we find that the factor (4.6) becomes

$$\beta \int_{-\beta/2}^{\beta/2} d\theta \frac{A_0}{2\pi} \cdots \ . \tag{4.7}$$

After the integration over the center of mass its position has to be fixed. For definiteness we will locate the instanton at $-\theta/2$ and the anti-instanton at $\theta/2$.

Next we evaluate the classical action for pseudoparticles fixed at these positions. For $n=2$ we obtain, for the action of the periodic configuration $X(t)$ of Eq. (4.5),

$$S_{\rm cl} = \tfrac{1}{2}\int dt [\dot{X}^2 + W'^2(X)] = \tfrac{1}{3} + S_{\rm int} \ . \tag{4.8}$$

The instanton interaction $S_{\rm int}$ is given by

$$\begin{aligned} S_{\rm int} = {} & 2\sum_{n=-\infty}^{\infty} \int_{-\beta/2}^{\beta/2} dt [x_I(n) + x_I^2(n) + x_A(n) + x_A^2(n)][1 + x_I(n)][1 + x_A(n)] \\ & + 2\sum_{n=-\infty}^{\infty} \int_{-\beta/2}^{\beta/2} dt [x_I(n) + x_I^2(n) + x_A(n) + x_A^2(n)][x_I(n) x_A(n-1) + x_A(n) x_I(n+1)] \\ & + O(e^{-2\theta}) + O(e^{-2(\beta-\theta)}) \ . \end{aligned} \tag{4.9}$$

where $x_I(t - n\beta)$ and $x_A(t - n\beta)$ are denoted by $x_I(n)$ and $x_A(n)$, respectively. The second term on the right-hand side results from the instanton–anti-instanton pairs in the intervals $[(n-1)\beta/2, n\beta/2]$, $n \in Z$, and the third term is from neighboring instantons and anti-instantons in intervals $[(n-1)\beta/2, n\beta/2]$ and $[n\beta/2, (n+1)\beta/2], n \in Z$. They are of $O(e^{-\theta})$ and $O(e^{-(\beta-\theta)})$, respectively. The sum over n and the integral over the interval $[-\beta/2, \beta/2]$ can be replaced by a single integral from $-\infty$ to ∞ without an additional summation over n. To leading order the evaluation of this integral is straightforward and we obtain, for the classical action,

$$S_{\rm cl} = \tfrac{1}{3} - 2[e^{-\theta} + e^{\theta - \beta}] + O(e^{-2\theta}) + O(e^{-2(\beta-\theta)}) \quad \text{for } n=2 \ . \tag{4.10}$$

The calculations for $n=3$ proceed along the same lines but are slightly more involved. In this case we obtain, for the classical action,

$$S_{\rm cl}=\tfrac{1}{2}\int dt[\dot{X}^2+W'^2(X)]=-\tfrac{1}{2}+S_{\rm int}\ , \tag{4.11}$$

where the instanton interaction $S_{\rm int}$ is given by

$$\begin{aligned}S_{\rm int}=3\sum_{n=-\infty}^{\infty}&\int_{-\beta/2}^{\beta/2}dt[x_I(n)+x_I^3(n)+x_A(n)+x_A^3(n)][x_I(n)-i][x_A(n)-i][x_I(n)+x_A(n)]\\ &+3\sum_{n=-\infty}^{\infty}\int_{-\beta/2}^{\beta/2}dt[x_I(n)+x_I^3(n)+x_A(n)+x_A^3(n)]\\ &\qquad\times\{x_A(n)x_I(n+1)[x_I(n+1)+x_A(n)]+x_A(n-1)x_I(n)[x_I(n)+x_A(n-1)]\}\\ &+O(e^{-4\theta})+O(e^{-2(\beta-\theta)})\ .\end{aligned} \tag{4.12}$$

Substitution of the explicit instanton and anti-instanton solutions (3.7a) and (3.7b) enables us to obtain $S_{\rm int}$ in a straightforward way. To leading order in the pseudoparticle separation we find

$$S_{\rm cl}=-\tfrac{1}{2}+(e^{-2\theta}+2e^{\theta-\beta})+O(e^{-4\theta})+O(e^{-2(\beta-\theta)})\quad\text{for } n=3\ . \tag{4.13}$$

Note that the exponents are proportional to the curvature of the potential. The action (4.10) and (4.13) will be used in the evaluation of the partition function according to Eq. (3.11)

V. FLUCTUATIONS ABOUT THE CLASSICAL PATH

We consider fluctuations about the path X of Eq. (4.5) on the interval $[-\beta/2,\beta/2]$. We study both the case of the double well and the triple well. A similar discussion for the double well can be found in Ref. 12. For an instanton–anti-instanton configuration on the interval $[-\beta/2,\beta/2]$ we have the approximate fermionic zero modes $\chi_I(t+\theta/2)$ and $\chi_A(t-\theta/2)$ [see Eq. (3.18)]. Their contribution to the fermion determinant is given by

$$[D_F^{\rm ZM}(x)]^{1/2}=\int d\zeta_A d\zeta_I \exp\left[-\tfrac{1}{2}\int_{-\beta/2}^{\beta/2}dt(\zeta_I\chi_I+\zeta_A\chi_A)(\partial_t+c\sigma_2W'')(\zeta_I\chi_I+\zeta_A\chi_A)\right]\ . \tag{5.1}$$

The Grassmann integration results in the square root of a determinant of a 2×2 matrix:

$$[D_F^{\rm ZM}(X)]^{1/2}=\det{}^{1/2}\begin{bmatrix}\langle\chi_I|\partial_t+c\sigma_2W''|\chi_I\rangle & \langle\chi_I|\partial_t+c\sigma_2W''|\chi_A\rangle\\ \langle\chi_A|\partial_t+c\sigma_2W''|\chi_I\rangle & \langle\chi_A|\partial_t+c\sigma_2W''|\chi_A\rangle\end{bmatrix}\ . \tag{5.2}$$

Note that this matrix is skew symmetric (its diagonal elements vanish). The contributions to the overlap matrix elements of the instanton–anti-instanton pairs that are not located in the interval $[-\beta/2,\beta/2]$ are subleading in $e^{-\beta}$. Therefore, we can replace $X(t)$ by $x_{IA}(t)$ and extend the range of the t integration from $-\infty$ to ∞ to first order in $e^{-\beta}$. The square root of its determinant is thus given by

$$[D_F^{\rm ZM}(X)]^{1/2}=\int_{-\infty}^{\infty}dt\,\chi_A(\partial_t+c\sigma_2W''(x_{IA}))\chi_I+O(e^{-\beta})\ . \tag{5.3}$$

To leading order in $e^{-\theta}$ we have

$$W''(x_{IA})=W''(x_I)+W''(x_A)+1+O(e^{-\theta})\quad\text{for } n=2\ , \tag{5.4a}$$

$$W''(x_{IA})=W''(x_I)+W''(x_A)+2+O(e^{-2\theta})\quad\text{for } n=3\ . \tag{5.4b}$$

By using the equations of motion for the fermionic zero modes χ_I and χ_A we can rewrite the overlap matrix element as

$$[D_F^{\rm ZM}(X)]^{1/2}=-\int_{-\infty}^{\infty}dt\,\chi_A(-\partial_t-c)\chi_I\quad\text{for } n=2\ , \tag{5.5a}$$

$$[D_F^{\rm ZM}(X)]^{1/2}=-\int_{-\infty}^{\infty}dt\,\chi_A(-\partial_t-2c)\chi_I\quad\text{for } n=3\ . \tag{5.5b}$$

In this equation we substitute the explicit expressions (3.18) for the zero modes and evaluate the integrals to leading order in $e^{-\theta}$. To leading order in $e^{-\theta}$ and $e^{-\beta}$ the final result for the contribution of the fermionic zero modes is given by

$$[D_F^{\rm ZM}(X)]^{1/2}=N^2 2^{4c}e^{-\theta c}\quad\text{for } n=2\ , \tag{5.6a}$$

$$[D_F^{\rm ZM}(X)]^{1/2}=N^2 e^{-2\theta c}\quad\text{for } n=3\ . \tag{5.6b}$$

where N is the normalization factor defined by Eq. (3.18).

The contribution of the fermionic nonzero modes is calculated by factorizing the determinant $D_F^{\rm NZM}$ in one-pseudoparticle contributions. They can be obtained conveniently by generalizing an identity derived by Salomonson and van Holten for the supersymmetric case ($c=1$) to arbitrary values of c. Using the technique discussed in Appendix A of Ref. 25 we have shown that

$$\det{}'(\partial_t + c\sigma_2 W'') = \frac{N}{[W'(x(\tau))]^c} \exp\left[-\frac{c}{2}\int_{T_1}^{\tau} d\tau' W''(x(\tau')) + \frac{c}{2}\int_{\tau}^{T_2} d\tau' W''(x(\tau'))\right]$$
$$= \frac{\left[\int_{T_1}^{T_2} [W'(x(\tau'))d\tau']^{2c}\right]^{1/2}}{[W'(x(T_1))W'(x(T_2))]^{c/2}} . \tag{5.7}$$

The determinant has been evaluated for fermionic paths $\psi(t)$ on the interval $[T_1, T_2]$ ($T_2 > T_1$) satisfying the boundary conditions $\psi(T_1) = -\psi(T_2)$. The final answer does not depend on the point τ inside this interval. The contributions of the instanton–antiinstanton pairs outside the interval $[-\beta/2, \beta/2]$ are subleading in $e^{-\beta}$, and to this order only the pair inside this interval has to be taken into account. The instanton happens at time $-\theta/2$ [see below Eq. (4.7)] between $-\beta/2$ and 0. In Eq. (5.7) we thus have $T_1 = -\beta/2$ and $T_2 = 0$. For the anti-instanton happening at time $\theta/2$ between times 0 and $\beta/2$ we have $T_1 = 0$ and $T_2 = \beta/2$. Using the explicit expressions for $W'(x_I)$ and $W'(x_A)$ and the factorization of the determinant $D_F^{\rm NZM}$ in a factor corresponding to the instanton and another factor corresponding to the anti-instanton, we obtain, as the leading-order contribution from the fermionic nonzero modes,

$$[D_F^{\rm NZM}(X)]^{1/2} = N^{-2} 2^{-4c} \exp\left[\frac{c}{2}\beta\right] \quad \text{for } n=2 , \tag{5.8a}$$

$$[D_F^{\rm NZM}(X)]^{1/2} = N^{-2} \exp\left[\frac{c}{2}(\beta+\theta)\right] \quad \text{for } n=3 . \tag{5.8b}$$

In our case the contribution of the bosonic nonzero modes for a single pseudoparticle is given by [see Eq. (A14) of Ref. 25] $[(2\omega/A_0)\exp(-2\omega T)]^{1/2}$, where A_0 is the absolute value of the classical action of an instanton, ω is the curvature at an extremum of the potential, and T is the total time spent near this extremum. By using the factorization of $D_B^{\rm NZM}(X)$ in an instanton contribution and an anti-instanton contribution we thus obtain

$$[D_B^{\rm NZM}(X)]^{-1/2} = 12 \exp\left[-\frac{\beta}{2}\right] \quad \text{for } n=2 , \tag{5.9a}$$

$$[D_B^{\rm NZM}(X)]^{-1/2} = 8\sqrt{2} \exp\left[-\frac{\beta}{2} - \frac{\theta}{2}\right] \quad \text{for } n=3 . \tag{5.9b}$$

Again this result is valid in the dilute instanton gas approximation, i.e., to leading order in $e^{-\theta}$ and $e^{-\beta}$.

Now we are in a position to collect all contributions (3.12) to the fluctuations about an instanton–anti-instanton pair. As a final result of this section we find, for the partition function,

$$\mathrm{Tr}\exp(-\beta H) = \frac{\beta}{\pi g}\int_{-\beta/2}^{\beta/2} d\theta \exp\left[-c\theta + \frac{\beta}{2}(c-1)\right] \exp\left[-\frac{S_{\rm cl}}{g}\right] \quad \text{for } n=2 , \tag{5.10a}$$

$$\mathrm{Tr}\exp(-\beta H) = \frac{\beta 2\sqrt{2}}{\pi g}\int_{-\beta/2}^{\beta/2} d\theta \exp\left[-\frac{3}{2}c\theta - \frac{\theta}{2} + \frac{\beta}{2}(c-1)\right] \exp\left[-\frac{S_{\rm cl}}{g}\right] \quad \text{for } n=3 , \tag{5.10b}$$

where $S_{\rm cl}$ is given in Eqs. (4.10) and (4.13). The preexponential factors have been evaluated to leading order in $e^{-\beta}$ and $e^{-\theta}$, and the classical action has been evaluated to next leading order in these parameters. Note that in the case $n=3$ there is an extra factor 2 due to the existence of two equivalent saddle points.

VI. LARGE-ORDER BEHAVIOR

In this section we derive the large-order behavior of the perturbation series of the eigenvalues by substituting the saddle-point approximation for $\exp(-\beta H)$, obtained in previous section, in Eq. (2.8). We obtain

$$\sum_l a_k^l \exp(-\beta E_l^0) = \frac{1}{2\pi i}\exp\left[\frac{\beta}{2}(c-1)\right]\int_a^b dg \frac{1}{g^{k+2}} \oint_C d\theta\, d_n(\theta) \exp\left[-\frac{S_{\rm cl}}{g}\right] , \tag{6.1}$$

where $S_{\rm cl}$ is given in Eqs. (4.10) and (4.13), and the functions $d_n(\theta)$ are defined by

$$d_n(\theta)=\frac{1}{\pi}\exp(-c\theta) \quad \text{for } n=2\ , \tag{6.2a}$$

$$d_n(\theta)=\frac{2\sqrt{2}}{\pi}\exp\left[-c\theta-\frac{\theta}{2}(c+1)\right] \quad \text{for } n=3\ . \tag{6.2b}$$

For $n=2$ the integration over g runs from 0 to ∞, and for $n=3$ the integration runs from $-\infty$ to 0. The θ integration runs over the contour C, which will be discussed in the next paragraph.

In Eq. (2.8) the sum over the nontrivial saddle points includes the integration over the collective coordinates. In our case the only relevant collective coordinate is θ. Originally, the θ integration is over the real axis between $-\beta/2$ and $\beta/2$. As discussed in Sec. II depending on the sign of the imaginary part of the coupling constant we have to deform the integration contour to C_+ or C_-. For $n=2$ a choice for C_+ that yields convergent integrals is obtained by rotating the contour in the variable $z\equiv e^{-\theta}$ about the trivial saddle point $z=\exp(-\beta/2)$ by $-\pi/4$. To obtain C_- we rotate the contour around the same point in the opposite direction by the same angle. For $n=3$ we introduce a new variable z by $z=\exp(-2\theta)$. The trivial saddle point is located at $z=\exp(-2\beta/3)$. Now, the contours C_+ and C_- are obtained by rotating about this point by the opposite angles as for $n=2$. The contributions from the trivial saddle point of the two contours cancel. Therefore, using the analyticity properties of the z integrand we can deform the contours in such a way that they can be combined in a single contour C_n encircling the negative real axis counterclockwise for $n=2$ and clockwise for $n=3$. The integration over g results in (for $n=2$)

$$\sum_l a_k^l\exp(-\beta E_l^0)=\frac{\Gamma(k+1)}{2\pi i}\exp\left[\frac{\beta}{2}(c-1)\right]\oint_{C_2}\frac{dz}{-z}\frac{d_n(-\ln z)}{[S_{\rm cl}(-\ln z)]^{k+1}}\ . \tag{6.3}$$

Finally, we perform the integration over z by exponentiating the denominator of the integrand and expanding the exponential function in powers of $e^{-\beta}$. For $n=2$ we obtain for the asymptotic values of the coefficients

$$\begin{aligned}\sum_l a_k^l\exp(-\beta E_l^0)&=-\frac{\Gamma(k+1)3^{k+1}}{2\pi^2 i}\exp\left[\frac{\beta}{2}(c-1)\right]\oint_{C_2}dz\frac{z^{c-1}}{[1-6(z+z^{-1}e^{-\beta})]^{k+1}}\\&=-\frac{\Gamma(k+1-c)}{\Gamma(1-c)}\frac{6^{-c}3^{k+1}}{\pi}\exp\left[\frac{c-1}{2}\beta\right]-\frac{\Gamma(k+3-c)}{\Gamma(2-c)}\frac{6^{2-c}3^{k+1}}{\pi}\exp\left[\frac{c-1}{2}\beta-\beta\right]-\cdots,\end{aligned} \tag{6.4}$$

which agrees with our numerical results. For $n=3$ a similar calculation leads to

$$\begin{aligned}\sum_l a_k^l\exp(-\beta E_l^0)&=-\frac{\Gamma(k+1)(-2)^{k+1}\sqrt{2}}{2\pi^2 i}\exp\left[\frac{\beta}{2}(c-1)\right]\oint_{C_3}dz\frac{z^{(3/4)(c-1)}}{[1-2(z+2z^{-1/2}e^{-\beta})]^{k+1}}\\&=\frac{2^{(1/4)-(3/4)c}}{\pi}\frac{\Gamma(k+\frac{3}{4}-\frac{3}{4}c)}{\Gamma(\frac{3}{4}-\frac{3}{4}c)}(-2)^{k+1}\exp\left[\frac{c-1}{2}\beta\right]\\&\quad+\frac{4\times2^{(3/4)-(3/4)c}}{\pi}\frac{\Gamma(k+\frac{9}{4}-\frac{3}{4}c)}{\Gamma(\frac{5}{4}-\frac{3}{4}c)}(-2)^{k+1}\exp\left[\frac{\beta}{2}(c-1)-\beta\right]\\&\quad+\frac{16\times2^{(1/4)-(3/4)c}}{\pi}\frac{\Gamma(k+\frac{15}{4}-\frac{3}{4}c)}{\Gamma(\frac{7}{4}-\frac{3}{4}c)}(-2)^{k+1}\exp\left[\frac{\beta}{2}(c-1)-2\beta\right]+\cdots\ .\end{aligned} \tag{6.5}$$

Note that the factors $e^{-\beta}$ resulting from the classical action are always accompanied by a factor k. Therefore, the $O(e^{-\beta})$ corrections to the determinants give rise to contributions of $O(1/k)$ and can be neglected for the calculation of the large-order behavior of the coefficients a_k^l. From the calculation of the integrals in Eqs. (6.4) and (6.5) it can be easily deduced that corrections to the preexponential factors of $O(e^{-\theta})$ also give rise to contributions of $O(1/k)$.

In Eq. (3.9) the fermions can be integrated out first. This yields the action[11]

$$S=\tfrac{1}{2}\dot{x}^2+\tfrac{1}{2}W'^2-\frac{c}{2}W''\ . \tag{6.6}$$

For zero coupling ($g=0$) the potential is given by $V=\frac{1}{2}x^2-c/2$. The energy levels of this shifted harmonic oscillator are given by $E_n=n+\frac{1}{2}-c/2$, $n=0,1,2,\ldots$, which explains the β dependence in Eqs. (6.4) and (6.5).

Indeed, we find that for the supersymmetric case ($c=1$) the large-order behavior of the coefficients a_k^0 of the ground state is suppressed by at least a factor $1/k$. Corrections of $O(e^{-\theta})$ to the preexponential factors only modify the z integrand by a regular function and do not have an effect on the zero result for the ground state. Corrections of $O(e^{-\beta})$ only affect the higher states and not the ground state. However, the large-order perturbation series is affected by corrections to the classical action, and by higher-order quantum corrections. For $c=2$ the potential $\frac{1}{2}W_{n=2}'^2-(c/2)W_{n=2}''-\frac{1}{2}$ is also supersymmetric. This can be seen by rewriting the potential in terms of the function $W' = W'_{n=2} - 1/x$. The zero-energy state of this potential corresponds with the first excited state of our original potential for $c=2$ and $n=2$, and therefore all coefficients of the perturbation series vanish. For $n=3$ the same argument can be used to show that the perturbative expansion of the first excited state vanishes for $c=\frac{5}{3}$.

VII. CONCLUSIONS

We have studied the large-order perturbation theory for excited states in the supersymmetric double and triple well, as well as, for variations about the supersymmetric point. In this paper we studied the effect of the strength of the "Yukawa-coupling" (denoted by c) and were able to reproduce analytically the results that were obtained numerically in an earlier paper.[11] The study of the excited states could be achieved in particular by constructing the analog of the caloron solution in QCD.[26]

The perturbation series for the ground-state energy vanishes in the supersymmetric case because of the analytic structure of the integration over the collective variables: at the supersymmetric point ($c=1$) the cut in the complex plane that exists for non-integer values of c disappears leading to vanishing coefficients. Since our calculations were performed in the dilute-instanton-gas approximation this actually was only shown to leading order in $1/k$ (k is the order of the perturbation theory). However, from numerical[11] work we know that the coefficients are equal to zero. From Eqs. (6.4) and (6.5) we find that the large-order perturbation series also vanishes for an infinite discrete series of c values different from $c=1$. In this case, numerical calculations show that subleading contributions in $1/k$ to the perturbation series do not vanish, and that the resulting series still diverges factorially. Apart from the exceptional case discussed above the same is true for the excited states. For all other c values the perturbation series diverges factorially both in the case of the ground state and the excited states.

In our previous paper, we also studied another variation about the supersymmetric point, i.e., the addition of the term $(d-1)gx^3$ and $(d-1)gx^4$ for the double well and the triple well, respectively. For $d=1$ the potential has two degenerate minima, and in this case the fermionic term, which is subleading in the coupling constant, can play an important role. For $d\neq 1$ the fermionic term only affects the overall constant of the large-order perturbation series. The d dependence of this series was studied analytically in Ref. 6 and their results can be applied and shown to agree with our numerical study.[11] In particular it was found that the large-order perturbation series has an oscillatory behavior for $d<1$, whereas it is monotonous or alternating for $d>1$. The reason is that for $d<1$ the classical action becomes complex. At $d=1$ the period of this oscillation becomes infinite. In this sense the supersymmetric point is a point of bifurcation.

Finally, we want to note that the analytic derivation of the large-order behavior given in this paper relies upon a sophisticated evaluation of the Feynman path integral. In this the comparison of our answers with numerical results has been an important check on the techniques we have made use of.

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[1] A. Vainshtein, *Decaying Systems and Divergence of Perturbative Series* (in Russian) (INP, Novosibirsk, 1964).
[2] C. M. Bender and T. T. Wu, Phys. Rev. **184**, 1232 (1969).
[3] C. M. Bender and T. T. Wu, Phys. Rev. D **7**, 1620 (1973).
[4] L. N. Lipatov, Pis'ma Zh. Eksp. Teor. Fiz. **24**, 179 (1976) [JETP Lett. **24**, 157 (1976)].
[5] L. N. Lipatov, Pis'ma Zh. Eksp. Teor. Fiz. **25**, 116 (1977) [JETP Lett. **25**, 104 (1977)].
[6] E. Brezin, J. C. Le Guillou, and J. Zinn-Justin, Phys. Rev. D **15**, 1544 (1977); **15**, 1558 (1977).
[7] E. Brezin, G. Parisi, and J. Zinn-Justin, Phys. Rev. D **16**, 408 (1977); **16**, 996 (1977).
[8] P. West, Nucl. Phys. **B106**, 219 (1976).
[9] B. Zumino, Nucl. Phys. **B89**, 535 (1975).
[10] J. J. M. Verbaarschot, P. West, and T. T. Wu, Phys. Rev. D **42**, 1276 (1990).
[11] J. J. M. Verbaarschot, P. West, and T. T. Wu, Phys. Lett. B **240**, 401 (1990).
[12] I. I. Balitsky and A. V. Yung, Nucl. Phys. **B274**, 475 (1986).
[13] I. I. Balitsky and A. V. Yung, Phys. Lett. **168B**, 113 (1986).
[14] E. V. Shuryak, in Proceedings of the Conference on Numerical Experiments in Quantum Field Theories, Alma Ata, USSR, 1985, edited by A. A. Migdal (in Russian) (unpublished).
[15] E. V. Shuryak, Nucl. Phys. **B302**, 621 (1988).
[16] D. I. Diakonov and V. Yu. Petrov, Zh. Eksp. Teor. Fiz. **89**, 361 (1985) [Sov. Phys. JETP **62**, 204 (1985)]; **89**, 751 (1985) [**62**, 431 (1985)].
[17] D. I. Diakonov and V. Yu. Petrov, Nucl. Phys. **B272**, 457 (1986).
[18] R. Kaul and L. Mizrachi, J. Phys. A **22**, 675 (1989).
[19] J. Zinn-Justin, Phys. Rep. **70**, 109 (1981).
[20] D. I. Kazakov and D. V. Shirkov, Fortsch. Phys. **28**, 465 (1980).
[21] E. B. Bogomolny, Phys. Lett. **91B**, 431 (1980).
[22] G. 't Hooft, Phys. Rev. D **14**, 3432 (1976).
[23] S. Coleman, in *The Why's of Subnuclear Physics,* proceedings of the International School, Erice, Italy, 1977, edited by A. Zichichi, Subnuclear Series, Vol. 15 (Plenum, New York, 1979).
[24] L. É. Gendenshtein and I. V. Krive, Usp. Fiz. Nauk **146**, 553 (1985) [Sov. Phys. Usp. **28**, 645 (1985)].
[25] E. Witten, Nucl. Phys. **B196**, 509 (1982).
[26] B. Harrington and H. Shepard, Phys. Rev. D **17**, 2122 (1978).