Gauge invariance of the phase diagram in quenched QED

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We incorporate an approximation to the full QED vertex containing both longitudinal and transverse pieces into the Schwinger-Dyson equation for the electron propagator in quenched QED, from which we find solutions to the dynamical electron mass $M(p)$. Throughout we use a general covariant gauge, and find that the resulting four-Fermi coupling versus gauge-coupling diagram separating the chiral-symmetric and dynamically broken phases of quenched massless QED is gauge invariant.

I. INTRODUCTION

The possible existence of a strong-coupling phase in QED, indicated by lattice calculations,[1] was suggested by Miransky[2] as a solution to the Landau ghost problem. The relevance of four-fermion operators to this nonperturbative phase in QED was pointed out by Bardeen, Leung, and Love.[3] By including four-fermion interactions in massless QED, Bardeen *et al.* showed that the running of the four-fermion coupling makes possible the transition from the chiral-symmetric to the chiral-symmetry-breaking phase. They examined the fermion propagator Schwinger-Dyson equation of a truncated model of QED known as quenched ladder QED; the full vertex and the full photon propagator are replaced by their bare counterparts, respectively, in the Landau gauge (the only gauge for which the approximation works). The resulting Schwinger-Dyson equation for the dynamical fermion mass $M(p)$ is only expected to be valid in the ultraviolet regime. By solving for $M(p)$, a critical curve of the four-fermion coupling versus the gauge coupling separating the massless (chiral-symmetric) and dynamically massive (chiral-symmetry-broken) phases of massless QED is found.[4] One expects the criticality relation between the four-fermion coupling and the gauge coupling to be gauge independent; a fact that is not obviated by the ladder approximation since other non-Landau gauges are not available for reproducing the same results.

An alternative method exists for solving the fermion propagator $S(p)$, Schwinger-Dyson equation that enables the derivation of the criticality curve in a general covariant gauge. To solve the Schwinger-Dyson equation for $S(p)$ we use a variation on the gauge technique, which differs mainly in its nonlinearization of the Schwinger-Dyson equation. The gauge technique is commonly implemented utilizing the Lehmann spectral representation, but here we will employ scalar functions. In the approach we pursue, the fermion propagator is written in terms of scalar functions to be determined by solving the fermion propagator Schwinger-Dyson equation. We use a bare photon propagator in a general covariant gauge, corresponding to quenched QED. The longitudinal part of the vertex appearing in the fermion propagator Schwinger-Dyson equation is found in terms of the same scalar functions by solving the Ward identity that relates the vertex to the fermion propagator. The transverse part of the vertex has zero divergence and hence remains undetermined. Omitting the transverse piece makes sense only in the infrared regime where it vanishes linearly in gauge-particle momentum. In the ultraviolet regime the transverse vertex has a significant contribution without which the fermion propagator Schwinger-Dyson equation loses multiplicative renormalizability. With only the longitudinal vertex appearing in the fermion propagator Schwinger-Dyson equation, overlapping divergences result that require the inclusion of the ultraviolet dominant part of the transverse vertex.

Inserting the longitudinal vertex in the photon proper self-energy $\Pi_{\mu\nu}(p)$ destroys the gauge invariance of $\Pi_{\mu\nu}(p)$ at intermediate and high energies, a problem again caused by the lack of a transverse vertex.[5] Using renormalizability and the gauge invariance of the photon proper self-energy in the ultraviolet regime, we construct an effective transverse vertex. Of course, our transverse vertex is only approximate at intermediate regimes. Incorporating the transverse vertex into the fermion propagator Schwinger-Dyson equation removes all gauge dependence from the two scalar functions that make up $S(p)$ leaving behind two equations identical to the ladder approximation equations, and thus yielding the same criticality curve as found in Ref. 4.

This paper consists of four parts. First we solve the Ward identity for the longitudinal vertex in terms of two scalar functions, and use the results to derive integral equations for the two scalar functions in a general covariant gauge. Second we derive a transverse vertex that removes all gauge parameter dependence from the photon proper self-energy, followed by a construction of the remainder of the transverse vertex necessary to reinstate multiplicative renormalizability to the fermion Schwinger-Dyson equation. In Sec. IV four-fermion interactions are introduced and the solution to the fermion propagator Schwinger-Dyson equation is presented. We end the paper with our conclusions.

II. THE LONGITUDINAL VERTEX

The Schwinger-Dyson equation for the inverse fermion propagator $S^{-1}(p)$ is

$$S^{-1}(p)=\not p-m_0-ig^2\int\frac{d^4k}{(2\pi)^4}\gamma_\mu S(k)\Gamma_\nu(k,p)D^{\mu\nu}(p-k)\ , \tag{2.1}$$

where

$$D_{\mu\nu}(k)=-\frac{1}{k^2}\left[g_{\mu\nu}-(1-\lambda)\frac{k_\mu k_\nu}{k^2}\right], \tag{2.2}$$

and Γ_ν is unknown. λ is the gauge parameter of a general covariant gauge. By writing $S^{-1}(p)$ in terms of scalar functions

$$S^{-1}(p)=A(p)\not p-B(p)\ , \tag{2.3}$$

we can solve the Ward identity for Γ_ν

$$(p-k)_\nu\Gamma^\nu(p,k)=S^{-1}(p)-S^{-1}(k)\ , \tag{2.4}$$

to find the longitudinal part of the quark-quark-gluon vertex Γ_ν^L, in terms of $A(p)$ and $B(p)$,

$$\Gamma_\nu^L(p,k)=\not p\gamma_\nu\not k\left[\frac{A(k^2)-A(p^2)}{k^2-p^2}\right]+(\not p\gamma_\nu+\gamma_\nu\not k)\left[\frac{B(p^2)-B(k^2)}{k^2-p^2}\right]+\gamma_\nu\frac{k^2A(k^2)-p^2A(p^2)}{k^2-p^2}\ . \tag{2.5}$$

Inserting Eq. (2.5) into Eq. (2.1), we find two coupled nonlinear integral equations:

$$\begin{aligned}\not p A(p^2)=\not p&+\frac{g^2}{16\pi^4}i\not p\int\frac{d^4k}{(p-k)^2}\frac{(3+\lambda)k^2A(k^2)}{A^2(k^2)k^2-B^2(k^2)}\frac{A(k^2)-A(p^2)}{k^2-p^2}\\&-\frac{g^2}{16\pi^4}i\int\frac{d^4k}{(p-k)^2}\left[2\not k-(3+\lambda)\not p+\frac{(1-\lambda)(\not p-\not k)\not k(\not p-\not k)}{(p-k)^2}\right]\frac{B(k^2)}{A^2(k^2)k^2-B^2(k^2)}\frac{B(p^2)-B(k^2)}{k^2-p^2}\\&-\frac{g^2}{16\pi^4}i\int\frac{d^4k}{(p-k)^2}\left[2\not k+(1-\lambda)\frac{(\not p-\not k)\not k(\not p-\not k)}{(p-k)^2}\right]\frac{A(k^2)}{A^2(k^2)k^2-B^2(k^2)}\frac{k^2A(k^2)-p^2A(p^2)}{k^2-p^2}\end{aligned} \tag{2.6}$$

and

$$\begin{aligned}B(p^2)=m_0&-\frac{g^2}{16\pi^4}i\int\frac{d^4k}{(p-k)^2}(3+\lambda)\frac{B(k^2)}{A^2(k^2)k^2-B^2(k^2)}\frac{A(k^2)k^2-p^2A(p^2)}{k^2-p^2}\\&+\frac{g^2}{16\pi^4}i\int\frac{d^4k}{(p-k)^2}\left[2\not k\not p+(1-\lambda)\frac{(\not p-\not k)\not k(\not p-\not k)\not p}{(p-k)^2}\right]\frac{B(k^2)}{A^2(k^2)k^2-B^2(k^2)}\frac{A(k^2)-A(p^2)}{k^2-p^2}\\&+\frac{g^2}{16\pi^4}i\int\frac{d^4k}{(p-k)^2}\left[2\not k\not p-(3+\lambda)k^2+(1-\lambda)\frac{(\not p-\not k)\not k(\not p-\not k)\not k}{(p-k)^2}\right]\frac{A(k^2)}{A^2(k^2)k^2-B^2(k^2)}\frac{B(p^2)-B(k^2)}{k^2-p^2}\ .\end{aligned} \tag{2.7}$$

The first integral on the right-hand side of Eq. (2.6), and the third integral on the right-hand side of Eq. (2.7) have overlapping divergences of the form

$$[\ln(\Lambda^2)]A(p^2)$$

and

$$[\ln(\Lambda^2)]B(p^2)\ ,$$

respectively, indicating the breakdown of renormalizability. The root of this problem is the neglect of the transverse vertex Γ_μ^T. Γ_μ^T is unimportant in the infrared regime where $\Gamma_\mu^T(p,k)\sim\sigma_{\mu\nu}(p-k)^\nu$, while its contribution in the ultraviolet is crucial to preserving multiplicative renormalizability.[6,7] Setting $\Gamma_\mu=\Gamma_\mu^L$ in the Schwinger-Dyson equation for the photon proper self-energy

$$\Pi_{\mu\nu}(p)=ig^2\mathrm{Tr}\int\frac{d^4k}{(2\pi)^4}\gamma_\mu S(k)\Gamma_\nu(k,p-k)S(p-k) \tag{2.8}$$

and taking all traces gives

$$\Pi_{\mu\nu}(p)=\frac{g^2}{4\pi^4}i\int\frac{d^4k}{(p-k)^2-k^2}\frac{1}{A^2(p-k)(p-k)^2-B^2(p-k)}\frac{1}{A^2(k)k^2-B^2(k)}$$
$$\times\{(p-k)^2k^2g_{\mu\nu}A(p-k)A(k)[A(p-k)-A(k)]$$
$$+[k^2g_{\mu\nu}+k_\mu(p-k)_\nu+(p-k)_\mu k_\nu-(p-k)\cdot kg_{\mu\nu}]B(p-k)A(k)[B(k)-B(p-k)]$$
$$-[k_\mu(p-k)_\nu+(p-k)_\mu k_\nu-(p-k)\cdot kg_{\mu\nu}]A(p-k)A(k)[(p-k)^2A(p-k)-k^2A(k)]$$
$$-[k_\mu(p-k)_\nu+(p-k)_\mu k_\nu-(p-k)\cdot kg_{\mu\nu}]B(p-k)B(k)[A(p-k)-A(k)]$$
$$-[k_\mu(p-k)_\nu+(p-k)_\mu k_\nu-(p-k)\cdot kg_{\mu\nu}-(p-k)^2g_{\mu\nu}]A(p-k)B(k)[B(k)-B(p-k)]$$
$$\times g_{\mu\nu}B(p-k)B(k)[(p-k)^2A(p-k)-k^2A(k)]\}\ . \tag{2.9}$$

From Eqs. (2.6) and (2.7) we see that not only are $A(p)$ and $B(p)$ dependent on the hard ultraviolet cutoff Λ, but they also each contain a dependence on the gauge parameter λ, leaving $\Pi_{\mu\nu}(p)$ cutoff and gauge dependent, and therefore unphysical. The absence of a transverse vertex in Eq. (2.9) is responsible for this problem. In the following sections we will construct a transverse vertex that removes the gauge dependence in Eq. (2.9), and restores renormalizability to Eqs. (2.6) and (2.7).

III. THE TRANSVERSE VERTEX

Before attempting to restore renormalizability to Eqs. (2.6) and (2.7), we will remove the gauge dependence from Eq. (2.9), which originates from $A(p)$ and $B(p)$. One way of eliminating the gauge dependence in $\Pi_{\mu\nu}$ is to rid Eqs. (2.6) and (2.7) of all explicit λ dependence. Both problems, the appearance of overlapping divergences and a gauge-dependent $\Pi_{\mu\nu}(p)$ originate in the use of $\Gamma_\nu=\Gamma_\nu^L$. We therefore construct an effective transverse vertex with a similar structure to Eq. (2.5) given by[8]

$$\Gamma_\mu^{T1}(k,p)=\{[F_1(p,k)\not p+F_2(p,k)\not k][\gamma_\mu,(\not p-\not k)]+[\gamma_\mu,(\not p-\not k)][F_3(p,k)\not p+F_4(p,k)\not k]\}\left[\frac{A(k^2)-A(p^2)}{k^2-p^2}\right]$$
$$+\left[[\gamma_\mu,(\not p-\not k)]F_5(p,k)+\left[\frac{(p-k)_\mu(k^2-p^2)-(p-k)^2(p+k)_\mu}{(p-k)^2}\right]F_6\right]\left[\frac{B(p^2)-B(k^2)}{k^2-p^2}\right]$$
$$+\{[F_7(p,k)\not p+F_8(p,k)\not k][\gamma_\mu,(\not p-\not k)]+[\gamma_\mu,(\not p-\not k)][F_9(p,k)\not p+F_{10}(p,k)\not k]\}\left[\frac{k^2A(k^2)-p^2A(p^2)}{k^2-p^2}\right], \tag{3.1}$$

where

$$F_1(p,k)=c_1\theta(p^2-k^2)+c_4\theta(k^2-p^2)\ , \tag{3.2}$$

$$F_2(p,k)=c_2\theta(p^2-k^2)+c_3\theta(k^2-p^2)\ , \tag{3.3}$$

$$F_3(p,k)=c_3\theta(p^2-k^2)+c_2\theta(k^2-p^2)\ , \tag{3.4}$$

$$F_4(p,k)=c_4\theta(p^2-k^2)+c_1\theta(k^2-p^2)\ , \tag{3.5}$$

$$F_5(p,k)=a_1+a_2\left[\frac{\theta(p^2-k^2)}{k^2}+\frac{\theta(k^2-p^2)}{p^2}\right], \tag{3.6}$$

$$F_6(p,k)=a_3+a_4\left[\frac{\theta(p^2-k^2)}{k^2}+\frac{\theta(k^2-p^2)}{p^2}\right], \tag{3.7}$$

$$F_7(p,k)=b_1\frac{\theta(p^2-k^2)}{p^2}+b_2\frac{\theta(k^2-p^2)}{k^2}+b_5\frac{\theta(p^2-k^2)}{p^2}\ , \tag{3.8}$$

$$F_8(p,k)=b_2\frac{\theta(p^2-k^2)}{p^2}+b_3\frac{\theta(k^2-p^2)}{k^2}\ , \tag{3.9}$$

$$F_9(p,k)=b_3\frac{\theta(p^2-k^2)}{p^2}+b_2\frac{\theta(k^2-p^2)}{k^2}\ , \tag{3.10}$$

$$F_{10}(p,k)=b_4\frac{\theta(p^2-k^2)}{p^2}+b_1\frac{\theta(k^2-p^2)}{k^2}+b_5\frac{\theta(p^2-k^2)}{k^2}\ , \tag{3.11}$$

and a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4, b_5, c_1, c_2, c_3, and c_4 are constants to be determined. Incorporating Eq. (3.1) into Eq. (2.1) by setting $\Gamma_\nu=\Gamma_\nu^L+\Gamma_\nu^{T1}$, we have (after rotating to Euclidean space and performing all angular integrations)

$$A(p^2)=1+\frac{g^2}{16\pi^2}\int_0^{p^2}\frac{dk^2}{A^2(k^2)k^2-B^2(k^2)}\frac{A(k^2)-A(p^2)}{k^2-p^2}A(k^2)\left[\frac{k^6}{p^4}\kappa_1+\frac{k^4}{p^2}(3+\kappa_2)\right]$$
$$+\frac{g^2}{16\pi^2}\int_{p^2}^{\Lambda^2}\frac{dk^2}{A^2(k^2)k^2-B^2(k^2)}\frac{A(k^2)-A(p^2)}{k^2-p^2}A(k^2)[k^2(3+\kappa_2)+p^2\kappa_1]$$
$$+\frac{g^2}{16\pi^2}\int_0^{p^2}\frac{dk^2}{A^2(k^2)k^2-B^2(k^2)}\frac{k^2A(k^2)-p^2A(p^2)}{k^2-p^2}A(k^2)\left[\frac{k^6}{p^6}\chi_1+\frac{k^4}{p^4}\chi_2\right]$$
$$+\frac{g^2}{16\pi^2}\int_{p^2}^{\Lambda^2}\frac{dk^2}{A^2(k^2)k^2-B^2(k^2)}\frac{k^2A(k^2)-p^2A(p^2)}{k^2-p^2}A(k^2)\left[\chi_2+\frac{p^2}{k^2}\chi_1\right]$$
$$+\frac{g^2}{16\pi^2}\int_0^{p^2}\frac{dk^2}{A^2(k^2)k^2-B^2(k^2)}\frac{B(p^2)-B(k^2)}{k^2-p^2}B(k^2)\left[\frac{k^2}{p^2}(3+\xi_1)+\frac{k^4}{p^4}\xi_2\right]$$
$$+\frac{g^2}{16\pi^2}\int_{p^2}^{\Lambda^2}\frac{dk^2}{A^2(k^2)k^2-B^2(k^2)}\frac{B(p^2)-B(k^2)}{k^2-p^2}B(k^2)\left[(3+\xi_3)+\frac{p^2}{k^2}\xi_4\right], \tag{3.12}$$

and

$$B(p^2)=m_0-\frac{g^2}{16\pi^2}\int_0^{p^2}\frac{dk^2}{A^2(k^2)k^2-B^2(k^2)}\frac{k^2A(k^2)-p^2A(p^2)}{k^2-p^2}B(k^2)\left[\frac{k^4}{p^4}\chi_3+\frac{k^2}{p^2}(3+\chi_4)\right]$$
$$-\frac{g^2}{16\pi^2}\int_{p^2}^{\Lambda^2}\frac{dk^2}{A^2(k^2)k^2-B^2(k^2)}\frac{k^2A(k^2)-p^2A(p^2)}{k^2-p^2}B(k^2)\left[(3+\chi_4)+\frac{p^2}{k^2}\chi_3\right]$$
$$-\frac{g^2}{16\pi^2}\int_0^{p^2}\frac{dk^2}{A^2(k^2)k^2-B^2(k^2)}\frac{A(k^2)-A(p^2)}{k^2-p^2}B(k^2)\left[\frac{k^4}{p^2}\kappa_3+k^2\kappa_4\right]$$
$$-\frac{g^2}{16\pi^2}\int_{p^2}^{\Lambda^2}\frac{dk^2}{A^2(k^2)k^2-B^2(k^2)}\frac{A(k^2)-A(p^2)}{k^2-p^2}B(k^2)(p^2\kappa_4+k^2\kappa_3)$$
$$-\frac{g^2}{16\pi^2}\int_0^{p^2}\frac{dk^2}{A^2(k^2)k^2-B^2(k^2)}A(k^2)\frac{B(p^2)-B(k^2)}{k^2-p^2}A(k^2)\left[\frac{k^4}{p^2}(3+\xi_3)+k^2\xi_4\right]$$
$$-\frac{g^2}{16\pi^2}\int_{p^2}^{\Lambda^2}\frac{dk^2}{A^2(k^2)k^2-B^2(k^2)}\frac{B(p^2)-B(k^2)}{k^2-p^2}A(k^2)[p^2\xi_2+k^2(3+\xi_1)], \tag{3.13}$$

where

$$\kappa_1=2c_1+3c_2-c_4, \tag{3.14}$$
$$\kappa_2=\lambda-3c_1-6c_2-3c_3, \tag{3.15}$$
$$\kappa_3=-\lambda-3c_1-3c_2+3c_3+3c_4, \tag{3.16}$$
$$\kappa_4=6c_1-6c_3, \tag{3.17}$$
$$\chi_1=2b_1+3b_2-b_4, \tag{3.18}$$
$$\chi_2=-\lambda-b_5-3b_1-6b_2-3b_3, \tag{3.19}$$
$$\chi_3=-3b_1-3b_2+3b_3+3b_4, \tag{3.20}$$
$$\chi_4=\lambda+6b_1-6b_3+3b_5, \tag{3.21}$$
$$\xi_1=\lambda+6a_1+\tfrac{3}{2}a_2-\tfrac{3}{4}a_3-\tfrac{1}{2}a_4, \tag{3.22}$$
$$\xi_2=-\lambda-3a_1-\tfrac{3}{2}a_2+\tfrac{1}{4}a_3+\tfrac{1}{8}a_4, \tag{3.23}$$
$$\xi_3=3a_5+\tfrac{3}{2}a_6+\tfrac{1}{4}a_7+\tfrac{1}{8}a_8 \tag{3.24}$$

and

$$\xi_4=-\tfrac{3}{2}a_6-\tfrac{3}{4}a_7-\tfrac{1}{2}a_8. \tag{3.25}$$

In order to cancel out the λ-dependent integrals on the right-hand sides of Eqs. (3.12) and (3.13) we set Eqs. (3.14) through (3.25) equal to zero and find the following values for the unknown constants appearing in Eq. (3.1):

$$a_4=-8\lambda+24a_1,\quad a_3=6\lambda-12a_1,\quad a_2=-\frac{\lambda}{3}-2a_1; \tag{3.26}$$

$$b_5=-\lambda,\quad b_4=b_1=\tfrac{1}{4}\lambda,\quad b_3=b_2=-\tfrac{1}{12}\lambda \tag{3.27}$$

and

$$c_4=-c_1+\frac{\lambda}{2},\quad c_3=c_1,\quad c_2=-c_1+\frac{\lambda}{6}. \tag{3.28}$$

Calculating the contribution of Γ_ν^{T1} to $\Pi_{\mu\nu}(p)$ gives

$$\Pi^{T1}_{\mu\nu}(p)=\frac{4g^2}{16\pi^4}\int\frac{d^4k}{[A^2(k^2)k^2-B(k^2)]^2}A(k^2)B(k^2)\times\left[\left[2(a_1+a_2)(p^2g_{\mu\nu}-p_\mu p_\nu)+2\frac{a_3+a_4}{k^2}(k\cdot pp_\mu k_\nu-p^2k_\mu k_\nu)\right]\right]\frac{B(k-p)-B(k)}{(k-p)^2-p^2}\ . \tag{3.29}$$

Where the contribution of the Dirac odd part of Eq. (3.1) to Eq. (3.29) cancels out for $k-p\approx k$ after the traces are taken, while for the remaining part we expand the kernel for $|k|>>|p|$, and drop all terms of $O((k-p)^2-k^2)$ and higher. The constant a_1 is determined by demanding that the left-hand side of Eq. (3.29) vanish, thus avoiding the introduction of a λ dependence in $\Pi_{\mu\nu}(p)$ via Γ^{T1}_ν, yielding

$$a_1=\frac{\lambda}{24}\ . \tag{3.30}$$

After setting $\Gamma_\nu=\Gamma^L_\nu+\Gamma^{T1}_\nu$ in Eq. (2.1), we find that Eqs. (2.6) and (2.7) no longer possess λ dependent pieces

$$\begin{aligned}\not{p}A(p^2)=\not{p}&+\frac{3g^2}{16\pi^4}i\not{p}\int\frac{d^4k}{(p-k)^2}\frac{k^2A(k^2)}{A^2(k^2)k^2-B^2(k^2)}\frac{A(k^2)-A(p^2)}{k^2-p^2}\\&-\frac{g^2}{16\pi^4}i\int\frac{d^4k}{(p-k)^2}\left[2\not{k}-3\not{p}+\frac{(\not{p}-\not{k})\not{k}(\not{p}-\not{k})}{(p-k)^2}\right]\frac{B(k^2)}{A^2(k^2)k^2-B^2(k^2)}\frac{B(p^2)-B(k^2)}{k^2-p^2}\\&-\frac{g^2}{16\pi^4}i\int\frac{d^4k}{(p-k)^2}\left[2\not{k}+\frac{(\not{p}-\not{k})\not{k}(\not{p}-\not{k})}{(p-k)^2}\right]\frac{A(k^2)}{A^2(k^2)k^2-B^2(k^2)}\frac{k^2A(k^2)-p^2A(p^2)}{k^2-p^2}\end{aligned} \tag{3.31}$$

and

$$\begin{aligned}B(p^2)=&-\frac{3g^2}{16\pi^4}i\int\frac{d^4k}{(p-k)^2}\frac{B(k^2)}{A^2(k^2)k^2-B^2(k^2)}\frac{A(k^2)k^2-p^2A(p^2)}{k^2-p^2}\\&+\frac{g^2}{16\pi^4}i\int\frac{d^4k}{(p-k)^2}\left[2\not{k}\not{p}+\frac{(\not{p}-\not{k})\not{k}(\not{p}-\not{k})\not{p}}{(p-k)^2}\right]\frac{B(k^2)}{A^2(k^2)k^2-B^2(k^2)}\frac{A(k^2)-A(p^2)}{k^2-p^2}\\&+\frac{g^2}{16\pi^4}i\int\frac{d^4k}{(p-k)^2}\left[2\not{k}\not{p}-3k^2+\frac{(\not{p}-\not{k})\not{k}(\not{p}-\not{k})\not{k}}{(p-k)^2}\right]\frac{A(k^2)}{A^2(k^2)k^2-B^2(k^2)}\frac{B(p^2)-B(k^2)}{k^2-p^2}\ ,\end{aligned} \tag{3.32}$$

but the problem of overlapping divergences remains. An appropriate transverse vertex for removing the overlapping divergence in third integral on the right-hand side of Eq. (3.32) is given by[9]

$$\Gamma^{T2}_\mu(k,p)=\left[f_1[\gamma_\mu,(\not{p}-\not{k})]+f_2\frac{(p-k)_\mu(k^2-p^2)-(p-k)^2(p+k)_\mu}{(p-k)^2}\right]\frac{B(p^2)-B(k^2)}{k^2-p^2}\ , \tag{3.33}$$

where f_1 and f_2 are constants determined by demanding that the overlapping divergences in Eq. (3.32) are canceled by those introduced when Γ^{T2}_ν is introduced into Eq. (2.1). Setting $\Gamma_\nu=\Gamma^L_\nu+\Gamma^{T1}_\nu+\Gamma^{T2}_\nu$ in Eq. (2.1) and taking the trace of the Dirac even part of the equation changes the third integral on the right-hand side of Eq. (3.32) to

$$+\frac{g^2}{16\pi^4}i\int\frac{d^4k}{(p-k)^2}\left[k\cdot p-3k^2+2\frac{(k\cdot p-k^2)(p^2-k\cdot p)}{k^2-p^2}+(k^2-k\cdot p)\left[6f_1-2\frac{p^2-k\cdot p}{(p-k)^2}f_2\right]\right]\times\frac{A(k^2)}{A^2(k^2)k^2-B^2(k^2)}\frac{B(p^2)-B(k^2)}{k^2-p^2}\ . \tag{3.34}$$

Requiring that the integral in (3.34) vanish and thus ridding Eq. (3.32) of overlapping divergences gives

$$f_1=\tfrac{1}{2} \tag{3.35}$$

and

$$f_2=-1\ . \tag{3.36}$$

With the choices of constants given in Eqs. (3.35) and (3.36) the contribution of Γ^{T2}_ν to Eq. (2.1) cancels the second integral on the right-hand side of Eq. (3.31), while the third integral on the right-hand side of Eq. (3.31) is identically zero, leaving

$$A(p^2)=1 \tag{3.37}$$

and

$$B(p^2)=m_0-\frac{3g^2}{16\pi^2}i\int\frac{d^4k}{(p-k)^2}\frac{B(k^2)}{k^2-B^2(k^2)}\ . \tag{3.38}$$

Even though Eq. (3.38) can also be obtained by means of the ladder approximation, here it was derived in a self-consistent manner rather than making a series of assumptions as required in the ladder approximation. Also Eq. (2.6) through Eq. (3.38) have been written in a general covariant gauge, while the ladder approximation relies on choosing the Landau gauge, i.e., $\lambda=0$.[10] The only approximation involved in our method stems from our use of the renormalizability of Eq. (2.1) and the gauge invariance of $\Pi_{\mu\nu}(p)$ as a guide to constructing Eqs. (3.1) and (3.33). We have no information about $\Gamma_\nu^T=\Gamma_\nu^{T1}+\Gamma_\nu^{T2}$ at intermediate energies, and therefore Eq. (3.38) is only accurate in the ultraviolet and infrared regimes. Now we are in a position to introduce the four-fermion interactions.

IV. FOUR-FERMION INTERACTIONS

With four-fermion interactions in the QED Lagrangian

$$\overline{\Psi}(i\gamma^\mu D_\mu-\mu_0)\Psi+\frac{G_0}{2}[(\overline{\Psi}\Psi)^2+(\overline{\Psi}i\gamma_5\Psi)^2] , \qquad (4.1)$$

we have a gauged Nambu–Jona-Lasinio model.[11] Γ_μ and $S(k)$ appearing in Eq. (2.1) are changed by the appearance of the four-fermion vertices in their skeleton graph expansions. Γ_μ^L, Γ_μ^T, and $S(k)$ all have been written in terms of the two scalar functions $A(p)$ and $B(p)$, via Eqs. (2.3), (2.5), (3.1), and (3.33); therefore ultimately introducing the four-fermion vertex only changes $A(p)$ and $B(p)$ through the addition of a term in the fermion propagator Schwinger-Dyson equation given by

$$iG_0\int\frac{d^4k}{(2\pi)^4}S(k)\Gamma , \qquad (4.2)$$

where Γ is the full four-fermion vertex of the pure four-fermionic theory (i.e., it has no gauge particle propagator dependence, the photon propagator in this case) and contains no explicit gauge dependence. Since Eq. (4.2) is not a momentum-dependent term, replacing Γ by 1 only changes Eq. (4.2) by a constant factor (which can be compensated for by rescaling G_0). Adding Eq. (4.2) to Eq. (3.38), rotating the result to Euclidean space and performing all the angular integrations gives

$$M(p^2)=\mu_0+\frac{3g^2}{16\pi^2}\int_0^{p^2}dk^2\frac{k^2}{p^2}\frac{M(k^2)}{k^2+M^2(k^2)}+\frac{3g^2}{16\pi^2}\int_{p^2}^{\mathcal{M}^2}dk^2\frac{M(k^2)}{k^2+M^2(k^2)}+G\int_0^{\mathcal{M}^2}dk^2\frac{k^2}{\mathcal{M}^2}\frac{M(k^2)}{k^2+M^2(k^2)} , \qquad (4.3)$$

where

$$M(p^2)\equiv\frac{B(p^2)}{A(p^2)} , \qquad (4.4)$$

$$G=\frac{G_0\mathcal{M}^2}{4\pi^2} \qquad (4.5)$$

and $\mathcal{M}$ is a physical cutoff above which the interactions responsible for the effective four-fermion piece of the Lagrangian are assumed to drop off sharply. Converting Eq. (4.4) into a differential equation yields[12–16,3]

$$p^2\frac{d}{dp^2}p^2\frac{dM(p^2)}{dp^2}+p^2\frac{dM(p^2)}{dp^2}+\frac{\alpha}{\alpha_c}\frac{p^2M(p^2)}{p^2+B^2(p^2)}=0 , \qquad (4.6)$$

with boundary conditions

$$\left[p^2\frac{dM(p^2)}{dp^2}\right]_{p^2=\mathcal{M}^2}+M(\mathcal{M}^2)=0 , \qquad (4.7)$$

$$\left[p^4\frac{dM(p^2)}{dp^2}\right]_{p^2=0}=0 , \qquad (4.8)$$

where $\alpha/\alpha_c=3g^2/16\pi^2$. Equation (4.6) has two solutions given by

$$M(p^2)=C\frac{M_0}{p}\sinh\left\{\left[1-\frac{\alpha}{\alpha_c}\right]^{1/2}\left[\ln\left[\frac{p}{M_0}\right]+\delta\right]\right\},\quad \alpha<\alpha_c \qquad (4.9)$$

and

$$M(p^2)=C\frac{M_0}{p}\sin\left\{\left[1-\frac{\alpha}{\alpha_c}\right]^{1/2}\left[\ln\left[\frac{p}{M_0}\right]+\delta\right]\right\},\quad \alpha>\alpha_c . \qquad (4.10)$$

For the case of no explicit chiral-symmetry breaking $\mu_0\mathcal{M}=0$, inserting Eq. (4.9) into the UV boundary condition Eq. (4.7) gives

$$\tanh\left\{\left[1-\frac{\alpha}{\alpha_c}\right]^{1/2}\left[\ln\left[\frac{\mathcal{M}}{M_0}\right]+\delta\right]\right\}=-\left[1-\frac{\alpha}{\alpha_c}\right]^{1/2}\left[\frac{\frac{\alpha}{4\alpha_c}+G}{\frac{\alpha}{4\alpha_c}-G}\right],\quad \alpha<\alpha_c . \qquad (4.11)$$

Equation (4.11) is not satisfied for $\alpha/\alpha_c > 4G$ and chiral symmetry is preserved, while dynamical chiral-symmetry breaking takes place for

$$-\left[1-\frac{\alpha}{\alpha_c}\right]^{1/2}\left[\frac{G+\frac{\alpha}{4\alpha_c}}{G-\frac{\alpha}{4\alpha_c}}\right]<1\ , \tag{4.12}$$

from which we can solve for the critical curve[4]

$$G_c=\left[\frac{1+\left[1-\frac{\alpha}{\alpha_c}\right]^{1/2}}{2}\right]^2<G\ . \tag{4.13}$$

On the other hand, for $\alpha > \alpha_c$, we have

$$\left\{\left[1-\frac{\alpha}{\alpha_c}\right]^{1/2}\left[\ln\left[\frac{\mathcal{M}}{M_0}\right]+\delta\right]\right\} = -\arctan\left[\left[1-\frac{\alpha}{\alpha_c}\right]^{1/2}\left[\frac{\frac{\alpha}{4\alpha_c}+G}{\frac{\alpha}{4\alpha_c}-G}\right]\right]+n\pi\ , \tag{4.14}$$

giving G_c = straight line for $0 < G_c < \frac{1}{4}$, and $\alpha = \alpha_c$. $G_c(\alpha)$ is displayed in Fig. 1.

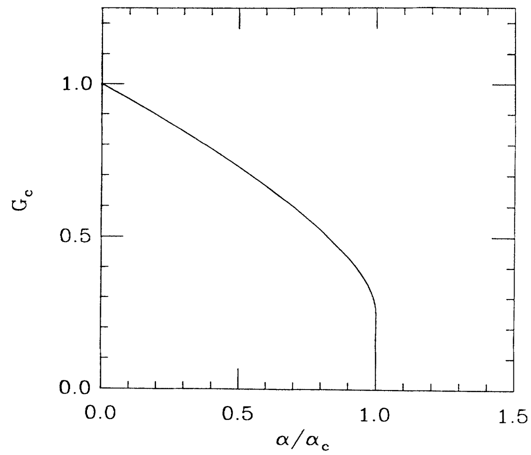

FIG. 1. Graph of the critical curve $G_c(\alpha)$.

V. CONCLUSION

We have derived the critical curve of fixed-point QED using a general covariant gauge by approximating the full vertex with a vertex that satisfies the Ward identity, leaves the photon proper self-energy gauge invariant, and restores multiplicative renormalizability to fermion propagator Schwinger-Dyson equation. Our approach has an advantage over the ladder approximation in that it does not rely on the use of the Landau gauge. Nowhere have we chosen any particular gauge, rather we have written the full vertex in terms of the fermion propagator and removed all gauge parameter dependence in $S(p)$ by introducing an explicitly gauge-dependent piece of an effective transverse vertex Γ_ν^{T1} in the fermion propagator Schwinger-Dyson equation. Meanwhile Γ_ν^{T1} gives no contribution to the photon proper self-energy. The rest of the full vertex is independent of the gauge parameter as is $S(p)$; therefore, $\Pi_{\mu\nu}(p)$ is gauge independent from the very start; as opposed to perturbation theory where $S(p)$ and Γ_ν are gauge dependent at each order in g^2 beginning with $O(g^2)$, and gauge-dependent terms cancel only after much labor as they must since only the gauge-independent parts of $S(p)$ and Γ_ν contribute.

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