Global structure of the standard model, anomalies, and charge quantization

Joseph Hucks*
Physics Department, University of California, Santa Barbara, California 93106
(Received 19 November 1990)

Lie groups with the same Lie algebra as the standard model but different global topology are considered, and the possibilities are reduced to four viable alternatives for the true symmetry group of the standard model: SU(3)×SU(2)×U(1), U(3)×SU(2), SU(3)×U(2), and S(U(3)×U(2)). It is demonstrated that the last three groups require hypercharge quantization for their allowable representations, and that S(U(3)×U(2)) is the most likely candidate for the true symmetry group of the standard model because it offers the best explanation of the observed hypercharges of the elementary fermions. Explicit S(U(3)×U(2)) tensor representations of the quarks and leptons are given and are compared to the standard SU(5) assignments. The spontaneous symmetry breaking of S(U(3)×U(2)) to an electrostrong U(3) is briefly discussed, and electric charge quantization follows from weak hypercharge quantization and the existence of the standard Higgs doublet with nonzero vacuum expectation value. Lastly, it is shown that combining the conditions imposed by anomaly cancellation with the S(U(3)×U(2)) hypercharge quantization condition uniquely determines the ratios of the hypercharges in a standard quark-lepton family.

I. INTRODUCTION

In this article, Lie groups with the same Lie algebra as the standard model but different topological structure are considered for the position of the true symmetry group of nature.

There are thirteen connected Lie groups[1-3] with the same Lie algebra as SU(3)×SU(2)×U(1). Four of these are noncompact:

1. $SU(3)\times SU(2)\times R$,
2. $[SU(3)/Z_3]\times SU(2)\times R$,
3. $SU(3)\times[SU(2)/Z_2]\times R$,
4. $[SU(3)/Z_3]\times[SU(2)/Z_2]\times R$;

and there are nine that are compact:

5. $SU(3)\times SU(2)\times[R/Z]=SU(3)\times SU(2)\times U(1)$,
6. $\{[SU(3)\times U(1)]/Z_3\}\times SU(2)=U(3)\times SU(2)$,
7. $SU(3)\times\{[SU(2)\times U(1)]/Z_2\}=SU(3)\times U(2)$,
8. $[SU(3)\times SU(2)\times U(1)]/(Z_3\times Z_2)$
   $=[SU(3)\times SU(2)\times U(1)]/Z_6=S(U(3)\times U(2))$,
9. $[SU(3)/Z_3]\times SU(2)\times U(1)$,
10. $SU(3)\times[SU(2)/Z_2]\times U(1)$,
11. $[SU(3)/Z_3]\times[SU(2)/Z_2]\times U(1)$,
12. $U(3)\times[SU(2)/Z_2]$,
13. $[SU(3)/Z_3]\times U(2)$.

We can eliminate choices 1–4 by demanding that the gauge group be compact. Choices 9–13 may be removed from consideration by using the fact that the true non-Abelian group of the quarks and leptons in the standard model is SU(3)×SU(2), since color triplet and weak doublet representations exist[4] in nature. Note that the simply connected universal covering group of all 13 groups above is $SU(3)\times SU(2)\times R$, while $SU(3)\times SU(2)\times U(1)$ is the covering group for groups 5–13.

As discussed previously by O'Raifeartaigh,[5] there then remain four possible true symmetry groups for the standard model:

1. $SU(3)\times SU(2)\times U(1)$,
2. $U(3)\times SU(2)$,
3. $SU(3)\times U(2)$,
4. $S(U(3)\times U(2))$.

The four groups above will each be discussed in this article. It will be shown that the allowable single-valued representations of the groups U(3)×SU(2), SU(3)×U(2), and S(U(3)×U(2)) must have quantized hypercharge. Hypercharge is quantized in nature, and a possible explanation of hypercharge quantization could be that the true group of the standard model is U(3)×SU(2), SU(3)×U(2), or S(U(3)×U(2)). It will be shown that the group which best explains the hypercharge assignments of the quarks and leptons is the last alternative, the group S(U(3)×U(2)). It will therefore be the focus of attention of this article.

The organization of this paper is as follows. Hypercharge quantization is shown to be required for the three groups U(3)×SU(2), SU(3)×U(2), and S(U(3)×U(2)). The quantization conditions for the groups U(3)×SU(2) and SU(3)×U(2) are demonstrated as being inadequate to explain the various hypercharge assignments of the quarks and leptons, whereas the quantization condition for the group $S(U(3)\times U(2))$ offers a simple explanation of the hypercharges based upon the given $SU(3)\times SU(2)$ representations of the elementary fermions. Explicit $S(U(3)\times U(2))$ tensor representations of the quarks and leptons are then given. Because $S(U(3)\times U(2))$ is a subgroup of SU(5), it is shown briefly how the $S(U(3)\times U(2))$ representations fit into the $\bar{5}+10$ representation of SU(5). It is also demonstrated that $S(U(3)\times U(2))$ is spontaneously broken to an electrostrong U(3) by the standard Higgs doublet with nonzero vacuum expectation value (VEV), and the electrostrong U(3) tensor representations of the quarks and leptons are presented. Lastly, it is shown how the conditions imposed by anomaly cancellation and the $S(U(3)\times U(2))$ hypercharge quantization condition yield uniquely the ratios of the hypercharges in a standard family of 15 left-handed (LH) fermions (a new result).

It should be understood from the beginning that the groups $U(3)\times SU(2)$, $SU(3)\times U(2)$, and $S(U(3)\times U(2))$ yield the same perturbative quantum field theory as $SU(3)\times SU(2)\times U(1)$, since perturbative effects depend only on the Lie algebra. Whether or not the four groups lead to different nonperturbative effects is to the author's knowledge still an open question. The group $S(U(3)\times U(2))$ still has three coupling constants and many other free parameters, but has the advantage of offering a possible explanation for the quantization of hypercharge and the hypercharge assignments of the elementary fermions (in conjunction with anomaly cancellation).

This work was based in part on conclusions by O'Raifeartaigh[6] about the global nature of the standard model. He suggested that the true symmetry group of nature was $S(U(3)\times U(2))$ and that the unbroken symmetry group was U(3) and not $SU(3)\times U(1)$. However, it was incorrectly stated that $S(U(3)\times U(2)) =[SU(3)\times SU(2)\times U(1)]/Z_5$, and no details of the $S(U(3)\times U(2))$ theory were presented. The details of the $S(U(3)\times U(2))$ theory are worked out explicitly here and the argument that the groups $U(3)\times SU(2)$, $SU(3)\times U(2)$, and $S(U(3)\times U(2))$ actually require hypercharge quantization is presented.

II. CONVENTIONS

In this paper we will only be dealing with left-handed fermion fields. Right-handed fields will be represented by the left-handed part of the charge-conjugate field,[7] and will use the same symbol as the left-handed part, but with an overbar to denote charge conjugation. This convention is used to avoid writing a lot of superfluous subscripts and superscripts.

The results of this paper are the same for all three generations so we will use the first family symbols to represent the corresponding elements of any of the families. Explicitly: e stands for e, μ, or τ; ν for ν_e, ν_μ, or ν_τ; d for d', s', or b' (the prime denotes the Kobayashi-Maskawa mixture of mass eigenstates[8]); and u for u, c, or t.

We will work in natural units ($\hbar=c=1$) and the convention for hypercharge (Y) is given in Table I below. Note that the convention for hypercharge is half that of some other references. In this paper we will only consider the standard model with three generations and no right-handed neutrinos.

III. THE GROUP $SU(3)\times SU(2)\times U(1)$

The group $SU(3)\times SU(2)\times U(1)$ is the covering group for the groups $U(3)\times SU(2)$, $SU(3)\times U(2)$, and $S(U(3)\times U(2))$ in the same way that SU(2) is the covering group for SO(3). It is well known that each element of SO(3) corresponds to two elements of SU(2). Similarly, each element of $U(3)\times SU(2)$ corresponds to three elements of $SU(3)\times SU(2)\times U(1)$; each element of $SU(3)\times U(2)$ corresponds to two elements of $SU(3)\times SU(2)\times U(1)$; and each element of $S(U(3)\times U(2))$ corresponds to six elements of $SU(3)\times SU(2)\times U(1)$. The details will be presented shortly.

Because $SU(3)\times SU(2)\times U(1)$ is the covering group for the other three groups, any allowable representation of the other groups must correspond to a unique single-valued representation of $SU(3)\times SU(2)\times U(1)$. However, every representation of $SU(3)\times SU(2)\times U(1)$ may not be an allowable representation of the other groups, as we shall see in the following sections.

The most general representation of $SU(3)\times SU(2)\times U(1)$ has the following index structure and transformation law [under an arbitrary $SU(3)\times SU(2)\times U(1)$ transformation]:[9]

$$\psi'^{a'_1\cdots a'_j A'_1\cdots A'_p}_{b'_1\cdots b'_k B'_1\cdots B'_q}=\tilde{U}^{a'_1}_{a_1}\cdots\tilde{U}^{a'_j}_{a_j}\tilde{U}^{b_1}_{b'_1}\cdots\tilde{U}^{b_k}_{b'_k}\tilde{V}^{A'_1}_{A_1}\cdots\tilde{V}^{A'_p}_{A_p}\tilde{V}^{B_1}_{B'_1}\cdots\tilde{V}^{B_q}_{B'_q}[\exp(iyg'\varphi)]^s\psi^{a_1\cdots a_j A_1\cdots A_p}_{b_1\cdots b_k B_1\cdots B_q}\ . \tag{1}$$

The s in Eq. (1) is an arbitrary real number, y is a real nonzero normalization constant to be determined later, and g' is the hypercharge coupling constant. The $\tilde{U}$'s above are elements of SU(3); the $\tilde{U}$'s with an upper primed index are the inverses of those with a lower primed index:

$$\tilde{U}^{a'}_{a}\tilde{U}^{a}_{b'}=\delta^{a'}_{b'}\quad\text{and}\quad\tilde{U}^{a}_{a'}\tilde{U}^{a'}_{b}=\delta^{a}_{b}\quad\text{with } a,a'=1,2,3\ . \tag{2}$$

Likewise, the $\tilde{V}$'s are elements of SU(2) and the $\tilde{V}$'s with an upper primed index are the inverses of those with a lower primed index:

$$\tilde{V}^{A'}_{A}\tilde{V}^{A}_{B'}=\delta^{A'}_{B'}\quad\text{and}\quad\tilde{V}^{A}_{A'}\tilde{V}^{A'}_{B}=\delta^{A}_{B}\quad\text{with } A,A'=4,5\ . \tag{3}$$

As indicated above, lower-case latin indices run from 1 to 3, and capital latin indices run from 4 to 5 [to avoid confusion and facilitate later comparison to the SU(5) theory of Georgi and Glashow]. The tildes over U's and V's denote the condition that they have unit determinant; when dealing with unitary matrices with nonunit determinant, the tildes will be dropped. As is apparent from Eqs. (1)–(3), we will be using a tensor method in the fundamental representations of the unitary groups. It is well known that there is no real distinction between upper and lower SU(2) indices because one can raise and lower indices with the SU(2)-invariant two-dimensional Levi-Civita symbol.[10] However, we will make the distinction in (1) since we will be dealing shortly with the group U(2), which does not preserve the two-dimensional Levi-Civita symbol, and gives opposite U(1) charges to upper and lower indices.

A field with hypercharge Y must be transformed by a factor $\exp(ig'Y\varphi)$ under a $U(1)$ transformation, so that we see the field in (1) has hypercharge

$$Y(\psi)=ys(\psi) \ . \tag{4}$$

Since the group $SU(3)\times SU(2)\times U(1)$ allows s to take on any real value, we see that hypercharge is not necessarily quantized in general if the true group is $SU(3)\times SU(2)\times U(1)$.

To summarize, every allowable representation of the groups $U(3)\times SU(2)$, $SU(3)\times U(2)$, or $S(U(3)\times U(2))$ must be given by a representation of $SU(3)\times SU(2)\times U(1)$, with a transformation law of the form (1).

IV. THE GROUPS U(3)×SU(2) AND SU(3)×U(2)

The group $U(3)=[SU(3)\times U(1)]/Z_3$ results from identifying the three elements of $SU(3)\times U(1)$

$$(e^{i2\pi m/3}\tilde{U}\,{}^{a'}_{a},\ e^{-i2\pi m/3}e^{iyg'\varphi}),\quad m=0,1,2\ , \tag{5}$$

as a single element $U_a^{a'}$ of $U(3)$:

$$\begin{aligned} U_a^{a'}=\tilde{U}\,{}^{a'}_{a}e^{iyg'\varphi}&=(e^{i2\pi/3}\tilde{U}\,{}^{a'}_{a})(e^{-i2\pi/3}e^{iyg'\varphi})\\ &=(e^{i4\pi/3}\tilde{U}\,{}^{a'}_{a})(e^{-i4\pi/3}e^{iyg'\varphi})\ . \end{aligned} \tag{6}$$

If the true group is $U(3)\times SU(2)$, then the three group elements in (5) together with an element of SU(2) correspond to the same group element of $U(3)\times SU(2)$, and no physical field can have a representation which may tell the difference between the three group elements; otherwise the true group of the physical fields would not be $U(3)\times SU(2)$. Thus the only allowable representations of the group $U(3)\times SU(2)$ are those which transform the same under the three elements in (5). A field has the same transformation property under the three elements in (5) if and only if it is invariant under the following global $SU(3)\times SU(2)\times U(1)$ transformation:

$$\tilde{U}\,{}^{a'}_{a}=e^{i2\pi/3}\delta^{a'}_{a}\ ,\quad \tilde{V}\,{}^{A'}_{A}=\delta^{A'}_{A},\quad e^{iyg'\varphi}=e^{-i2\pi/3} \tag{7}$$

This is just using the fact that a subgroup H of a group G acts as the identity element for the quotient group G/H.[11] If we substitute (7) in (1) and demand that the field ψ be invariant, we then obtain the following condition on s:

$$e^{i2\pi(j-k-s)/3}=1\ , \tag{8}$$

which implies that s is quantized:

$$s=-(k-j)+3m,\quad m=0,\pm1,\pm2,\ldots\ . \tag{9}$$

It will turn out to be convenient to define the net 3-covariance of the field ψ, which is just the number of lower (covariant) minus the number of upper (contravariant) indices:

$$\mathrm{cov}_3(\psi)=k-j\ . \tag{10}$$

Using (4) and (10), we see that the quantization of hypercharge for the group $U(3)\times SU(2)$ is given by

$$Y(\psi)=y[-\mathrm{cov}_3(\psi)+3m],\quad m=0,\pm1,\pm2,\ldots\ . \tag{11}$$

With our conventions, hypercharge of the known quarks and leptons is quantized in units of $\frac{1}{6}$ (see Table I), so that in order to have (11) agree with the known assignments we would have to pick $y=\pm\frac{1}{6}$ in (11). The hypercharge quantization condition (11) only depends on the way the field ψ transforms under SU(3), so that the hypercharges of the color singlets (leptons) are quantized, but unexplained and somewhat abitrary. We then conclude that the group $U(3)\times SU(2)$ is not a satisfactory choice for the true symmetry group.

The group $U(2)=[SU(2)\times U(1)]/Z_2$ results from identifying the two elements of $SU(2)\times U(1)$

$$(e^{-i2\pi n/2}\tilde{V}\,{}^{A'}_{A},\ e^{i2\pi n/2}e^{iyg'\varphi})\quad n=0,1 \tag{12}$$

as a single element of U(2):

$$V_A^{A'}=\tilde{V}\,{}^{A'}_{A}e^{iyg'\varphi}=(-\tilde{V}\,{}^{A'}_{A})(-e^{iyg'\varphi})\ . \tag{13}$$

By the same argument as before, if the true group is $SU(3)\times U(2)$, then no representation must be able to distinguish between the two group elements in (12). Thus its transformation law must be the same for both elements. This condition is equivalent to the invariance of the field under the following global $SU(3)\times SU(2)\times U(1)$ transformation:

$$\tilde{U}\,{}^{a'}_{a}=\delta^{a'}_{a},\quad \tilde{V}\,{}^{A'}_{A}=e^{-i2\pi/2}\delta^{A'}_{A},\quad e^{iyg'\varphi}=e^{i2\pi/2}\ . \tag{14}$$

As before, upon substituting (14) in (1), we obtain the following condition[12] on s:

$$e^{i2\pi(q-p+s)/2}=1\ , \tag{15}$$

which yields a quantization condition for s:

$$s=-(q-p)+2n,\quad n=0,\pm1,\pm2,\ldots\ . \tag{16}$$

Analogously to (10), we define the 2-covariance

$$\mathrm{cov}_2(\psi)=q-p\ . \tag{17}$$

TABLE I. $SU(3)\times SU(2)\times U(1)$ assignments of a family.

Field	Color	Weak Isospin I^w	I_3^w	Y	$Q=I_3^w+Y$
LH quark doublet $\binom{u}{d}$	triplet	$\frac{1}{2}$	$+\frac{1}{2}$ $-\frac{1}{2}$	$+\frac{1}{6}$	$+\frac{2}{3}$ $-\frac{1}{3}$
LH anti-"down" quark $\bar{d}$	antitriplet	0	0	$+\frac{1}{3}$	$+\frac{1}{3}$
LH anti-"up" quark $\bar{u}$	antitriplet	0	0	$-\frac{2}{3}$	$-\frac{2}{3}$
LH lepton doublet $\binom{\nu}{e}$	singlet	$\frac{1}{2}$	$+\frac{1}{2}$ $-\frac{1}{2}$	$-\frac{1}{2}$	0 -1
LH "positron" $\bar{e}$	singlet	0	0	$+1$	$+1$
Higgs doublet $\binom{\phi_+}{\phi_0}$	singlet	$\frac{1}{2}$	$+\frac{1}{2}$ $-\frac{1}{2}$	$+\frac{1}{2}$	$+1$ 0

With this definition, and using (4), we see the quantization of hypercharge for the group $SU(3)\times U(2)$ is given by

$$Y(\psi)=y[-\mathrm{cov}_2(\psi)+2n]\ , \quad n=0,\pm1,\pm2.\ldots \tag{18}$$

To agree with the observed hypercharges of the quarks and leptons, we would have to take $y=\pm\frac{1}{6}$ again. However, this quantization condition sheds no light on the different hypercharge assignments of the quark and lepton doublets, nor on the different hypercharges of the left-handed antiquarks. On the other hand, if there were no quarks, and we took $y=\pm\frac{1}{2}$, then we would have a satisfactory explanation of the different hypercharges of the leptons. As both quarks and leptons exist in nature, we see that the group $SU(3)\times U(2)$ gives an unsatisfactory explanation of the various hypercharges of the elementary fermions.

In summary, we see that the groups $U(3)\times SU(2)$ and $SU(3)\times U(2)$ require weak hypercharge quantization, but do not seem to offer much insight into the various hypercharges of the quarks and leptons and are thus unsatisfactory choices for the true symmetry group.

V. THE GROUP $S(U(3)\times U(2))$

The group $S(U(3)\times U(2))=[SU(3)\times SU(2)\times U(1)]/Z_6$ is obtained by identifying the following six elements of $SU(3)\times SU(2)\times U(1)$:

$$(e^{i2\pi p/3}\tilde{U}_a^{\ a'},\ e^{-i2\pi p/2}\tilde{V}_A^{\ A'},\ e^{-i2\pi p/6}e^{iyg'\varphi})\ ,$$
$$p=0,1,2,3,4,5\ . \tag{19}$$

If the true symmetry group is $S(U(3)\times U(2))$, then the six $SU(3)\times SU(2)\times U(1)$ group elements in (19) correspond to the same element of the group $S(U(3)\times U(2))$, and any allowable representation of $S(U(3)\times U(2))$ must have the same transformation property under all six elements in (19). This is equivalent to the condition that allowable representations of $S(U(3)\times U(2))$ be invariant under the following global $SU(3)\times SU(2)\times U(1)$ transformation:

$$\tilde{U}_a^{\ a'}=e^{i2\pi/3}\delta_a^{a'},\quad \tilde{V}_A^{\ A'}=e^{-i2\pi/2}\delta_A^{A'}\ ,$$
$$e^{iyg'\varphi}=e^{-i2\pi/6}\ , \tag{20}$$

which is just the statement that the group Z_6 must act as the identity of the group $[SU(3)\times SU(2)\times U(1)]/Z_6$, since the transformation in (20) is a generator of Z_6 transformations.

Upon inserting (20) in (1) and demanding that the field ψ be invariant, we obtain the following condition on s:

$$\exp\left[-\frac{i2\pi[s+2(k-j)-3(q-p)]}{6}\right]=1\ , \tag{21}$$

which implies that s is quantized:

$$s=-2(k-j)+3(q-p)+6h,\quad h=0,\pm1,\pm2,\ldots\ . \tag{22}$$

Using (4), (10), and (17), the quantization of hypercharge for the group $S(U(3)\times U(2))$ is given by

$$Y(\psi)=y[-2\,\mathrm{cov}_3(\psi)+3\,\mathrm{cov}_2(\psi)+6h]\ ,$$
$$h=0,\pm1,\pm2\ ,\ldots\ . \tag{23}$$

In order that the quantization condition above yield hypercharges in units of $\frac{1}{6}$, we must take $y=\pm\frac{1}{6}$. We make the convention that $y=+\frac{1}{6}$. Equation (23) then becomes

$$Y(\psi)=-\tfrac{1}{3}\mathrm{cov}_3(\psi)+\tfrac{1}{2}\mathrm{cov}_2(\psi)+h\ ,$$
$$h=0,\pm1,\pm2,\ldots\ . \tag{24}$$

If we insert the expression (22) for s and $y=+\frac{1}{6}$ into (1), after a little manipulation we obtain the following transformation law for the field ψ:

$$\psi'^{a'_1\cdots a'_j A'_1\cdots A'_p}_{b'_1\cdots b'_k B'_1\cdots B'_q}=U^{a'_1}_{a_1}\cdots U^{a'_j}_{a_j}U^{b_1}_{b'_1}\cdots U^{b_k}_{b'_k}V^{A'_1}_{A_1}\cdots V^{A'_p}_{A_p}V^{B_1}_{B'_1}\cdots V^{B_q}_{B'_q}(\det U)^h\psi^{a_1\cdots a_j A_1\cdots A_p}_{b_1\cdots b_k B_1\cdots B_q}\ , \tag{25}$$

where

$$U_a^{a'}=e^{ig'\varphi/3}\tilde{U}\,{}_a^{a'},\quad V_A^{A'}=e^{-ig'\varphi/2}\tilde{V}\,{}_A^{A'}\ ,$$
$$\det U=\det(U_a^{a'})=(\det V)^{-1}=[\det(V_A^{A'})]^{-1}=e^{ig'\varphi}\ , \tag{26}$$

and the inverses are defined by taking relations (2) and (3) to hold without the tildes.

Note that S(U(3)×U(2)) is defined to be the group which consists of an element of U(3) and an element of U(2) such that the determinants cancel:

$$\mathrm{S(U(3)\times U(2))}\equiv\{U\in \mathrm{U}(3),\ V\in \mathrm{U}(2)\,|\,\det U\det V=1\}\ . \tag{27}$$

From (26) we see that any element of $[\mathrm{SU(3)\times SU(2)\times U(1)}]/Z_6$ is an element of S(U(3)×U(2)); it is also true that any element of S(U(3)×U(2)) is an element of $[\mathrm{SU(3)\times SU(2)\times U(1)}]/Z_6$ and may be written in the form (26), so that the two groups are equal. For example, an element of S(U(3)×U(2)) that is not obviously of the form (26) is

$$U_a^{a'}=e^{i2\pi/5}\delta_a^{a'},\quad V_A^{A'}=e^{i2\pi/5}\delta_A^{A'},\quad \det U\det V=1\ . \tag{28}$$

However, the choices

$$\tilde{U}\,{}_a^{a'}=\delta_a^{a'},\quad \tilde{V}\,{}_A^{A'}=\delta_A^{A'},\quad g'\varphi=\frac{36\pi}{5} \tag{29}$$

inserted in (26) yield (28). This result can be used in the proof of equality of the two groups. The element in (28) actually generates the Z_5 center of SU(5) under the embedding discussed in Sec. VII below.

VI. S(U(3)×U(2)) TENSOR REPRESENTATIONS OF THE ELEMENTARY FERMIONS

It is natural to define S(U(3)×U(2)) tensors to be quantities which transform as (25) with $h=0$. Quantities which transform as (25) with h having a nonzero integral value are defined to be S(U(3)×U(2)) tensor densities[13] of weight h. For S(U(3)×U(2)) tensor representations ($h=0$), Eq. (24) means the following.

(1) Each upper U(3) index carries a hypercharge of $+\frac{1}{3}$.

(2) Each lower U(3) index carries a hypercharge of $-\frac{1}{3}$.

(3) Each upper U(2) index carries a hypercharge of $-\frac{1}{2}$.

(4) Each lower U(2) index carries a hypercharge of $+\frac{1}{2}$.

With the above rules, finding the hypercharge of a tensor representation of S(U(3)×U(2)) is a matter of just counting indices. The group U(3) describes a hyperstrong force, and the group U(2) describes a hyperweak force. In a sense, the U(1) belongs to both the strong and weak interactions in a very symmetrical manner.

It is a simple exercise to show that the leptons, quarks, and Higgs doublet may be represented as S(U(3)×U(2)) tensors of rank 1 or 2. These representations are given in Table II below.

A few remarks about the representation given in Table II are in order. Note that the lepton doublet is represented as a $\bar{2}$ of U(2) and the quark doublet as a 2. Under SU(2) the $\bar{2}$ and 2 are equivalent, but under U(2) they carry opposite hypercharges. The inequivalence of the $\bar{2}$ and 2 under U(2) is due to the fact that the two-dimensional Levi-Civita symbol is invariant under SU(2) but not under U(2) (it is a tensor *density*, as is well known from ten-

TABLE II. S(U(3)×U(2)) tensor representations of the quarks, leptons, and Higgs doublet.

Field	S(U(3)×U(2)) representation	cov_3	cov_2	$Y=-\frac{1}{3}\mathrm{cov}_3+\frac{1}{2}\mathrm{cov}_2$
LH anti-"down" quark	$\bar{d}^a$	-1	0	$+\frac{1}{3}$
LH lepton doublet	$l^A=(e,-\nu)$	0	-1	$-\frac{1}{2}$
LH anti-"up" quark	$\bar{u}_{[ab]}=\epsilon_{abc}\bar{u}^c$	2	0	$-\frac{2}{3}$
LH quark doublet	$q_{Aa}=\begin{bmatrix}u_a\\ d_a\end{bmatrix}$	1	1	$+\frac{1}{6}$
LH "positron"	$\bar{e}_{[AB]}=\bar{e}\epsilon_{AB}$ or $\bar{e}^{[abc]}=\bar{e}\epsilon^{abc}$	0 / -3	2 / 0	$+1$
Higgs doublet	$\phi_A=\begin{bmatrix}\phi_+\\ \phi_0\end{bmatrix}$	0	1	$+\frac{1}{2}$

sor calculus).

Note also that it is necessary to multiply the LH "positron" field by a Levi-Civita symbol to be able to write it as a tensor; otherwise it may be written as a scalar density of weight $+1$. It is not hard to show that any allowable tensor density may be represented as a tensor by appropriate outer multiplication with one or more Levi-Civita symbols, so that with no loss of generality we could only consider $S(U(3)\times U(2))$ tensors.

The $+\frac{1}{6}$ hypercharge of the quark doublet follows simply from the fact that $+\frac{1}{6}=-\frac{1}{3}+\frac{1}{2}$. The reasons that the LH antiquarks have different hypercharges comes from the fact that under SU(3) there are two equivalent ways to write a $\mathbf{\bar{3}}$ (or a $\mathbf{3}$) by use of the SU(3) invariant three-dimensional Levi-Civita symbol. However, under the group U(3) these two representations are inequivalent (the hypercharges differ by a factor of -2) since the three-dimensional Levi-Civita symbol is not invariant under U(3) transformations (it is a tensor *density*).

We thus see that the group $S(U(3)\times U(2))$ provides simple reasons for the various hypercharges of the quarks and leptons, as opposed to the group $SU(3)\times SU(2)\times U(1)$, which is mute on the subject of hypercharge quantization and assignments.

As for the gauge bosons, it is well known that the adjoint representation of a given group transforms under the symmetry group with its center quotiented out. As the center of $SU(3)\times SU(2)\times U(1)$ is Z_6, we see that the gauge bosons also transform as $S(U(3)\times U(2))$ tensors.

We thus come to the conclusion that every physical field may be represented as an $S(U(3)\times U(2))$ tensor of appropriate rank.

VII. COMPARISON TO THE SU(5) THEORY OF GEORGI AND GLASHOW (REF. 14)

The group $S(U(3)\times U(2))$ may be embedded in SU(5) in the following way:

$$W=\begin{bmatrix} U & 0 \\ 0 & V \end{bmatrix}\in SU(5) \quad \text{since } \det W=\det U\ \det V=1\ . \tag{30}$$

The generators of $S(U(3)\times U(2))$ may be written as 5×5 matrices. We would have the eight SU(3) generators in the upper left-hand corner with zeros everywhere else, the three SU(2) generators in the lower right-hand corner with zeros everywhere else, and the additional 5×5 traceless diagonal generator:

$$Y=\begin{bmatrix} -\frac{1}{3} & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{3} & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{3} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} \end{bmatrix} . \tag{31}$$

The 24th generator of SU(5) is usually taken to be proportional to (31).

The LH anti-"down" quark and the lepton doublet may be embedded in the $\mathbf{\bar{5}}$ of SU(5):

$$\psi^i=(\bar{d}^{\,a}\ l^A)=(\bar{d}^{\,1}\ \bar{d}^{\,2}\ \bar{d}^{\,3}\ l^4\ l^5)=(\bar{d}^{\,1}\ \bar{d}^{\,2}\ \bar{d}^{\,3}\ e\ -\nu)\ , \tag{32}$$

where in this section lower case latin indices from the middle of the alphabet run from 1 to 5.

The LH anti-"up" quark, the LH quark doublet, and the LH "positron" may be embedded in the $\mathbf{10}$ of SU(5):

$$\chi_{ij}=-\chi_{ji} \quad \text{and}$$
$$\chi_{ab}=\bar{u}_{[ab]},\quad \chi_{aA}=q_{aA},\quad \chi_{AB}=\bar{e}_{[AB]}\ , \tag{33}$$

or more explicitly:

$$\chi=\begin{bmatrix} 0 & \bar{u}^{\,3} & -\bar{u}^{\,2} & u^1 & d^1 \\ -\bar{u}^{\,3} & 0 & \bar{u}^{\,1} & u^2 & d^2 \\ \bar{u}^{\,2} & -\bar{u}^{\,1} & 0 & u^3 & d^3 \\ -u^1 & -u^2 & -u^3 & 0 & \bar{e} \\ -d^1 & -d^2 & -d^3 & -\bar{e} & 0 \end{bmatrix} . \tag{34}$$

Since the group $S(U(3)\times U(2))$ does not mix leptons and quarks, perturbatively there is no proton decay, and there are also still three independent coupling constants.

VIII. SPONTANEOUS SYMMETRY BREAKING OF S(U(3)×U(2)) TO AN ELECTROSTRONG U(3)

When the Higgs field has a nonzero vacuum expectation value (VEV) proportional to $\binom{0}{1}$, the unbroken generators are those of SU(3) and the electric charge generator:

$$\begin{aligned} Q=Y+I_w^3&=\mathrm{diag}(-\tfrac{1}{3},-\tfrac{1}{3},-\tfrac{1}{3},\tfrac{1}{2},\tfrac{1}{2}) \\ &\quad+\mathrm{diag}(0,0,0,\tfrac{1}{2},-\tfrac{1}{2}) \\ &=\mathrm{diag}(-\tfrac{1}{3},-\tfrac{1}{3},-\tfrac{1}{3},1,0)\ . \end{aligned} \tag{35}$$

The unbroken SU(3) generators and the electric charge generator above generate the following subgroup of SU(5):

$$\begin{bmatrix} & & & 0 & 0 \\ & U & & 0 & 0 \\ & & & 0 & 0 \\ 0 & 0 & 0 & (\det U)^{-1} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad \text{where } U\in U(3)\ . \tag{36}$$

The above group is just $S(U(3)\times U(1))$, which is isomorphic to U(3) by taking the upper left-hand corner. Therefore the unbroken group[15] is an electrostrong U(3), and by the results of Sec. IV the electric charge is quantized in units of $\frac{1}{3}e$ [note that this already followed from hypercharge quantization and the definition (35) of electric charge].

We see that the group in (36) would treat the 5th component of a contravariant or covariant U(2) vector as a scalar, and leave invariant the hyperstrong U(3) index structure. It turns out that all the known quarks and leptons may be written as either hyperstrong U(3) tensors with no hyperweak U(2) indices, or as 5th components of hyperweak U(2) vectors with possibly other hyperstrong

TABLE III. Electrostrong tensor representations of the elementary fermions.

Field	Electrostrong U(3) representation	Relation to S(U(3)×U(2)) representation
LH "down" quark	d_a	$d_a = q_{a5}$
LH "up" quark	$u^{[ab]}$	$u^{[ab]} = -q^{[ab]5} \equiv -\epsilon^{abc}\epsilon^{5B}q_{cB} = -\epsilon^{abc}\epsilon^{54}q_{c4} = \epsilon^{abc}u_c$
LH anti-"down" quark	$\bar{d}^{\,a}$	$\bar{d}^{\,a}$
LH anti-"up" quark	$\bar{u}_{[ab]}$	$\bar{u}_{[ab]}$
LH "electron"	$e_{[abc]}$	$e_{[abc]} = l_{[abc]5} \equiv \epsilon_{abc}\epsilon_{5A}l^A = \epsilon_{abc}\epsilon_{54}l^4 = \epsilon_{abc}(+1)e = e\epsilon_{abc}$
LH "positron"	$\bar{e}^{\,[abc]}$	$\bar{e}^{\,[abc]} \equiv -\frac{1}{2}\epsilon^{abc}\epsilon^{AB}\bar{e}_{[AB]} = -\frac{1}{2}\epsilon^{abc}\epsilon^{AB}\bar{e}\epsilon_{AB} = \bar{e}\epsilon^{abc}$
LH neutrino	ν	$\nu = -l^5$

indices. The resulting electrostrong U(3) index structure of the elementary fermions is presented in Table III.

We have already remarked that the three- and two-dimensional Levi-Civita symbols are not invariant S(U(3)×U(2)) tensors. However, from the properties of Levi-Civita symbols,[16] it may be shown that the following two outer products of the three- and two-dimensional Levi-Civita symbols are invariant S(U(3)×U(2)) tensors:

$$\epsilon_{abc}\epsilon_{AB} \quad \text{and} \quad \epsilon^{abc}\epsilon^{AB} \; . \tag{37}$$

These symbols are used extensively in Table III to derive equivalent representations of the S(U(3)×U(2)) tensors from which the electrostrong index structure may be easily deduced.

For electrostrong U(3) tensors, we have the simple rule for the electric charge of the representation: each upper electrostrong U(3) index carries an electric charge of $+\frac{1}{3}$ and each lower electrostrong U(3) index carries an electric charge of $-\frac{1}{3}$.

IX. ANOMALIES

There have been arguments in the literature for charge quantization based on anomaly cancellation[17–22] rather than on group-theoretical considerations. It is the purpose of this section to combine the constraints imposed by anomaly cancellation with the hypercharge quantization condition (derived in Sec. V) to show that the ratios of the hypercharges of a family are then uniquely determined.

We assume that we have a family of 15 LH fermions (no RH neutrino) with the following SU(3)×SU(2)×U(1) representations:

$$\begin{aligned} &(3,2,Y(q)),\quad (\bar{3},1,Y(\bar{d})),\quad (\bar{3},1,Y(\bar{u}))\;,\\ &(1,2,Y(l)),\quad (1,1,Y(\bar{e}))\;, \end{aligned} \tag{38}$$

where we are using the same symbols for the fermions as before. The Witten global SU(2) anomaly is automatically satisfied because the family in (38) contains an even number of SU(2) doublets.[23]

We then impose the vanishing of the gauge and mixed gauge-gravitational anomalies[17–22] within the family:

$$\begin{aligned} &[\mathrm{SU}(3)]^2\mathrm{U}(1): && 2Y(q)+Y(\bar{d})+Y(\bar{u})=0;\\ &[\mathrm{SU}(2)]^2\mathrm{U}(1): && Y(l)+3Y(q)=0;\\ &[\mathrm{U}(1)]^3: && 6Y(q)^3+3Y(\bar{d})^3+3Y(\bar{u})^3+2Y(l)^3+Y(\bar{e})^3=0;\\ &[\mathrm{graviton}]^2\mathrm{U}(1): && 3(2Y(q)+Y(\bar{d})+Y(\bar{u}))+2Y(l)+Y(\bar{e})=0. \end{aligned} \tag{39}$$

The four equations above have two types of nontrivial solutions:[18–22]

$$\begin{aligned} &Y(q)\neq 0,\quad Y(\bar{d})=2Y(q),\quad Y(\bar{u})=-4Y(q)\;,\\ &Y(l)=-3Y(q),\quad Y(\bar{e})=6Y(q)\;; \end{aligned} \tag{40a}$$

$$Y(q)=Y(l)=Y(\bar{e})=0,\quad Y(\bar{d})=-Y(\bar{u})\neq 0\;. \tag{40b}$$

In solution (40a), we have defined $\bar{d}$ to be the field with the smaller absolute hypercharge of the pair $\bar{d}$ and $\bar{u}$.

Solution (40a) is proportional to the standard hypercharge assignments of a family (see Table I), and is compatible with the hypercharge quantization condition for the group S(U(3)×U(2)) (see Secs. V and VI and Table II). However, the solution (40b) is not compatible with the S(U(3)×U(2)) hypercharge quantization condition, which is the main result of this section.

The incompatibility of solution (40b) and the hypercharge quantization condition is most easily seen by considering the fields $\bar{d}$ and $\bar{u}$. By use of the Levi-Civita symbols, they may be represented as tensors or tensor densities with the following covariances:

$$\mathrm{cov}_2(\bar{d})=\mathrm{cov}_2(\bar{u})=0,\quad \mathrm{cov}_3(\bar{d})=-1,\quad \mathrm{cov}_3(\bar{u})=2\ , \tag{41}$$

exactly as in Table II. The hypercharge quantization condition [Eq. (23)] then tells us that the hypercharges are given by

$$Y(\bar{d})=y(-2\,\mathrm{cov}_3(\bar{d})+3\,\mathrm{cov}_2(\bar{d})+6h_1)=y(2+6h_1)\ ,\quad h_1\in Z\ ,$$
$$Y(\bar{u})=y(-2\,\mathrm{cov}_3(\bar{u})+3\,\mathrm{cov}_2(\bar{u})+6h_2)=y(-4+6h_2)\ ,\quad h_2\in Z\ . \tag{42}$$

Together with (40b), this implies that (using the fact that y is nonzero)

$$h_1+h_2=\tfrac{1}{3} \tag{43}$$

which is impossible since h_1 and h_2 are integers. Solution (40b) is thereby eliminated on group-theoretical grounds. We thus see that anomaly cancellation together with the S(U(3)×U(2)) hypercharge quantization condition uniquely specifies the ratios of the hypercharges within a family.

If we normalize hypercharge as in Sec. V, we see that anomaly cancellation and the S(U(3)×U(2)) hypercharge quantization condition allow the following hypercharge assignments for the quark doublet in a family:

$$Y(q)=\frac{6k+1}{6},\quad k\in Z\ , \tag{44}$$

with the hypercharges of the other members of the family given by (40a). There is no *a priori* reason why all three families in the standard model must have $k=0$. Some other condition is needed to eliminate the nonzero k families, or one may impose a "generations as copies" rule. Clearly, quark mixing is only possible if all three families have the same value of k.

X. SUMMARY AND DISCUSSION

We have seen that of the four viable Lie groups with the same Lie algebra as SU(3)×SU(2)×U(1), the group S(U(3)×U(2)) provides the best explanation of hypercharge and electric charge quantization, and offers the most insight into the different hypercharge assignments of the elementary fermions.

We have also seen how the index structure of the S(U(3)×U(2)) tensor representations makes it obvious that SU(5) can contain the standard model, and the fitting of the elementary fermions into the $\bar{\mathbf{5}}+\mathbf{10}$ of SU(5) seems less miraculous.

We have seen that the unbroken symmetry group of nature is not $\mathrm{SU}(3)_{\mathrm{color}}\times\mathrm{U}(1)_{\mathrm{em}}$, but an electrostrong $\mathrm{U}(3)_{\mathrm{cem}}=[\mathrm{SU}(3)_{\mathrm{color}}\times\mathrm{U}(1)_{\mathrm{em}}]/Z_3$ for which electric charge must be quantized.

Lastly, we have seen how the S(U(3)×U(2)) hypercharge quantization condition together with the constraints imposed by anomaly cancellation uniquely determine the hypercharge ratios within a family.

Even though the group S(U(3)×U(2)) is a convincing candidate for the true symmetry group of nature, it gives the same perturbative quantum field theory as the group SU(3)×SU(2)×U(1). Should non-perturbative effects be different for the two groups, observation of such effects could be used to experimentally determine the true symmetry group.

ACKNOWLEDGMENTS

This work could not have been accomplished if not for the many discussions on this topic with Doug Eardley of the Institute of Theoretical Physics at Santa Barbara. I would like to thank Mark Srednicki of UCSB for reading through preliminary drafts and giving helpful criticisms. I also wish to thank L. O'Raifeartaigh for an enlightening conversation, and John Bisognano for fruitful correspondence. This work was partially supported by NSF Grant No. PHY85-06686.

---

*Electronic address: JCHucks@Voodoo.bitnet.

[1]R. Gilmore, *Lie Groups, Lie Algebras, and Some of Their Applications* (Wiley, New York, 1974). See pp. 108–110, especially p. 110. The key result is that if $D_1,\ldots,D_r$ is the complete set of discrete invariant subgroups of the simply connected covering group G [in our case $G=\mathrm{SU}(3)\times\mathrm{SU}(2)\times R$] then the complete set of Lie groups with the same Lie algebra (as G) is given by $G/D_1,\ldots,G/D_r$. See also Ref. 2 below.

[2]L. Michel, in *Group Theoretical Concepts and Methods in Elementary Particle Physics,* edited by F. Gürsey (Gordon and Breach, New York, 1964). On pp. 140 and 141 there is a clear exposition of the fact that Lie groups which share the same Lie algebra are quotients of the simply connected covering Lie group by discrete subgroups of its center. On pp. 142–145, the Lie groups with the same Lie algebra as SU(2)×U(1) are discussed (see especially p. 144).

[3]H. B. Nielsen and N. Brene, Phys. Lett. B **223**, 399 (1989). The authors also discuss the uniqueness of the group S(U(3)×U(2)) based upon characteristic group properties.

[4]L. O'Raifeartaigh, *Group Structure of Gauge Theories* (Cambridge University Press, Cambridge, England, 1986). On pp. 53 and 54 the true groups of representations of SU(n) are discussed.

[5]On p. 55 of O'Raifeartaigh, *Group Structure of Gauge Theories* (Ref. 4), it is stated that for a local $G=\mathrm{SU}(p)\times\mathrm{SU}(q)\times\mathrm{U}(1)$ gauge group (with p and q unequal and prime) the possible global groups are G, $\mathrm{U}(p)\times\mathrm{SU}(q)=G/Z_p$, $\mathrm{SU}(p)\times\mathrm{U}(q)=G/Z_q$, and $\mathrm{S}(\mathrm{U}(p)\times\mathrm{U}(q))=G/Z_{p+q}$. However, as corroborated in a private communication with Dr. O'Raifeartaigh, we actually have $\mathrm{S}(\mathrm{U}(p)\times\mathrm{U}(q))=G/Z_{pq}$.

[6]Please refer to Chapters 5 and 9 of O'Raifeartaigh, *Group Structure of Gauge Theories* (Ref. 4), especially pp. 56 and

107.
[7]P. D. B. Collins, A. D. Martin, and E. J. Squires, *Particle Physics and Cosmology* (Wiley, New York, 1989). For an example of this practice, see pp. 158 and 159. If one wishes to put different fermions into the same representation of a grand-unified-theory group, they must all be in the same representation of the Lorentz group (such as all LH fields) to preserve Lorentz invariance.
[8]K. Huang, *Quarks, Leptons & Gauge Fields* (World Scientific, Singapore, 1982). See pp. 117–119 for a more complete discussion of Kobayashi-Maskawa mixing of quark flavors (in this reference the u, c, and t quarks are mixed).
[9]S. Coleman, *Aspects of Symmetry* (Cambridge University Press, Cambridge, England, 1985). See pp. 5–8 for a discussion of the representations of SU(3) and SU(2).
[10]See pp. 6 and 7 of Coleman, *Aspects of Symmetry* (Ref. 9).
[11]See p. 5 of O'Raifeartaigh, *Group Structure of Gauge Theories* (Ref. 4).
[12]For a similar condition on the U(1) charges of U(2), see pp. 144 and 145 of Michel, *Group Theoretical Concepts and Methods in Elementary Particle Physics* (Ref. 2).
[13]D. Lovelock and H. Rund, *Tensors, Differential Forms, and Variational Principles* (Wiley, New York, 1975; Dover, New York, 1989). Chapter 4 completely covers Levi-Civita symbols and tensor densities. This is a very thorough reference on tensor calculus.
[14]For example, see pp. 167–181 of Collins, Martin, and Squires, *Particle Physics and Cosmology* (Ref. 7), for a more complete discussion of the SU(5) theory of Georgi and Glashow (with slightly different conventions from this article).
[15]On p. 107 of O'Raifeartaigh, *Group Structure of Gauge Theories* (Ref. 4), it is concluded that the unbroken symmetry group of nature is actually U(3). See also, e.g., W. Nahm, in *Theory and Detection of Magnetic Monopoles in Gauge Theories*, edited by N. Craigie (World Scientific, Singapore, 1986).
[16]See pp. 113 and 114 of Lovelock and Rund, *Tensors, Differential Forms, and Variational Principles* (Ref. 13), for detailed properties of Levi-Civita symbols.
[17]C. Q. Geng and R. E. Marshak, Phys. Rev. D **39**, 693 (1989).
[18]J. A. Minahan, P. Ramond, and R. C. Warner, Phys. Rev. D **41**, 715 (1990). This is a comment on Ref. 17.
[19]C. Q. Geng and R. E. Marshak, Phys. Rev. D **41**, 717 (1990). This is a reply to Ref. 18.
[20]X.-G. He, G. C. Joshi, and R. R. Volkas, Phys. Rev. D **41**, 278 (1990).
[21]C. Q. Geng, Phys. Rev. D **41**, 1292 (1990).
[22]R. Foot, G. C. Joshi, H. Lew, and R. R. Volkas, Mod. Phys. Lett. A **5**, 2721 (1990). This article contains many references on the subject of anomaly cancellation and charge quantization, as well as a review of the different approaches taken in the literature.
[23]E. Witten, Phys. Lett. **117B**, 324 (1982).