Global spacetime symmetries in the functional Schrödinger picture

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(Received 10 January 1991)

In the conventional functional Schrödinger quantization of field theory, the background spacetime manifold is foliated into a set of three-surfaces and the quantum state of the field is represented by a wave functional of the field configurations on each three-surface. Although this procedure may be covariantly described, the wave functionals generally fail to carry a representation of the complete spacetime symmetry group of the background, such as the Poincaré group in Minkowski spacetime, because spacetime symmetries generally involve distortions or motions of the three-surfaces themselves within that spacetime. In this paper, we show that global spacetime symmetries in the functional Schrödinger picture may be represented by *parametrizing* the field theory—raising to the status of dynamical variables the *embedding* variables describing the spacetime location of each three-surface. In particular, we show that the embedding variables provide a connection between the purely geometrical operation of an isometry group on the spacetime and the operation of the usual global symmetry generators (constructed from the energy-momentum tensor) on the wave functionals of the theory. We study the path-integral representation of the wave functionals of the parametrized field theory. We show how to construct, from the path integral, wave functionals that are annihilated by the global symmetry generators, i.e., that are invariant under global spacetime symmetry groups. The invariance of the class of histories summed over in the path integral is identified as the source of the invariance of the wave functionals. We apply this understanding to a study of vacuum states in the de Sitter spacetime. We make mathematically precise a previously given heuristic argument for the de Sitter invariance of the matter wave functionals defined by the no-boundary proposal of Hartle and Hawking. The treatment is largely formal, but a brief discussion of anomalies is given.

I. INTRODUCTION

Many kinematical issues in quantum field theory assume a particularly transparent form in the functional Schrödinger picture.[1] There, the quantum state of the field is represented by a wave functional of the field configuration on a surface of constant time, and evolution of the state is governed by the functional Schrödinger equation. One of the appeals of this method is that many field-theoretic problems assume the form of elementary quantum-mechanical problems. This method has been profitably applied, for example, to Yang-Mills theory and Chern-Simons theory.[2] It has also been of great use in quantum field theory in curved spacetime, especially in de Sitter space, where it affords a convenient method of studying vacuum states.[3–7]

Perhaps the main drawback of this method, however, is its lack of covariance, or at least the manifest exhibition thereof. This apparent lack of covariance stems from the fact that quantum states in the functional Schrödinger formalism are represented by wave functionals of the fields on spacelike surfaces, and to discuss these it is necessary to break the manifest covariance of the theory by foliating the spacetime into spacelike slices. Although the field configurations on a spacelike surface typically carry representations of internal symmetry groups, and of isometry groups of the surfaces themselves, they generally fail to carry complete representations of the symmetry groups of the background spacetime, such as Poincaré symmetry. This is because spacetime symmetry groups typically involve motions or distortions of the surfaces themselves in the spacetime, but the fields on the surface generally carry no information about the location or orientation of that surface with respect to the background spacetime. This complicates the search, in the functional Schrödinger picture, for quantum states of the field invariant under *spacetime* symmetries.

Because the functional Schrödinger quantization is derived most directly from the Hamiltonian formulation of classical field theory, various attempts have been made to develop alternative Hamiltonian formulations of classical field theories which avoid breaking covariance.[8–10] One particular formulation, due to Dirac, is known as parametrized field theory.[10] The basic idea is that one raises to the status of dynamical variables the so-called *embedding* variables X^α describing the location of the three-surfaces in the spacetime background. In the quantized theory, the wave functionals are functions of the field configurations on a spacelike surface, *and* the embeddings X^α. One can then talk about symmetry groups that involve motions of the three-surfaces themselves.

The prime aim of this paper is to explore some of the virtues of taking the quantized parametrized field theory seriously as a way to do quantum field theory in curved spacetime. In particular, we shall argue that it is not only useful, but in some situations essential, in discussions of global spacetime symmetries in the functional Schrödinger picture.

Work on parametrized field theory has previously been inspired by studies in quantum gravity, as a result of various formal similarities.[11] For example, the algebra of the constraints of the parametrized field theory—the Dirac algebra—is identical to the algebra of the constraints of the Hamiltonian and momentum constraints of general relativity.[11,12] The emphasis of the present paper, however, is rather different.

Two particular related issues motivated the present work, and will form the ultimate focus of our discussion.

(i) In Ref. 13, the derivation of the operator constraints of the Dirac quantization procedure from the path-integral representation of the wave function was considered, for a broad class of theories with symmetry. That work was primarily concerned with local symmetries, and in particular, with the derivation of the Wheeler-DeWitt equation and momentum constraints for the wave function of the Universe. A more detailed treatment of the case of constraints arising from global symmetries is called for. In particular, for the case of global *spacetime* symmetries, it turns out that the operator constraints on the wave function cannot be derived from the path integral using the usual functional Schrödinger formalism. A covariant method, such as the one described in this paper, is needed.

(ii) In Ref. 14, the quantum state of matter fields in de Sitter space defined by the no-boundary proposal of Hartle and Hawking[15] was studied. A heuristic argument for the de Sitter invariance of this state was given. However, a more formal version of this argument is definitely called for, and this can be achieved through the introduction of the embedding variables.

We begin, in Sec. II, by describing some basic properties of the embedding variables, and discussing the Hamiltonian formulation and quantization of standard scalar field theory in an arbitrary background spacetime. Noting the shortcomings of the standard approach, we then discuss the Hamiltonian formulation and functional Schrödinger quantization of the parametrized field theory, in which the embedding variables (essentially the spacetime coordinates of the three-surfaces) are adjoined to the matter fields as basic dynamical variables. Various technicalities, such as the algebra of the constraints of the parametrized field theory, are discussed in Sec. III.

In Sec. IV, we discuss global spacetime symmetry groups, and construct their generators. A wave functional of the parametrized field theory is said to be invariant under a global spacetime symmetry if it is annihilated by the appropriate generator. In Sec. V, we recall the results of Ref. 13, in which it was shown how to generate wave functions satisfying global and/or local constraints from a path integral. These results are elaborated on, paying particular attention to the case of global symmetries. In Sec. VI, we use the results of Sec. V to demonstrate that wave functions of the parametrized field theory generated from a path integral will satisfy various operator constraints. The invariance of the class of paths summed over is identified as the source of the global invariance of the quantum state. We also emphasize the *necessity* of using the embeddings to derive constraints in the case of global spacetime symmetries. In Sec. VII, we apply the understanding gained to a discussion of vacuum states in de Sitter space. We use the embeddings approach to make more mathematically precise a previously given argument[14] for the de Sitter invariance of the state defined by the "no-boundary" proposal of Hartle and Hawking.[15]

The level of this paper, like that of Ref. 13, is largely formal, in that we do not directly address the issues of operator ordering, regularization, etc. However, in Sec. VIII, we offer some heuristic remarks about anomalies in the algebra of the various constraints. Our conclusions are presented in Sec. IX.

It was in fact Dirac who, in the case of field theory in Minkowski spacetime, first realized that the functional Schrödinger picture may be made manifestly Lorentz invariant using parametrized field theory.[10] Parts of the present work may be therefore thought of as a generalization of Dirac's ideas to more general curved-spacetime backgrounds. As explained above, however, the main thrust of this work is to study invariant states generated from a path integral, and this was not discussed by Dirac.

II. HAMILTONIAN FORMULATION OF PARAMETRIZED FIELD THEORY

We begin, in this section, by describing the properties of the embedding variables, and the Hamiltonian formulation and functional Schrödinger quantization of standard scalar field theory, and of parametrized scalar field theory.

The first step in the construction of a Hamiltonian formulation of a field theory is to foliate the spacetime manifold $\mathcal{M}$. To do this it is necessary to assume that the manifold $\mathcal{M}$ is of the form $\mathcal{M}=\mathbb{R}\times\Sigma$. We will largely be concerned with the case of closed three-surfaces Σ. If the spacetime manifold has coordinates X^α, it is normally foliated by taking the leaves of the foliation to be surfaces of constant X^0, and the intrinsic coordinates within each surface to be the coordinates $\mathrm{X}^1,\mathrm{X}^2,\mathrm{X}^3$ inherited directly from the spacetime. More generally, however, the spacetime may be foliated in an arbitrary fashion by introducing the so-called *embedding* variables X. The embeddings are maps $X:\Sigma\rightarrow\mathcal{M}$ which take a point $\mathbf{x}$ in the surface Σ, to a point in the spacetime, $\mathrm{X}^\alpha=X^\alpha(\mathbf{x},t)$, where t labels the leaves of the foliation. We need to devote some space to summarizing the properties of the embedding variables.[16–18]

Using the embeddings X^α one may construct projections of spacetime quantities normal and tangential to Σ. The tangential projections are defined by

$$X_i^\alpha \equiv \frac{\partial X^\alpha}{\partial x^i} . \tag{2.1}$$

The normal to Σ is uniquely defined by the relations

$$n_\alpha X_i^\alpha = 0, \quad g^{\alpha\beta} n_\alpha n_\beta = -1 , \tag{2.2}$$

where $g^{\alpha\beta}$ is the inverse spacetime metric and the signature of spacetime is taken to be $(-,+,+,+)$. The normal depends on the embeddings, as may be seen from the explicit expression

$$n_\alpha = k \epsilon_{\alpha\beta\gamma\sigma} X_i^\beta X_j^\gamma X_k^\sigma \epsilon^{ijk} , \tag{2.3}$$

where k is a factor ensuring the correct normalization (see Appendix).

In terms of n_α and X_i^α, any tensor on the spacetime $\mathcal{M}$ may be written in terms of its projections normal and tangential to Σ. For example, a spacetime vector V^α may be written

$$V^\alpha = V^\perp n^\alpha + V^i X_i^\alpha , \tag{2.4}$$

where

$$V^\perp = -n_\alpha V^\alpha, \quad V^i = h^{ij} g_{\alpha\beta} X_j^\alpha V^\beta . \tag{2.5}$$

Here, we have introduced the induced metric on Σ:

$$h_{ij} = X_i^\alpha X_j^\beta g_{\alpha\beta} . \tag{2.6}$$

From here onwards, we will raise and lower indices on all quantities using the metrics h_{ij} and $g_{\alpha\beta}$.

The *deformation vector* of the foliation is defined by

$$N^\alpha \equiv \frac{\partial X^\alpha}{\partial t} \equiv \dot{X}^\alpha . \tag{2.7}$$

Lapse and shift arise as the coefficients in the tangential and normal projections of $\dot{X}^\alpha$:

$$\dot{X}^\alpha \equiv N n^\alpha + N^i X_i^\alpha . \tag{2.8}$$

One thus has

$$N = -n_\alpha \dot{X}^\alpha, \quad N^i = X_\alpha^i \dot{X}^\alpha . \tag{2.9}$$

By writing out the usual expression for the four-metric, it is readily verified that it takes the familiar $3+1$ form:

$$\begin{aligned} ds^2 &= g_{\alpha\beta} dX^\alpha dX^\beta \\ &= g_{\alpha\beta}(\dot{X}^\alpha \dot{X}^\beta dt^2 + 2\dot{X}^\alpha X_j^\beta dt\, dx^j + X_i^\alpha X_j^\beta dx^i dx^j) \\ &= (-N^2 + N_i N^i) dt^2 + 2N_i dx^i dt + h_{ij} dx^i dx^j . \end{aligned} \tag{2.10}$$

From Eq. (2.8), it also follows that

$$\frac{\partial t}{\partial X^\alpha} = -\frac{n_\alpha}{N}, \quad \frac{\partial x^i}{\partial X^\alpha} = \frac{n_\alpha}{N} N^i + X_\alpha^i . \tag{2.11}$$

Finally, one has

$$\sqrt{-g(X)} \det \left[\frac{\partial X^\alpha}{\partial (x^i, t)} \right] = N h^{1/2} \tag{2.12}$$

(see Appendix).

Now we may consider the Hamiltonian formulation of scalar field theory in the spacetime $\mathcal{M}$. For simplicity we consider only the case of a massive, minimally coupled field. We take as our starting point the action

$$S = \frac{1}{2} \int_{\mathcal{M}} d^4X \sqrt{-g(X)} \left[g^{\alpha\beta} \frac{\partial\phi}{\partial X^\alpha} \frac{\partial\phi}{\partial X^\beta} + m^2\phi^2 \right] . \tag{2.13}$$

Foliating $\mathcal{M}$ as $\mathbb{R} \times \Sigma$, we rewrite this as

$$\begin{aligned} S = -\frac{1}{2} \int_{\mathbb{R}} dt \int_\Sigma d^3\mathbf{x} \det \left[\frac{\partial X^\alpha}{\partial (x^i, t)} \right] \sqrt{-g(X)} \\ \times \left[g^{\alpha\beta} \left[\frac{\partial t}{\partial X^\alpha} \frac{\partial t}{\partial X^\beta} \dot{\phi}^2 + 2 \frac{\partial t}{\partial X^\alpha} \frac{\partial x^j}{\partial X^\beta} \dot{\phi} \partial_j \phi \right.\right. \\ \left.\left. + \frac{\partial x^i}{\partial X^\alpha} \frac{\partial x^j}{\partial X^\beta} \partial_i \phi \partial_j \phi \right] + m^2 \phi^2 \right] . \end{aligned} \tag{2.14}$$

Using the above results, Eq. (2.14) becomes

$$\begin{aligned} S &= \frac{1}{2} \int_{\mathbb{R}} dt \int_\Sigma d^3\mathbf{x}\, N h^{1/2} \left[\frac{\dot{\phi}^2}{N^2} - 2\frac{N^i}{N^2} \partial_i \phi \dot{\phi} \right. \\ &\qquad \left. - \left[h^{ij} - \frac{N^i N^j}{N^2} \right] \partial_i \phi \partial_j \phi - m^2 \phi^2 \right] \\ &\equiv \int d^3\mathbf{x}\, dt \mathcal{L} . \end{aligned} \tag{2.15}$$

Canonical momenta are defined in the usual way,

$$\pi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \frac{h^{1/2}}{N} (\dot{\phi} - N^i \partial_i \phi) , \tag{2.16}$$

and one readily obtains the Hamiltonian form of the action

$$S = \int_{\mathbb{R}} dt \int_\Sigma d^3\mathbf{x} (\dot{\phi}\pi - N\mathcal{H} - N^i \mathcal{H}_i) \tag{2.17}$$

where

$$\mathcal{H} = \frac{1}{2} h^{1/2} (h^{-1}\pi^2 + h^{ij} \partial_i \phi \partial_j \phi + m^2 \phi^2) , \tag{2.18}$$

$$\mathcal{H}_i = \partial_i \phi \pi . \tag{2.19}$$

We note for future reference that when (2.16) holds, (2.18) and (2.19) are projections of the energy-momentum tensor

$$\mathcal{H} = h^{1/2} n^\alpha n^\beta T_{\alpha\beta}, \quad \mathcal{H}_i = h^{1/2} X_i^\alpha n^\beta T_{\alpha\beta} \tag{2.20}$$

where

$$T_{\alpha\beta} = \partial_\alpha \phi \partial_\beta \phi - \frac{1}{2} g_{\alpha\beta} (g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + m^2 \phi^2) . \tag{2.21}$$

In the usual functional Schrödinger quantization of scalar field theory, the quantum state of the field is represented by a wave functional $\Psi[\phi(\mathbf{x}), t]$, a functional of the field configuration $\phi(\mathbf{x})$ on the surface Σ labeled by t. The evolution of the wave functional along the foliation is described by the functional Schrödinger equation

$$i \frac{\partial}{\partial t} \Psi[\phi(\mathbf{x}), t] = \int_\Sigma d^3\mathbf{x} (N \hat{\mathcal{H}} + N^i \hat{\mathcal{H}}_i) \Psi[\phi(\mathbf{x}), t] , \tag{2.22}$$

where $\hat{\mathcal{H}}$ and $\hat{\mathcal{H}}_i$ are the quantities (2.18) and (2.19) with the momenta replaced by operators using the usual substitutions. For practical purposes the foliation is typically chosen so that $N=1$ and $N^i=0$. Although this quantization scheme[6] is covariant, in the sense that it does not refer to a particular set of coordinates in the spacetime or in the hypersurfaces of the foliation, it is important to emphasize that the foliation is completely fixed. It may be chosen arbitrarily, but once chosen it cannot be changed. In particular, the Schrödinger equation (2.22) describes only evolution along the fixed foliation: it tells us nothing about the response of the wave function to a *change of foliation.*

Recall that the primary concern of this paper is to understand how spacetime symmetries are represented in the functional Schrödinger picture. Such symmetries will typically not act transitively in the three-surfaces of a given foliation; rather, they will generally involve distortions or motions of the hypersurface in spacetime. Spacetime symmetries in the functional Schrödinger picture will therefore be most transparently represented in a formalism which permits the foliation to be changed at will.

As outlined in the Introduction, there exists a formalism in which this may be achieved, and is known as *parametrized field theory.*[10] The idea is that one returns to the action (2.15), but one allows the embedding variables X^α to become dynamical. Let us therefore repeat the Hamiltonian analysis of (2.15), now regarding not only $\phi(\mathbf{x},t)$ but also $X^\alpha(\mathbf{x},t)$ as dynamical variables. The lapse, shift, and three-metric, N, N^i, and h_{ij}, are regarded as fixed functions of the embeddings, as described above. The momentum conjugate to ϕ is as before, Eq. (2.16). To find the momenta conjugate to the embeddings X^α, note that $\dot{X}^\alpha$ occurs only through the lapse and shift, and is linear in them. One thus has

$$P_\alpha=\frac{\partial\mathcal{L}}{\partial\dot{X}^\alpha}=\frac{\partial\mathcal{L}}{\partial N}\frac{\partial N}{\partial\dot{X}^\alpha}+\frac{\partial\mathcal{L}}{\partial N^i}\frac{\partial N^i}{\partial\dot{X}^\alpha}$$
$$=n^\alpha\mathcal{H}-X^i_\alpha\mathcal{H}_i\ . \tag{2.23}$$

There is therefore a primary constraint

$$\Pi_\alpha\equiv P_\alpha-n_\alpha\mathcal{H}+X^i_\alpha\mathcal{H}_i\equiv P_\alpha+\mathcal{H}_\alpha\approx 0\ . \tag{2.24}$$

In the extended phase space $(\phi,\pi,X^\alpha,P_\alpha)$ it is readily shown that the canonical Hamiltonian vanishes, and the Hamiltonian form of the action is therefore given by

$$S=\int_{\mathbb{R}}dt\int_\Sigma d^3\mathbf{x}(\dot{\phi}\pi+\dot{X}^\alpha P_\alpha-N^\alpha\Pi_\alpha)\ . \tag{2.25}$$

This action defines the parametrized field theory. Here, N^α is regarded as a Lagrange multiplier which is to be varied independently of the rest of the variables. It is only after extremization with respect to P_α that it is equated with $\dot{X}^\alpha$, as in Eqs. (2.7) and (2.8).

The constraints of the theory, (2.24), may be shown to be first class. Indeed, their algebra is Abelian:

$$\{\Pi_\alpha(\mathbf{x}),\Pi_\beta(\mathbf{x}')\}=0\ . \tag{2.26}$$

The rather lengthy calculation necessary to demonstrate this is outlined in the next section. Let us consider the symmetry generated by these constraints. It is straightforward to show that under the transformations

$$\delta F(\phi,\pi,X^\alpha,P_\alpha)=\int d^3\mathbf{x}\,\epsilon^\alpha(\mathbf{x})\{F,\Pi_\alpha(\mathbf{x})\}\ , \tag{2.27}$$

$$\delta N^\alpha=\dot{\epsilon}^\alpha(\mathbf{x})\ , \tag{2.28}$$

the action is invariant if $\epsilon^\alpha(\mathbf{x},t)$ is chosen to vanish at both end points, $t=t',t=t''$.

Equation (2.26) actually implies that the constraints generate the algebra of four-dimensional diffeomorphisms, in the following sense.[19] Suppose we smear the constraints with a spacetime vector field U^α restricted to the embeddings; i.e., define

$$\Pi(U)=\int d^3\mathbf{x}\,U^\alpha(X^\beta(\mathbf{x}))\Pi_\alpha(\mathbf{x})\ . \tag{2.29}$$

Then it immediately follows from (2.26) that

$$\{\Pi(U),\Pi(V)\}=-\Pi([U,V])\ , \tag{2.30}$$

where [,] denotes the Lie bracket:

$$[U,V]^\beta\equiv U^\alpha\partial_\alpha V^\beta-V^\alpha\partial_\alpha U^\beta\ . \tag{2.31}$$

Equation (2.30) is essentially the algebra of four-dimensional diffeomorphisms.

In the quantization of the parametrized field theory, the quantum state of the field is represented by a wave functional $\Psi[\phi(\mathbf{x}),X^\alpha(\mathbf{x})]$, a functional of both the scalar field $\phi(\mathbf{x})$ and the embedding variables $X^\alpha(\mathbf{x})$ on a three-surface. Because the theory is a parametrized field theory, there is not explicit dependence on the time label t, and the evolution of the wave functional is described entirely by the operator version of the constraint equation (2.24):

$$\hat{\Pi}_\alpha(\mathbf{x})\Psi[\phi(\mathbf{x}),X^\alpha(\mathbf{x})]=0\ , \tag{2.32}$$

which may also be written

$$i\frac{\delta}{\delta X^\alpha(\mathbf{x})}\Psi[\phi(\mathbf{x}),X^\alpha(\mathbf{x})]=\hat{\mathcal{H}}_\alpha\Psi[\phi(\mathbf{x}),X^\alpha(\mathbf{x})]\ . \tag{2.33}$$

Equation (2.33) has the form of a generalized Tomonaga-Schwinger equation.[20] The more familiar Tomonaga-Schwinger equation is obtained by projecting (2.30) in a particular direction. The Schrödinger equation (2.22) may be recovered by projecting (2.33) along the deformation vector $\dot{X}^\alpha$ and integrating over spatial coordinates $\mathbf{x}$. To do either of these things, however, would be to lose a very desirable property of this quantization scheme: Equation (2.33) describes the response of the wave function to arbitrary changes in foliation. In particular, unlike their predecessors satisfying (2.22), the wave functions $\Psi[\phi(\mathbf{x}),X^\alpha(\mathbf{x})]$ may carry representations of spacetime symmetry groups.[21] We will go on to exploit this aspect of the parametrized field theory in later sections, but first it is necessary to present some technical details.

III. POISSON-BRACKETS RELATIONS AND THE CONSTRAINT ALGEBRA

It is now necessary to review and develop some Poisson-brackets relations for the embedding variables, and then to discuss the algebra of the constraints and related quantities.

A. Poisson-brackets relations

We have the basic Poisson-brackets relations

$$\{X^\alpha(\mathbf{x}),P_\beta(\mathbf{x}')\}=\delta^\alpha_\beta\delta(\mathbf{x},\mathbf{x}') \tag{3.1}$$

from which it follows that

$$\{X^\alpha_i(\mathbf{x}),P_\beta(\mathbf{x}')\}=\delta^\alpha_\beta\delta_{,i}(\mathbf{x},\mathbf{x}') . \tag{3.2}$$

The δ function $\delta(\mathbf{x},\mathbf{x}')$ is scalar with respect to its first argument and a density of weight one with respect to its second. $\delta_i(\mathbf{x},\mathbf{x}')$ will always denote a derivative with respect to the first argument. For convenience we record the useful identity

$$f(\mathbf{x}')\delta_{,i}(\mathbf{x},\mathbf{x}')=f(\mathbf{x})\delta_{,i}(\mathbf{x},\mathbf{x}')+f_{,i}(\mathbf{x})\delta(\mathbf{x},\mathbf{x}') . \tag{3.3}$$

It follows from (3.1) that

$$\{g_{\alpha\beta}(X(\mathbf{x})),P_\gamma(\mathbf{x}')\}=g_{\alpha\beta,\gamma}(X(\mathbf{x}))\delta(\mathbf{x},\mathbf{x}') \tag{3.4}$$

and from the definition of the induced three-metric $h_{ij}(\mathbf{x})$ one has

$$\{h_{ij}(\mathbf{x}),P_\gamma(\mathbf{x}')\}=2X_{\gamma(i}(\mathbf{x})\delta_{,j)}(\mathbf{x},\mathbf{x}') + g_{\alpha\beta,\gamma}(X(\mathbf{x}))X^\alpha_i(\mathbf{x})X^\beta_j(\mathbf{x})\delta(\mathbf{x},\mathbf{x}') \tag{3.5}$$

where subscript parentheses denote symmetrization: $T_{(ab)}=\frac{1}{2}(T_{ab}+T_{ba})$.

We will need the Poisson brackets of the projected embedding momenta:

$$P\equiv n^\alpha P_\alpha, \quad P_i\equiv X^\alpha_i P_\alpha . \tag{3.6}$$

From Eq. (3.5) it follows that

$$\{h_{ij}(\mathbf{x}),P_k(\mathbf{x}')\}=2h_{k(i}(\mathbf{x})\delta_{,j)}(\mathbf{x},\mathbf{x}')+h_{ij,k}(\mathbf{x})\delta(\mathbf{x},\mathbf{x}') \tag{3.7}$$

and

$$\{h_{ij}(\mathbf{x}),P(\mathbf{x}')\}=-2K_{ij}(\mathbf{x})\delta(\mathbf{x},\mathbf{x}') , \tag{3.8}$$

where $K_{ij}\equiv n_\alpha X^\alpha_{i;j}$ is the extrinsic curvature, and the semicolon denotes the spacetime covariant derivative.

It is useful to define the smeared embedding momentum

$$P(\xi)=\int d^3\mathbf{x}\, \xi^\alpha(X(\mathbf{x}))P_\alpha(\mathbf{x}) , \tag{3.9}$$

where $\xi^\alpha(X(\mathbf{x}))$ is an arbitrary spacetime vector field restricted to the embeddings. Smearing Eq. (3.5), one thus obtains the useful result

$$\{h_{ij}(\mathbf{x}),P(\xi)\}=X^\alpha_i X^\beta_j(\xi_{\alpha;\beta}+\xi_{\beta;\alpha}) . \tag{3.10}$$

Equation (3.10) implies, in particular, that $P(\xi)$ has vanishing Poisson brackets with any function of the three-metric if ξ^α is a Killing vector field.

From (3.4) and (3.5) one may derive

$$\{X^i_\alpha(\mathbf{x}),P_\beta(\mathbf{x}')\}=(X^i_\beta(\mathbf{x})X^j_\alpha(\mathbf{x})+h^{ij}(\mathbf{x})n_\beta(\mathbf{x})n_\alpha(\mathbf{x}))\delta_{,j}(\mathbf{x},\mathbf{x}')-g_{\mu\nu,\beta}(X(\mathbf{x}))X^{\mu i}(\mathbf{x})n^\nu(\mathbf{x})n_\alpha(\mathbf{x})\delta(\mathbf{x},\mathbf{x}') . \tag{3.11}$$

Smearing P_β with ξ^β this yields

$$\{X^i_\alpha(\mathbf{x}),P(\xi)\}=-n_\alpha X^{i\mu}n^\nu(\xi_{\mu;\nu}+\xi_{\nu;\mu})-X^i_\beta\xi^\beta_{,\alpha} . \tag{3.12}$$

From the definition of the normal, one finds

$$\{n_\alpha(\mathbf{x}),P_\beta(\mathbf{x}')\}=-X^i_\alpha(\mathbf{x})n_\beta(\mathbf{x})\delta_{,i}(\mathbf{x},\mathbf{x}')-\tfrac{1}{2}g_{\mu\nu,\beta}(X(\mathbf{x}))n^\mu(\mathbf{x})n^\nu(\mathbf{x})n_\alpha(\mathbf{x})\delta(\mathbf{x},\mathbf{x}') \tag{3.13}$$

and smeared with ξ^β this yields

$$\{n_\alpha(\mathbf{x}),P(\xi)\}=-n_\alpha n^\mu n^\nu\xi_{(\mu;\nu)}-n_\beta\xi^\beta_{,\alpha} . \tag{3.14}$$

The simplifications occurring in (3.12) and (3.14) when ξ^α is a Killing vector should be noted.

B. Covariant derivatives and Lie brackets

Next we record some useful results about the normal and tangential projections of spacetime covariant derivatives. For an arbitrary spacetime covector W_α, one has the following projections of its covariant derivative, $W_{\alpha;\beta}$ (Ref. 18):

$$n^\alpha n^\beta W_{\alpha;\beta}=-\frac{1}{N}\delta_N W_\perp-\frac{1}{N}W^i\partial_i N , \tag{3.15}$$

$$X^\alpha_i n^\beta W_{\alpha;\beta}=\frac{1}{N}\delta_N W_i+K_{ij}W^j+\frac{W_\perp}{N}\partial_i N , \tag{3.16}$$

$$n^\alpha X^\beta_j W_{\alpha;\beta}=-W_{\perp|j}+K_{jk}W^k , \tag{3.17}$$

$$X^\alpha_i X^\beta_j W_{\alpha;\beta}=W_{i|j}-W_\perp K_{ij} . \tag{3.18}$$

Here, N is the lapse, a vertical bar denotes the three-dimensional covariant derivative with respect to the three-metric h_{ij}, and

$$\delta_N=-\int d^3\mathbf{x}\, N(\mathbf{x})n^\alpha(\mathbf{x})\frac{\delta}{\delta X^\alpha} . \tag{3.19}$$

Using the above results one may derive the projections of the Lie brackets between spacetime vectors U^α, V^α, Eq. (2.31). Its tangential and normal projections may be shown to be

$$[U,V]^i=h^{ij}(U^\perp\partial_j V^\perp-V^\perp\partial_j U^\perp)+[\mathbf{U},\mathbf{V}]^i + 2n^\alpha X^{i\beta}n^\gamma(U_\gamma V_{(\alpha;\beta)}-V_\gamma U_{(\alpha;\beta)}) , \tag{3.20}$$

$$[U,V]^\perp=U^i\partial_i V^\perp-V^i\partial_i U^\perp - n^\alpha n^\beta n^\gamma(U_\gamma V_{(\alpha;\beta)}-V_\gamma U_{(\alpha;\beta)}) . \tag{3.21}$$

Here, $[\mathbf{U},\mathbf{V}]^i$ denotes the three-dimensional Lie brackets between three-vectors U^i, V^i. Again the simplifications arising when U^α and V^α are Killing vectors should be noted.

C. The algebra of the constraints

Central to what follows is the Dirac algebra. For a set of generators $C(\mathbf{x}), C_i(\mathbf{x})$ it is

$$\{C_i(\mathbf{x}), C_j(\mathbf{x}')\} = C_j(\mathbf{x})\delta_{,i}(\mathbf{x},\mathbf{x}') - C_i(\mathbf{x}')\delta_{,j}(\mathbf{x}',\mathbf{x}) , \tag{3.22}$$

$$\{C_i(\mathbf{x}), C(\mathbf{x}')\} = C(\mathbf{x})\delta_{,i}(\mathbf{x},\mathbf{x}') , \tag{3.23}$$

$$\{C(\mathbf{x}), C(\mathbf{x}')\} = C_i(\mathbf{x})h^{ij}(\mathbf{x})\delta_{,j}(\mathbf{x},\mathbf{x}') - C_i(\mathbf{x}')h^{ij}(\mathbf{x}')\delta_{,j}(\mathbf{x}',\mathbf{x}) . \tag{3.24}$$

The Dirac relations are also conveniently written in smeared form. Define

$$C(N) = \int d^3\mathbf{x}\, N(\mathbf{x})C(\mathbf{x}), \quad C(\mathbf{N}) = \int d^3\mathbf{x}\, N^i(\mathbf{x})C_i(\mathbf{x}). \tag{3.25}$$

We will sometimes write $C(N^i)$ instead of $C(\mathbf{N})$. In smeared form Eqs. (3.22)–(3.24) become

$$\{C(\mathbf{M}), C(\mathbf{N})\} = C([\mathbf{M},\mathbf{N}]) , \tag{3.26}$$

$$\{C(\mathbf{M}), C(N)\} = C(M^i\partial_i N) , \tag{3.27}$$

$$\{C(M), C(N)\} = C(h^{ij}(N\partial_j M - M\partial_j N)) . \tag{3.28}$$

By direct computation, using the results of Sec. III A, it may be shown that the projected embedding momenta $P(\mathbf{x}), P_i(\mathbf{x})$ obey the Dirac relations. Similarly, by direct computation, the scalar field Hamiltonians $\mathcal{H}(\mathbf{x}), \mathcal{H}_i(\mathbf{x})$ may be shown to obey the Dirac relations (3.26) and (3.28), but not (3.27). Instead of (3.27), one has the result

$$\{P(\mathbf{M}) + \mathcal{H}(\mathbf{M}), \mathcal{H}(N)\} = \mathcal{H}(M^i\partial_i N) . \tag{3.29}$$

Consider next the constraints of the parametrized field theory $\Pi_\alpha(\mathbf{x})$. Define the projected quantities

$$\Pi \equiv n^\alpha \Pi_\alpha = P + \mathcal{H} , \tag{3.30}$$

$$\Pi_i \equiv X_i^\alpha \Pi_\alpha = P_i + \mathcal{H}_i . \tag{3.31}$$

Then it readily follows from the above that the projected constraints $\Pi(\mathbf{x}), \Pi_i(\mathbf{x})$ obey the Dirac relations. Finally, one can use these results to establish the Poisson-brackets algebra of the unprojected constraints. One finds

$$\{\Pi_\alpha(\mathbf{x}), \Pi_\beta(\mathbf{x}')\} = 0 . \tag{3.32}$$

This is most easily established by considering each projection of (3.32) and showing that it is zero, which follows from the fact that the projected constraints obey the Dirac algebra, and from the results of Sec. III A.

IV. REPRESENTATION OF SPACETIME ISOMETRIES

One of our ultimate aims is to find quantum states invariant under the isometry groups of spacetime. A state will be invariant if it is annihilated by an appropriate generator of the isometry. Our first task, therefore, is to construct the generators of spacetime isometries.

A spacetime has an isometry if it possesses one or more Killing vectors; i.e., vectors $k^\alpha(\mathbf{X})$ satisfying

$$k_{\alpha;\beta} + k_{\beta;\alpha} = 0 . \tag{4.1}$$

We shall assume that the spacetime has n Killing vectors, k_A^α, $A = 1, 2, \ldots, n$. They will obey the algebra

$$[k_A, k_B] = K_{AB}^C k_C , \tag{4.2}$$

where [,] is the Lie brackets and K_{AB}^C are the structure constants of the isometry group. de Sitter space is the main example in mind, for which the isometry group is SO(4,1), but we will not specialize to this case until much later.

A. Global symmetry generators

At the classical level, we seek the generators of the isometry group acting on the enlarged phase space $(\phi(\mathbf{x}), \pi(\mathbf{x}), X^\alpha(\mathbf{x}), P_\alpha(\mathbf{x}))$, and at the quantum level, we seek their operator counterparts acting on wave functions $\Psi[\phi(\mathbf{x}), X^\alpha(\mathbf{x})]$. Because the quantities $\Pi(U)$ defined by Eq. (2.27) generate the full diffeomorphism group, it trivially follows that the quantities $-\Pi(k_A)$ are generators of the global spacetime isometry group, whose Poisson-brackets algebra closes with structure coefficients K_{AB}^C. However, these are not in fact the objects we need. There are two more sets of generators on the enlarged phase space. The first set is[22]

$$G_A \equiv -\int d^3\mathbf{x}\, k_A^\alpha(X(\mathbf{x}))P_\alpha(\mathbf{x}) = -P(k_A) , \tag{4.3}$$

for which it readily follows that

$$\{G_A, G_B\} = \{P(k_A), P(k_B)\} = -P([k_A, k_B]) = K_{AB}^C G_C . \tag{4.4}$$

These generators act only on (X^α, P_α), not on (ϕ, π). The second set of generators, acting only on (ϕ, π), are

$$Q_A \equiv \int d^3\mathbf{x}\, k_A^\alpha(X(\mathbf{x}))\mathcal{H}_\alpha(\mathbf{x}) = \int d^3\mathbf{x}\, k_A^\alpha(X(\mathbf{x}))n^\beta(\mathbf{x})T_{\alpha\beta}(\mathbf{x}) . \tag{4.5}$$

These generators are of course the familiar conserved charges constructed from the canonical energy-momentum tensor, (2.21). Classically, one has $Q_A = G_A$ when the constraints $\Pi_\alpha = 0$ hold.

Let us now compute the algebra of the generators Q_A. To do this, first note that Q_A may be written

$$Q_A = \mathcal{H}(k_A^\perp) + \mathcal{H}(\mathbf{k}_A) . \tag{4.6}$$

Using the fact that $\mathcal{H}$ and $\mathcal{H}_i$ obey the Dirac relations (3.26) and (3.28), and the relation (3.29), one thus obtains

$$\{Q_A, Q_B\} = \mathcal{H}([\mathbf{k_A}, \mathbf{k_B}]^i + h^{ij}(k_A^\perp \partial_j k_B^\perp - k_B^\perp \partial_j k_A^\perp)) + \mathcal{H}(k_A^i \partial_i k_B^\perp - k_B^i \partial_i k_A^\perp) - \{P(\mathbf{k_A}), \mathcal{H}(k_B^\perp)\} + \{P(\mathbf{k_B}), \mathcal{H}(k_A^\perp)\} . \tag{4.7}$$

Consider the last two terms in (4.7). Because k_A is a Killing vector, it follows from Eq. (3.10) that

$$\{P(k_A),\mathcal{H}(k_B)\}=0\ , \tag{4.8}$$

where, recall, $P(k_A)$ is the embedding momentum smeared with the four-dimensional vector k_A^α. We may, however, write

$$P(k_A)=P(k_A^\perp)+P(\mathbf{k_A})\ , \tag{4.9}$$

where the terms on the right denote the projected embedding momenta P and P_i smeared with $k_A^\perp$ and k_A^i, respectively. Using (4.8) and (4.9), the last two terms in Eq. (4.7) are therefore equal to

$$\{P(k_A^\perp),\mathcal{H}(k_B^\perp)\}-\{P(k_B^\perp),\mathcal{H}(k_A^\perp)\}\ , \tag{4.10}$$

but these two terms cancel, because the Poisson bracket $\{P(\mathbf{x}),\mathcal{H}(\mathbf{x}')\}$ is ultralocal [see Eq. (3.8)].

Consider then the remaining terms in Eq. (4.7). These terms cannot in general be equated with the quantity $\mathcal{H}_\alpha(\mathbf{x})$ smeared with the four-dimensional Lie brackets $[k_A,k_B]$. It is for this reason that the Dirac "algebra" is not a true algebra. Consider, however, the projections of the four-dimensional Lie brackets, (3.20) and (3.21). For Killing vectors k_A,k_B one has the result

$$[k_A,k_B]^i=h^{ij}(k_A^\perp\partial_j k_B^\perp-k_B^\perp\partial_j k_A^\perp)+[\mathbf{k_A},\mathbf{k_B}]^i\ , \tag{4.11}$$

$$[k_A,k_B]^\perp=k_A^i\partial_i k_B^\perp-k_B^i\partial_i k_A^\perp\ . \tag{4.12}$$

Comparing (4.11) and (4.12) with Eq. (4.7), one may see immediately that

$$\begin{aligned}\{Q_A,Q_B\}&=\mathcal{H}([k_A,k_B]^\perp)+\mathcal{H}([k_A,k_B]^i)\\&=\int d^3x[k_A,k_B]^\alpha\mathcal{H}_\alpha\ .\end{aligned} \tag{4.13}$$

That is, the Dirac relations *do* form a true algebra when the smearing functions are projections of spacetime Killing vectors. We thus arrive at the desired result

$$\{Q_A,Q_B\}=K_{AB}^C Q_C\ . \tag{4.14}$$

B. Invariance of the action

We study next the behavior of the action (2.25) under the global transformations generated by G_A and Q_A. These results will be needed when we come to study the path integral in the next section. Consider first G_A. Any function $F(\phi,\pi,X^\alpha,P_\alpha)$ changes under the global transformation generated by G_A according to

$$\delta F=\epsilon^A\{F,G_A\}=-\epsilon^A\{F,P(k_A)\}\ , \tag{4.15}$$

where ϵ^A is a small arbitrary constant parameter. The transformation on the Lagrange multiplier N^α is yet to be determined. Because the transformation is canonical, the change in the kinetic term in (2.25) is given by

$$\delta\int d^3\mathbf{x}\,dt(\dot{X}^\alpha P_\alpha+\dot\phi\pi)=\epsilon^A\left[\int d^3\mathbf{x}\,P_\alpha(\mathbf{x})\frac{\delta G_A}{\delta P_\alpha(\mathbf{x})}-G_A\right]_{t'}^{t''}\ . \tag{4.16}$$

But this vanishes, because G_A is linear in P_α. The change in the action (2.25) is thus given by

$$\delta S=\int d^3\mathbf{x}\,dt[-\delta N^\alpha(P_\alpha+\mathcal{H}_\alpha)-N^\alpha(\delta P_\alpha+\delta\mathcal{H}_\alpha)]\ . \tag{4.17}$$

From (4.15) it readily follows that

$$\delta P_\alpha=\epsilon^A k^\beta_{A,\alpha}P_\beta\ . \tag{4.18}$$

Similarly,

$$\delta\mathcal{H}_\alpha=-\epsilon^A\{-n_\alpha\mathcal{H}+X^i_\alpha\mathcal{H}_i,P(k_A)\}=\epsilon^A k^\beta_{A,\alpha}\mathcal{H}_\beta \tag{4.19}$$

using Eqs. (3.10), (3.12), and (3.14). It is then readily seen that $\delta S=0$ if the transformation of the Lagrange multiplier N^α is taken to be

$$\delta N^\alpha=-\epsilon^A k^\alpha_{A,\beta}N^\beta\ . \tag{4.20}$$

The action is therefore invariant under the global symmetry generated by G_A, (4.15) and (4.20).

Next consider the global transformation generated by Q_A. For the kinetic term, again because the transformation is a canonical one, one has

$$\delta\int d^3\mathbf{x}\,dt(\dot{X}^\alpha P_\alpha+\dot\phi\pi)=\epsilon^A\left[\int d^3\mathbf{x}\,\pi(\mathbf{x})\frac{\delta Q_A}{\delta\pi(\mathbf{x})}-Q_A\right]_{t'}^{t''}\ , \tag{4.21}$$

but this is not zero, because Q_A is quadratic in the momenta $\pi(\mathbf{x})$. The change in the remaining part of the action is again of the form (4.17), and one has

$$\begin{aligned}\delta P_\alpha(\mathbf{x})+\delta\mathcal{H}_\alpha(\mathbf{x})&=\epsilon^A\{\Pi_\alpha(\mathbf{x}),Q_A\}\\&=\epsilon^A\int d^3\mathbf{x}'[\{\Pi_\alpha(\mathbf{x}),k_A^\beta(X(\mathbf{x}'))\}\mathcal{H}_\beta(\mathbf{x}')+\{\Pi_\alpha(\mathbf{x}),\mathcal{H}_\beta(\mathbf{x}')\}k_A^\beta(X(\mathbf{x}'))]\ .\end{aligned} \tag{4.22}$$

Evaluating the Poisson bracket in the first term and using the fact that the Π_α's commute to simplify the second term, one thus obtains

$$\delta P_\alpha(\mathbf{x})+\delta\mathcal{H}_\alpha(\mathbf{x})=-\epsilon^A k^\beta_{A,\alpha}\mathcal{H}_\beta-\epsilon^A\{\mathcal{H}_\alpha(\mathbf{x}),P(k_A)\}\ . \tag{4.23}$$

The second term in (4.23) we have, however, evaluated already in Eq. (4.19) [but note the extra minus sign in (4.19)], and one discovers that the two terms in (4.23) cancel, yielding the result

$$\delta P_\alpha(\mathbf{x})+\delta\mathcal{H}_\alpha(\mathbf{x})=0\ . \tag{4.24}$$

We therefore finally need to choose

$$\delta N^{\alpha}=0 \tag{4.25}$$

for the other terms to vanish. The action is therefore changed by no more than the boundary term (4.21) under the transformation generated by Q_A, supplemented by (4.25).

C. Quantum constraints

In the quantum theory, the quantum states of the field are represented by wave functionals $\Psi[\phi(\mathbf{x}),X^{\alpha}(\mathbf{x})]$ satisfying the generalized functional Schrödinger equation (2.32). An inner product between states may be introduced. It has the form

$$(\Psi_1,\Psi_2)=\int \mathcal{D}\phi\, \Psi_1^*[\phi(\mathbf{x}),X^{\alpha}(\mathbf{x})]\Psi_2[\phi(\mathbf{x}),X^{\alpha}(\mathbf{x})] \; . \tag{4.26}$$

There is no integral over the embeddings, X^{α}, in (4.26), since they really play the role of a "many-fingered" time parameter, rather than that of true dynamical variables. A globally invariant state is one annihilated by the operator versions of G_A or Q_A:

$$\hat{G}_A\Psi=i\int d^3\mathbf{x}\, k_A^{\alpha}\frac{\delta\Psi}{\delta X^{\alpha}}=0 \tag{4.27}$$

or

$$\hat{Q}_A\Psi=\int d^3\mathbf{x}\, k_A^{\alpha}\mathcal{H}_{\alpha}\left[X^{\alpha},\phi,-i\frac{\delta}{\delta\phi}\right]\Psi=0 \; , \tag{4.28}$$

where, for the moment, we are ignoring operator ordering issues. Equations (4.27) and (4.28) are equivalent for states satisfying (2.32). However, of these two it is only through Eq. (4.27) that one may see directly the connection between the action of the isometries on the fields, and the operation of the generators on the wave function. For under a transformation generated by G_A one has

$$\delta X^{\alpha}(\mathbf{x})=\{X^{\alpha}(\mathbf{x}),\epsilon^A G_A\}=-\epsilon^A k_A^{\alpha}, \quad \delta\phi(x)=0 \tag{4.29}$$

for some constant parameter ϵ^A. The change in the wave function under this transformation is

$$\Psi[\phi,X^{\alpha}+\delta X^{\alpha}]=\Psi[\phi,X^{\alpha}]+\int d^3\mathbf{x}\,\delta X^{\alpha}\frac{\delta\Psi}{\delta X^{\alpha}}$$
$$=(1-i\epsilon^A\hat{G}_A)\Psi[\phi,X^{\alpha}] \; . \tag{4.30}$$

From this one clearly sees the connection between the geometrical change in the foliation and the operation of $\hat{G}_A$ on the wave function. In particular, states which for independent reasons can be argued to be independent of such changes of foliation will be annihilated by $\hat{G}_A$.

No such immediate geometric picture exists for the more standard generator $\hat{Q}_A$. Under a transformation generated by it, one has

$$\delta X^{\alpha}(\mathbf{x})=0, \quad \delta\phi(\mathbf{x})=\epsilon^A\{\phi(\mathbf{x}),Q_A\} \; . \tag{4.31}$$

Because Q_A is quadratic in the momenta $\pi(\mathbf{x})$, $\delta\phi(\mathbf{x})$ involves $\pi(\mathbf{x})$ and it is not possible to relate directly the change in the wave function generated by $\delta\phi$ to the operation of $\hat{Q}_A$. It is, however, possible to do this indirectly for wave functionals Ψ satisfying Eq. (2.32), in that for such wave functionals, one has

$$\hat{Q}_A\Psi=\hat{G}_A\Psi \tag{4.32}$$

and (4.30) may then be written

$$\Psi[\phi,X^{\alpha}-\epsilon^A k_A^{\alpha}]=(1-i\epsilon^A\hat{Q}_A)\Psi[\phi,X^{\alpha}] \; . \tag{4.33}$$

Equation (4.33) concisely underscores the utility of the embeddings in this context: the purely geometrical operation of moving or distorting the hypersurfaces, on the left-hand side, is equated with, on the right-hand side, an operation on the wave function involving only the field ϕ.

V. PATH-INTEGRAL REPRESENTATION OF INVARIANT WAVE FUNCTIONALS

Our next task is the construction from a path integral of wave functionals of the parametrized field theory that satisfy the Schrödinger equation (2.33), and that are also invariant under the global symmetry, i.e., satisfy (4.27) or (4.28). What follows is a direct application of the general results of Ref. 13. There, it was shown that for a very general class of theories with global and/or local symmetries, wave functions generated by a path integral are annihilated by the corresponding constraints *provided* the path-integral construction is *invariant* under the symmetry in question. Since it is central to what follows, we now summarize and elaborate on the results of Ref. 13 as they concern the present paper, and explain what is meant by an invariant path-integral construction.

Reference 13 was concerned with a large class of theories with symmetry for which the following two very general requirements are true. First of all, the theory is described by a configuration space consisting of n coordinates q^i which have nonvanishing conjuate momenta p_i and multipliers λ^{α} with vanishing momenta. This includes theories described by a Lagrangian action $S[q^i,\lambda^{\alpha}]$ containing no more than first derivatives in time, or described by a Hamiltonian action $S[p_i,q^i,\lambda^{\alpha}]$. We shall write $S[z^A]$ with z^A being either the set (q^i,λ^{α}) or $(p_i,q^i,\lambda^{\alpha})$. For the case at hand, the q^i's are $\phi(x)$ and $X^{\alpha}(\mathbf{x})$, the p_i's are $\pi(\mathbf{x})$ and $P_{\alpha}(\mathbf{x})$, and λ^{α} is $N^{\alpha}(\mathbf{x})$. We shall devote our attention almost exclusively to the Hamiltonian form of the theory.

The second requirement is that the theory must possess a global or local invariance under which

$$z^A\rightarrow z^A+\delta z^A \; , \tag{5.1}$$

where δz^A depends linearly on m parameters ϵ^{α} and their derivatives $\dot{\epsilon}^{\alpha}$, with the ϵ^{α} freely specifiable functions of t in the case of local symmetries, and freely specifiable constants in the case of global symmetries. Further, we require

$$\delta q^i=\epsilon^{\alpha}f_{\alpha}^i(p_i,q^i) \tag{5.2}$$

for some functions f_{α}^i which depend only on q^i and p_i in the Hamiltonian form or on q^i and $p_i=p_i(\dot{q}^i,q^i)$ in the

Lagrangian one. In particular, δq^i is independent of any multipliers or $\dot{\epsilon}^\alpha$. We assume that the action changes under (5.1) by at most a boundary term independent of $\dot{\epsilon}^\alpha$ and the multipliers. That is, δS has the form

$$\delta S = [\epsilon^\alpha F_\alpha(p_i, q^i)]_{t'}^{t''} \ . \tag{5.3}$$

Condition (5.3) implies the invariance of the equations of motion. It also implies the invariance of the action if the conditions $\epsilon^\alpha(t')=0=\epsilon^\alpha(t'')$ are imposed for those α for which F_α does not vanish identically. It is readily shown that conditions (5.2) and (5.3) hold for constrained Hamiltonian systems, such as the parametrized field theory discussed in this paper.

Now consider wave functions $\Psi(q^i)$ constructed as invariant path integrals of the form

$$\Psi(q^{i''}) = \int_{\mathcal{C}} \mathcal{D}z^A \delta(q^i(t'') - q^{i''}) \delta(q^i(t') - q^{i'}) \times \Delta_\Phi[z^A] \delta[\Phi^\alpha(z^A)] \exp(iS[z^A]) \ . \tag{5.4}$$

The ingredients of this formula are as follows: The variables z^A are the configuration-space or phase-space coordinates defined above. $S[z^A]$ is the Lagrangian or Hamiltonian action satisfying (5.3). $\mathcal{C}$ denotes the class of paths that are integrated over. This integration *includes* an integration over the initial and final values $q^i(t')$, $q^i(t'')$. It is the surface δ functions $\delta(q^i(t')-q^{i'})$ and $\delta(q^i(t'')-q^{i''})$ that ensure that all paths end at the point $q^i(t'')=q^{i''}$, which is the argument of the wave function, and that they begin at the point $q^i(t')=q^{i'}$. We have chosen not to label the wave function by the initial conditions $q^{i'}$ because they will eventually be set to particular values. For definiteness we have fixed $q^i(t)$ initially, but one could equally well fix other dynamical variables there, such as momenta. Other choices of boundary conditions are discussed at the end of the section. $\Phi^\alpha(z^A)$ are a set of gauge-fixing conditions and $\Delta_\Phi[z^A]$ are associated weight factors discussed below. In the case of gauge theories they are Faddeev-Popov determinants. More generally, they are integrals over the ghosts of the exponential of a suitable ghost action and may not always be interpreted as determinants.[13] They are, however, determinants for the system considered in this paper.

In Reference 13 it was shown that if a path-integral construction of the form (5.4) is invariant with a symmetry satisfying (5.2) and (5.3) then the resulting wave function will satisfy operator constraints. To derive this, it is necessary to assume that the action, measure, and class of paths summed over satisfy the following four properties under the transformation (5.1).

(1) The action S changes at the most by a surface term of the form (5.3). It is thus strictly invariant under transformations (5.1) where ϵ vanishes at the end points.

(2) The class of paths, $\mathcal{C}$, is invariant. This condition concerns the ranges of integration of the Lagrange multipliers, but in the case of global symmetries, it also concerns the conditions at the initial point of the paths.

(3) The path integral (5.4) is independent of the choice of gauge conditions Φ^α in a class which includes those generated from a defining Φ^α by a symmetry transformation, that is, at least all $\Phi^\alpha_\epsilon[z^A]$ of the form

$$\Phi^\alpha_\epsilon[z^A] = \Phi^\alpha[z^A + \delta z^A] \ . \tag{5.5}$$

(4) The combination of the measure and the gauge-fixing weight factor transform under a symmetry transformation (5.1) according to

$$\mathcal{D}z^A \Delta_\Phi[z^A] \rightarrow \mathcal{D}z^A \Delta_{\Phi_\epsilon}[z^A] \ . \tag{5.6}$$

In this paper, it will be sufficient to use the Faddeev-Popov construction for gauge-fixing machinery, for which requirement (3) certainly holds. Requirement (4) may be shown to be a property of the Faddeev-Popov construction, in the case where the transformation δ is the local symmetry transformation that the gauge-fixing machinery is breaking, but we will need to check it in the case of global symmetries.

In addition to these four properties characterizing the invariance of the path integral we will also need to assume the following about the implementation of the sum over histories.

(5) Integrals of the form (5.4) weighted by functions of p_i and q^i on the final surface are equal to corresponding, appropriately ordered operators acting on $\Psi(q^{i''})$. That is, for given $\mathcal{F}(p_i, q^i)$,

$$\int_{\mathcal{C}} \mathcal{D}z^A \mathcal{F}(p_i(t''), q^i(t'')) \delta(q^i(t'') - q^{i''}) \Delta_\Phi[z^A] \delta[\Phi^\alpha(z^A)] \exp(iS[z^A]) = \mathcal{F}\left[-i\frac{\partial}{\partial q^{i''}}, q^{i''} \right] \Psi(q^{i''}) \ . \tag{5.7}$$

We shall also assume that a similar result holds for the initial surface, but with a crucial sign difference, namely, that p_i' is effectively replaced by $+i(\partial/\partial q^{i'})$.

Requirement (5) is a property, for example, of a time-slicing implementation of the path integral. It will be assumed without further comment. The task of the next sections will be to show explicitly that the other assumptions hold.

Now we briefly review the derivation of the constraints, paying particular attention to the case of global symmetries. The idea is to perform a change of variables in the path integral consisting of a transformation (5.1), which, in the local case, is taken to be such that $\epsilon^\alpha(t')=0$, but with $\epsilon^\alpha(t'')\neq 0$. The overall integral is unchanged because we are just performing a change of variables. By requirement (1), the action changes by no more than a surface term of the form (5.3). The class of paths $\mathcal{C}$ is unchanged because it is invariant, requirement (2). The measure and gauge-fixing machinery change by no more than a change of gauge-fixing machinery, requirement (4), but by requirement (3), the path integral is insensitive to such changes. One thus has

$$\Psi(q^{i''}) = \int_{\mathcal{C}} \mathcal{D}z^A \delta(q^i(t'') + \delta q^i(t'') - q^{i''}) \delta(q^i(t') + \delta q^i(t') - q^{i'}) \Delta_\Phi[z^A] \delta[\Phi^\alpha(z^A)] \exp(i(S[z^A] + \delta S[z^A])) \ . \tag{5.8}$$

Then subtracting (5.4) from (5.8) and expanding to first order in ϵ^α, one has

$$0=\int_{\mathcal{C}}\mathcal{D}z^A\left[-\delta q^i(t'')\frac{\partial}{\partial q^{i''}}-\delta q^i(t')\frac{\partial}{\partial q^{i'}}+i\delta S\right]\delta(q^i(t'')-q^{i''})\delta(q^i(t')-q^{i'})\Delta_\Phi[z^A]\delta[\Phi^\alpha(z^A)]\exp(iS[z^A]) . \quad (5.9)$$

Using (5.2), (5.3), and assumption (5), this may be written

$$0=-i\int_{\mathcal{C}}\mathcal{D}z^A[\epsilon^\alpha(t'')A_\alpha(p_i(t''),q^i(t''))-\epsilon^\alpha(t')A_\alpha(p_i(t'),q^i(t'))]$$
$$\times\delta(q^i(t'')-q^{i''})\delta(q^i(t')-q^{i'})\Delta_\Phi[z^A]\delta[\Phi^\alpha(z^A)]\exp(iS[z^A]) , \quad (5.10)$$

where we have introduced

$$A_\alpha(p_i,q^i)\equiv f^i_\alpha(p_i,q^i)p_i-F_\alpha(p_i,q^i) . \quad (5.11)$$

For the case of local symmetries, recall that we are taking $\epsilon^\alpha(t)$ to be nonzero at $t=t''$ but to vanish at $t=t'$. Only the terms at the final surface contribute in (5.10), and using assumption (5), we derive the constraints

$$A_\alpha(\hat{p}_i,\hat{q}^i)\Psi(q^i)=0 \quad (5.12)$$

in some operator ordering, where

$$\hat{p}_i=-i\frac{\partial}{\partial q^i},\ \ \hat{q}^i=q^i . \quad (5.13)$$

For the case of global symmetries ϵ^α is a constant, and we cannot dispose of the initial surface terms in (5.10) by imposing suitable boundary conditions on ϵ^α. In cases where one has evolution under a time-dependent Schrödinger equation, assumption (5) applied to both end points in (5.10) would lead to the conclusion that if Ψ is an eigenstate of the operator A_α at $t=t'$, then it remains so under time evolution. In this paper, however, we are concerned with constructing wave functions *invariant* under global symmetries; i.e., Ψ must be annihilated by A_α. Clearly in the global case this can only be achieved *if the initial point of the class of paths summed over satisfies the conditions*

$$A_\alpha(p_i(t'),q^i(t'))=0 . \quad (5.14)$$

Then we derive constraint equations of the form (5.12) in the global case also. The conditions (5.14) are in a sense already subsumed under requirement (2) above—that the class of histories be invariant—but it is useful to be explicit about this to draw the distinction between the case of global and local symmetries. We therefore identify the invariance of the class of histories summed over as the source of the invariance of the quantum state generated by the path integral.

Finally, it will be important for what follows to consider how the above derivation, and in particular Eq. (5.14), may be modified with choices of initial conditions other than fixed initial $q^i(t)$. To this end, let us divide the variables q^i, where $i=1,\ldots,m$ say, into (q^a,q^s), with conjugate momenta (p_a,p_s), where $a=1,\ldots,\tilde{m}$ and $s=\tilde{m}+1,\ldots,m$. Let the variables fixed at the initial point of the paths be p_a and q^s. In the path integral (5.4), the action S is that appropriate to fixed initial and final q^i. However, with fixed initial p_a, this action must be modified by a boundary term, and one works with a new action

$$\tilde{S}=S+p_a(t')q^a(t') . \quad (5.15)$$

It will also be necessary to make an assumption very similar to (5.7), namely, that any function of $q^a(t')$ on the initial slice in the path integral may be replaced by the same function of the operator $-i(\partial/\partial p_a')$. Like assumption (5), this will hold in a time-slicing implementation.

Now consider again the derivation of the constraints, but with this new set of boundary conditions. The derivation is as before, except that the terms in the large parentheses in (5.9) are now

$$-i[\delta q^i(t'')p_i(t'')-\delta q^s(t')p_s(t')+\delta p_a(t')q^a(t')-\delta\tilde{S}] , \quad (5.16)$$

where we have used assumption (5) and the above extension of it to replace the partial derivatives with momenta or coordinates. From (5.15), one has

$$\delta\tilde{S}=\delta S+\delta p_a(t')q^a(t')+p_a(t')\delta q^a(t') . \quad (5.17)$$

The crucial point now is that the terms proportional to $\delta p_a(t')$ cancel in Eq. (5.16), and the calculation is exactly as before: we again set Eq. (5.14), even when some of the momenta are fixed initially.

We conclude, therefore, that in the case of global symmetries we derive the constraints on the wave function (5.12), provided that the combination of coordinates and momenta fixed at the initial point of the histories are such that (5.14) holds.

VI. DERIVATION OF THE CONSTRAINTS FOR PARAMETRIZED FIELD THEORY

Now we apply the results of the preceding section, to the path-integral derivation of global and local constraints on the wave functions of the parametrized field theory. We begin by writing down a path integral for the wave functions of the theory. It is

$$\Psi[\phi''(\mathbf{x}),X^{\alpha''}(\mathbf{x})]$$
$$=\int\mathcal{D}\phi\,\mathcal{D}\pi\,\mathcal{D}X^\alpha\mathcal{D}P_\alpha\mathcal{D}N^\alpha$$
$$\times\delta[\phi(\mathbf{x},t'')-\phi''(\mathbf{x})]\delta[X^\alpha(\mathbf{x},t'')-X^{\alpha''}(\mathbf{x})]$$
$$\times\delta[\Phi^\alpha]\Delta_\Phi\exp(iS) . \quad (6.1)$$

Here, S is the action

$$S=\int_{\mathbb{R}}dt\int_\Sigma d^3\mathbf{x}(\dot{\phi}\pi+\dot{X}^\alpha P_\alpha-N^\alpha\Pi_\alpha) . \quad (6.2)$$

The sum is over the class of histories

$$(\phi(\mathbf{x},t),\pi(\mathbf{x},t),X^{\alpha}(\mathbf{x},t),P_{\alpha}(\mathbf{x},t),N^{\alpha}(\mathbf{x},t)) , \quad t'\le t\le t'' \tag{6.3}$$

satisfying the final conditions

$$\phi(\mathbf{x},t'')=\phi''(\mathbf{x}), \quad X^{\alpha}(\mathbf{x},t'')=X^{\alpha''}(\mathbf{x}) \tag{6.4}$$

(as enforced by the δ functions) with $\pi(\mathbf{x},t'')$, $P_\alpha(\mathbf{x},t'')$, and $N^\alpha(\mathbf{x},t'')$ free, and some initial conditions that we leave for the moment unspecified. The measure in (6.1) is the canonically invariant Liouville measure on $\phi,\pi,X^\alpha,P_\alpha$, and the measure on N^α is just the simple flat measure. The gauge-fixing condition $\Phi^\alpha=0$ must break the local symmetry generated by Π_α, (2.27) and (2.28).

Because the algebra of the constraints is Abelian, the construction of the Faddeev-Popov factor Δ_Φ is comparatively straightforward (in contrast with the case of gravity discussed in Ref. 13). Explicitly, it may be written

$$\Delta_\Phi=\int \mathcal{D}c^\alpha \mathcal{D}\bar{c}_\alpha \exp(iS_{\rm ghost}) , \tag{6.5}$$

where $c^\alpha(\mathbf{x},t)$ and $\bar{c}_\alpha(\mathbf{x},t)$ are anticommuting ghost fields. The ghost action $S_{\rm ghost}$ is given by

$$S_{\rm ghost}=\int d^3x\, dt\, \bar{c}_\alpha \delta_c \Phi^\alpha , \tag{6.6}$$

where δ_c denotes a local symmetry transformation (2.27),(2.28), but with the parameter $\epsilon^\alpha(\mathbf{x},t)$ replaced by the ghost field $c^\alpha(\mathbf{x},t)$.

Given this explicit path integral, we may now consider the derivation of the local and global constraints on the wave function.

A. Local constraints

We first use the local invariance of the sum over histories to derive the generalized functional Schrödinger equation (2.30). Recall that the action is invariant under the diffeomorphisms generated by the constraints Π_α. In particular, under the transformations

$$\delta F(\phi,\pi,X^\alpha,P_\alpha)=\int d^3\mathbf{x}\, \epsilon^\alpha(\mathbf{x})\{F,\Pi_\alpha(\mathbf{x})\} , \tag{6.7}$$

$$\delta N^\alpha=\dot{\epsilon}^\alpha(\mathbf{x}) , \tag{6.8}$$

the action changes by an amount

$$\delta S=\Bigg[\int d^3\mathbf{x}\, \epsilon^\alpha(\mathbf{x},t) \times\Bigg[\int d^3\mathbf{y}\, \pi(\mathbf{y})\frac{\delta\Pi_\alpha(\mathbf{x})}{\delta\pi(\mathbf{y})} +\int d^3\mathbf{y}\, P_\beta(\mathbf{y})\frac{\delta\Pi_\alpha(\mathbf{x})}{\delta P_\beta(\mathbf{y})}-\Pi_\alpha(\mathbf{x})\Bigg]\Bigg]_{t'}^{t''} . \tag{6.9}$$

The action is therefore strictly unchanged if $\epsilon^\alpha(\mathbf{x},t)$ vanishes at both end points. The changes in ϕ and X^α are, explicitly,

$$\begin{aligned}\delta\phi(\mathbf{x})&=\epsilon^\alpha(\mathbf{x})\int d^3\mathbf{y}\, \pi(\mathbf{y})\frac{\delta\Pi_\alpha(\mathbf{x})}{\delta\pi(\mathbf{y})} ,\\ \delta X^\alpha(\mathbf{x})&=\epsilon^\alpha(\mathbf{x})\int d^3\mathbf{y}\, P_\beta(\mathbf{y})\frac{\delta\Pi_\alpha(\mathbf{x})}{\delta P_\beta(\mathbf{y})} .\end{aligned} \tag{6.10}$$

The transformation is therefore of the form (5.2),(5.3), and we may apply the results of the preceding section to derive the constraints.

Let us now consider whether requirements (1)–(4) of the lemma are satisfied. (1) is satisfied by virtue of (6.9). For the class of histories to be invariant, requirement (2), the main restriction is that the Lagrange multipliers N^α must be integrated over an infinite range [cf. Eq. (6.8)]. The measure is invariant because (6.7) is canonical and (6.8) is just a simple shift of N^α. The independence of gauge fixing, requirement (3), is a property of the Faddeev-Popov construction. The only thing left to demonstrate, therefore, is the invariance of the Faddeev-Popov determinant, which we now do.

Under a transformation (6.7),(6.8) with parameter ϵ^α, denote it δ_ϵ, the only change in Δ_Φ, Eq. (6.5), is in the ghost action, for which the change may be written

$$\delta S_{\rm ghost}=\int d^3\mathbf{x}\, dt(\bar{c}_\alpha\delta_c\delta_\epsilon\Phi^\alpha+\bar{c}_\alpha[\delta_\epsilon,\delta_c]\Phi^\alpha) . \tag{6.11}$$

Here, [,] denotes the commutator. The first term in (6.11) is just a change of gauge-fixing function. Since Φ^α can be any function of the dynamical variables, to calculate the second term we need to find the commutator of two local symmetry transformations on each of the dynamical variables. One has

$$[\delta_\epsilon,\delta_c]N^\alpha=\delta_\epsilon\dot{c}^\alpha-\delta_c\dot{\epsilon}^\alpha=0 \tag{6.12}$$

and

$$[\delta_\epsilon,\delta_c]F(\phi,\pi,X^\alpha,P_\alpha) =-\int d^3\mathbf{x}\, d^3\mathbf{y}\, c^\alpha(\mathbf{x})\epsilon^\beta(\mathbf{y})\{\{\Pi_\alpha(\mathbf{x}),\Pi_\beta(\mathbf{y})\},F\}=0 \tag{6.13}$$

by virtue of the Jacobi identity and the fact that the algebra of the Π_α's is Abelian. The second term in (6.11) therefore vanishes, so Δ_Φ changes by no more than a change of gauge-fixing function and requirement (4) is satisfied.

All the assumptions of the lemma are now satisfied, and we therefore derive the constraint (5.12), which reads

$$\hat{\Pi}_\alpha\Psi[\phi(\mathbf{x}),X^\alpha(\mathbf{x})]=\left[-i\frac{\delta}{\delta X^\alpha}+\hat{\mathcal{H}}_\alpha\left[X^\alpha,\phi,-i\frac{\delta}{\delta\phi}\right]\right] \times\Psi[\phi(\mathbf{x}),X^\alpha(\mathbf{x})]=0 . \tag{6.14}$$

B. Global constraints Q_A

Next we consider the derivation of the global constraints corresponding to the symmetry generated by Q_A. Recall that under a symmetry transformation generated by Q_A with parameter ϵ^A, which we denote $\tilde{\delta}_\epsilon$, one has

$$\tilde{\delta}_\epsilon F(\phi,\pi,X^\alpha,P_\alpha)=\epsilon^A\{F,Q_A\} , \tag{6.15}$$

$$\tilde{\delta}_\epsilon N^\alpha = 0 \ . \tag{6.16}$$

Explicitly, the transformations on the configuration-space variables are

$$\tilde{\delta}_\epsilon \phi(\mathbf{x}) = \epsilon^A \frac{\delta Q_A}{\delta \pi(\mathbf{x})}, \quad \tilde{\delta}_\epsilon X^\alpha(\mathbf{x}) = 0 \ , \tag{6.17}$$

so condition (5.2) holds. Under the transformation (6.15), (6.16), the action changes by an amount

$$\tilde{\delta}_\epsilon S = \epsilon^A \left[\int d^3\mathbf{x}\, \pi(\mathbf{x}) \frac{\delta Q_A}{\delta \pi(\mathbf{x})} - Q_A \right]_{t'}^{t''} , \tag{6.18}$$

so condition (5.3) holds.

Now let us check the requirements of the lemma of the preceding section. (1) holds by virtue of (6.18). Requirement (2) necessitates an important set of initial conditions on the histories summed over, and is discussed below. As before, (3) holds. The measure is invariant because (6.15) is canonical, and thus to satisfy (4) we only need to demonstrate the correct transformation properties of the Faddeev-Popov factor Δ_Φ. As in the derivation of the local constraints, this immediately boils down to the issue of calculating the quantity $[\tilde{\delta}_\epsilon, \delta_c]\Phi$ in (6.11). To this end, one has

$$[\tilde{\delta}_\epsilon, \delta_c] N^\alpha = \tilde{\delta}_\epsilon \dot{c}^{\,\alpha} - 0 = 0 \tag{6.19}$$

and

$$[\tilde{\delta}_\epsilon, \delta_c] F(\phi, \pi, X^\alpha, P_\alpha) = \int d^3\mathbf{x}\, c^\alpha(\mathbf{x}) \epsilon^A \{\{Q_A, \Pi_\alpha(\mathbf{x})\}, F\} = 0 \tag{6.20}$$

because $\{Q_A, \Pi_\alpha(\mathbf{x})\} = 0$, by Eq. (4.24). Δ_Φ is therefore changed by no more than a gauge-fixing term, as required. The quantity (5.11) for this case is readily calculated, and is found to be quite simply Q_A. We therefore derive the operator constraints on the wave function,

$$\hat{Q}_A \Psi[\phi(\mathbf{x}), X^\alpha(\mathbf{x})] = 0 \ , \tag{6.21}$$

provided that the histories summed over satisfy the initial condition (5.14), which in this case turns out to be the condition $Q_A = 0$ at $t = t'$.

C. Global constraints G_A

Now we repeat the above derivation, but this time using the global symmetry generator G_A. We again denote the symmetry transformation generated by G_A as $\tilde{\delta}_\epsilon$. Under this transformation one has

$$\tilde{\delta}_\epsilon F(\phi, \pi, X^\alpha, P_\alpha) = \epsilon^A \{F, G_A\} \ , \tag{6.22}$$

$$\tilde{\delta}_\epsilon N^\alpha = -\epsilon^A k^{\alpha}_{A,\beta} N^\beta \ . \tag{6.23}$$

The explicit transformations on the configuration-space variables are

$$\tilde{\delta}_\epsilon \phi(\mathbf{x}) = 0, \quad \tilde{\delta}_\epsilon X^\alpha(\mathbf{x}) = \epsilon^A \frac{\delta G_A}{\delta P_\alpha(\mathbf{x})} = -\epsilon^A k^\alpha_A \ , \tag{6.24}$$

so condition (5.2) holds. The action is strictly invariant under (6.22) and (6.23): $\tilde{\delta}_\epsilon S = 0$ with no boundary terms.

Next we must check the requirements of the lemma. (1) holds because $\tilde{\delta}_\epsilon S = 0$. Again (2) necessitates certain initial conditions on the set of histories summed over. (3) is a property of the Faddeev-Popov construction. As before, the main calculational issue is in requirement (4). The measure $\mathcal{D}\phi \mathcal{D}\pi \mathcal{D}X^\alpha \mathcal{D}P_\alpha$ is again invariant because (6.22) is canonical. The measure $\mathcal{D}N^\alpha$ is not invariant, however, because the transformation (6.23) is nontrivial and a Jacobian factor arises.

Now consider the transformation of the Faddeev-Popov term. Again we find we have to calculate the quantity $[\tilde{\delta}_\epsilon, \delta_c]\Phi^\alpha$. One readily finds

$$[\tilde{\delta}_\epsilon, \delta_c] F(\phi, \pi, X^\alpha) = -\int d^3\mathbf{x}\, \epsilon^A c^\alpha(\mathbf{x}) k^\beta_{A,\alpha} \{\Pi_\beta(\mathbf{x}), F\} \ , \tag{6.25}$$

where note that for simplicity we have taken F to be independent of P_α. We are therefore restricting attention to gauge-fixing functions Φ^α independent of P_α, but this is not a serious restriction. Introducing δc^β, defined by

$$\delta c^\beta = \epsilon^A c^\alpha k^\beta_{A,\alpha} \ , \tag{6.26}$$

(6.25) may be written

$$[\tilde{\delta}_\epsilon, \delta_c] F = \delta_{\delta c} F \ , \tag{6.27}$$

where $\delta_{\delta c}$ denotes a local symmetry transformation (6.7), with parameter δc. Similarly for N^α one has

$$[\tilde{\delta}_\epsilon, \delta_c] N^\alpha = \epsilon^A (k^\beta_{A,\alpha\gamma} N^\gamma + k^\beta_{A,\alpha} \dot{c}^{\,\alpha}) \ . \tag{6.28}$$

The right-hand side may be equated with $\delta_{\delta c} N^\alpha$ once it is observed that one may write $N^\alpha = \dot{X}^\alpha$. One may do this because the P_α integral in the path integral brings down a δ function enforcing this equality, provided that, as we have already assumed, the gauge-fixing function is independent of P_α. We have now proved the result

$$[\tilde{\delta}_\epsilon, \delta_c] \Phi^\alpha = \delta_{\delta c} \Phi^\alpha \ . \tag{6.29}$$

The term (6.29) appearing in the transformed ghost action (6.11) may be eliminated by performing the change of variables

$$c^\alpha \to c^\alpha - \delta c^\alpha = c^\alpha - \epsilon^A c^\beta k^\alpha_{A,\beta} \tag{6.30}$$

in expression (6.5) for Δ_Φ, leading to a nontrivial Jacobian factor coming from the measure $\mathcal{D}c^\alpha$. Compare the transformation (6.30) with the transformation on N^α, Eq. (6.23), which also leads to a nontrivial Jacobian factor in the measure $\mathcal{D}N^\alpha$. The two transformations are in fact identical. However, because the c^α's are *anticommuting* variables, the Jacobian arising from (6.30) is precisely the *inverse* of that arising from (6.23), and thus they cancel. So although the measure and Faddeev-Popov factor do not separately have the desired transformation properties, together they do—their combination changes by no more than a change of gauge-fixing function and requirement (4) is satisfied.

The quantity (5.11) is readily calculated and is found to be G_A. Once again, therefore, we derive the operator constraints on the wave function

$$\hat{G}_A \Psi[\phi(\mathbf{x}), X^\alpha(\mathbf{x})] = 0 \qquad (6.31)$$

provided that the histories summed over have vanishing G_A initially.

D. The importance of being embedded

Since it is perhaps not completely obvious from the preceding discussion, it is important to emphasize the *necessity* of using the embedding variables in the derivation of these constraints from the path integral. One might, for example, have thought that it is possible to derive the global constraints using the usual path-integral representation of the wave function without the embeddings, on a fixed foliation. This path integral takes the form

$$\Psi[\phi''(\mathbf{x}), t] = \int \mathcal{D}\phi \, \mathcal{D}\pi \, \delta[\phi(\mathbf{x}, t'') - \phi''(\mathbf{x})] e^{iS} , \qquad (6.32)$$

where S is the usual action

$$S = \int d^3\mathbf{x} \, dt (\pi \dot{\phi} - N\mathcal{H} - N^i \mathcal{H}_i) . \qquad (6.33)$$

Here the lapse and shift N and N^i are totally fixed quantities which are not integrated over. In attempting to apply the lemma to derive the constraints, using the symmetry generated by Q_A (since G_A is no longer available) one quickly discovers that although the measure is invariant, the action is not, due to the noninvariance of the Hamiltonian:

$$\{Q_A, N(\mathbf{x})\mathcal{H}(\mathbf{x}) + N^i(\mathbf{x})\mathcal{H}_i(\mathbf{x})\} = -\int d^3\mathbf{y} \, k_A^\alpha(\mathbf{x}) N^\beta(\mathbf{y}) \{\mathcal{H}_\alpha(\mathbf{x}), P_\beta(\mathbf{y})\} \neq 0 . \qquad (6.34)$$

It is for this reason that one fails to derive the constraints.

Is the noninvariance of the action (6.33) in conflict with one's expectation that, in Minkowski space, for example, the scalar field action should be Lorentz invariant? The answer is no. The action (6.33) is a function of the dynamical fields ϕ, π, but it is also a function of nondynamical, prescribed background fields, namely the metric components. The generators of the Lorentz transformation, Q_A, act only on the dynamical fields ϕ, π, not on the background fields. Lorentz invariance of the action is in fact attained if, by hand, one supplements the canonical transformations on ϕ and π with appropriate transformations on the metric. The derivation of the constraints, however (from the Hamiltonian form of the path integral), relied crucially on the existence of a *canonical* transformation under which the action is invariant, and the necessary transformation on the metric is not canonical. It is by making the embeddings dynamical that the metric becomes a function of the dynamical variables, and the appropriate transformation on it then *is* a canonical transformation.

Put differently, the difficulty is due to the fact that a fixed foliation is taken in Eq. (6.32), while global spacetime isometries typically involve motions or distortions of the foliation itself. Indeed, the amount (6.34) by which the Hamiltonian fails to be invariant is nothing more than a transformation of the foliation itself. By making the embeddings describing the foliation dynamical, as is done here in the parametrized field theory, changes in the foliation such as that incurred in (6.34) may be absorbed by symmetry transformations of the embeddings.

It should also be noted that this inability to represent spacetime symmetries is not necessarily a consequence of basing our analysis on the Hamiltonian form of the path integral—it would also arise in the equivalent Lagrangian form of the path integral. Rather, the difficulty is intrinsic to the functional Schrödinger picture in which the central notion is that of a wave function on a spacelike surface.

To end this section, we make some remarks about the main result of this section—that summing over histories with vanishing initial global charges leads to invariant wave functionals. This result may perhaps seem obvious, or even trivial, in that it is essentially equivalent to the statement that a zero-eigenvalue eigenstate of the charge operator will remain so under time evolution. However, in this paper we are not merely using the path integral to construct a propagation amplitude which respects the global invariance of the theory. Rather, we are giving the path integral a more fundamental role as a *generator* of invariant wave functionals, and attempting to identify those aspects of the sum-over-histories construction that lead to invariant states. This endeavor is not without motivation in that, as we shall see in the next section, there exist proposals for the quantum state of matter modes in certain spacetimes that are given in path-integral form.

VII. VACUUM STATES IN de SITTER SPACE

We now apply the results of the preceding sections to the case of de Sitter space. Recall that de Sitter space is a solution to the Einstein equations with positive cosmological constant Λ. It may be thought of as a four-dimensional hyperboloid embedded in five-dimensional Minkowski space. It is maximally symmetric, with ten Killing vectors and isometry group SO(4,1), the five-dimensional Lorentz group. The metric on de Sitter space may be written

$$ds^2 = -dt^2 + \frac{1}{H^2}\cosh^2(Ht) d\Omega_3^2 , \qquad (7.1)$$

where $d\Omega_3^2$ is the metric on the unit three-sphere, and $3H^2 = \Lambda$. Its Euclidean section is the four-sphere S^4 with isometry group SO(5), and metric

$$ds^2 = d\tau^2 + \frac{1}{H^2}\sin^2(H\tau) d\Omega_3^2 . \qquad (7.2)$$

Scalar field theory in a background such as de Sitter space[23] is normally quantized in the Heisenberg picture by first introducing a set of mode functions $u_k(\mathbf{x}, t)$ satisfying a wave equation of the form

$$(\Box - m^2) u_k(\mathbf{x}, t) = 0 . \qquad (7.3)$$

The field operator $\hat{\Phi}$ is then expanded in terms of these mode functions

$$\hat{\Phi}(\mathbf{x}, t) = \sum_k [\hat{a}_k u_k(\mathbf{x}, t) + \hat{a}_k^\dagger u_k^*(\mathbf{x}, t)] , \qquad (7.4)$$

where $\hat{a}_k^\dagger$ and $\hat{a}_k$ are the usual creation and annihilation operators. The vacuum state is then defined to be the state $|0\rangle$ for which

$$\hat{a}_k|0\rangle=0 \tag{7.5}$$

for all k. The vacuum state is determined by the choice of mode functions u_k.

In Minkowski space, there is a unique vacuum state which is invariant under the Poincaré group; so is the agreed vacuum state for all inertial observers. For an arbitrary curved spacetime, however, there is generally no unique natural choice of vacuum state. For spacetimes with isometries, it is natural to look for states which are invariant under the isometry group. In particular, in de Sitter space, one is interested in de Sitter-invariant states.[24]

The traditional way of studying invariant vacua is through the symmetric two-point function in a state $|\lambda\rangle$:

$$G_\lambda(x,y)=\langle\lambda|[\hat{\Phi}(x)\hat{\Phi}(y)+\hat{\Phi}(y)\hat{\Phi}(x)]|\lambda\rangle\ , \tag{7.6}$$

where $\hat{\Phi}(x)$ is the scalar field operator in the Heisenberg picture. The state $|\lambda\rangle$ is then said to be de Sitter invariant if the two-point function depends on x and y only through $\mu(x,y)$, the geodesic distance between x and y:

$$G_\lambda(x,y)=f_\lambda(\mu)\ . \tag{7.7}$$

Using the fact that $\hat{\Phi}$ obeys the Klein-Gordon equation, a second order ordinary differential equation for $f_\lambda(\mu)$ is readily derived. From it, it may be shown that there is not just one de Sitter-invariant vacuum, according to the above definition, but there is a one-parameter family of inequivalent de Sitter-invariant vacua.

For this one-parameter family, the function $f_\lambda(\mu)$ generally has two poles: one when y is on the light cone of x, the other when y is on the light cone of $\bar{x}$, the point in de Sitter space antipodal to x. However, among the one-parameter family, there is one member for which $f_\lambda(\mu)$ has just one pole, when y is on the light cone of x. This member is called the "Euclidean" or "Bunch-Davies" vacuum, and has the nicest analytic properties. This is often the one that is used in calculations of density fluctuations in inflationary universe models, for example.

A second way of characterizing the Euclidean vacuum is in terms of the mode expansion (7.4). The Euclidean vacuum is defined to be the vacuum state corresponding to the set of mode functions $u_k(\mathbf{x},t)$ which are *regular* on the Euclidean section of de Sitter space, i.e., regular on the entire four-sphere.

In the context of this paper, a third definition of de Sitter-invariant state is appropriate: a de Sitter-invariant state is one represented by a wave functional $\Psi[\phi(\mathbf{x}),X^\alpha(\mathbf{x})]$ which is annihilated by the de Sitter generators, i.e., satisfies (4.27) or (4.28) with the Killing vectors k_A^α taken to be the ten Killing vectors of de Sitter space. This definition has previously been studied by Burges,[4] and by Floreanini *et al.*,[5] but without using the parametrized field theory. Floreanini *et al.* argued that by this definition of invariant state there is in fact only one state that is truly invariant under the de Sitter-invariant group, namely the Euclidean vacuum. The other "invariant" states in the one-parameter family $|\lambda\rangle$ as defined above are in fact eigenstates of the de Sitter generators, and are therefore changed by a phase under the action of the de Sitter operator $\exp(-i\epsilon^A\hat{Q}_A)$.

Here, we will discuss de Sitter-invariant states in the context of a path-integral representation of the wave functionals of scalar field theory. If we were starting from scratch, then such an endeavor would involve identifying the class of histories that it would be necessary to sum over to generate de Sitter-invariant states. However, it turns out that studies in quantum cosmology have already led to the identification of the appropriate class of histories: the "no-boundary" path-integral proposal of Hartle and Hawking[15] for the wave function of the Universe has been argued to lead, in the semiclassical approximation, to de Sitter-invariant states for matter wave functionals.[14,25] The argument of Ref. 25 used the second of the above three definitions of de Sitter invariance. Reference 14 used a very heuristic argument that hinted at the third definition. Our task to review the argument of Ref. 14, and present it in a mathematically precise fashion, using the results of the preceding sections.

We begin by reviewing the calculation of the no-boundary wave function for the Universe. We are interested in the no-boundary wave function for a three-surface Σ of three-sphere topology on which the three-metric is $h''_{ij}(\mathbf{x})$ and the matter-field configuration is $\phi''(\mathbf{x})$. It is defined by a path-integral expression of the form

$$\Psi_{NB}[h''_{ij}(\mathbf{x}),\phi''(\mathbf{x})] = \int \mathcal{D}g_{\mu\nu}\mathcal{D}\phi\,\exp(-I_g[g_{\mu\nu}]-I_m[g_{\mu\nu},\phi])\ , \tag{7.8}$$

where I_g is the Euclidean action for gravity,

$$I_g[g_{\mu\nu}]=\int d^4x\sqrt{g}\,(-R+2\Lambda)\ , \tag{7.9}$$

and I_m the Euclidean version of the matter action, (2.13). The integral (7.8) is taken over metrics $g_{\mu\nu}$ and matter fields ϕ on *compact* four-manifolds $\mathcal{M}$ whose only boundary is the three-surface Σ, on which $g_{\mu\nu}$ and ϕ must match the arguments of the wave function, $h''_{ij}(\mathbf{x}),\phi''(\mathbf{x})$. One is also supposed to sum over all compact manifolds M with boundary Σ. However, it is not known exactly how to define this sum, so in practice one considers each term in the sum over manifolds separately. Here, we will consider only the case in which the four-manifold M is the four-ball B^4. In the standard $3+1$ decomposition we are using here (that is, *without* the embedding variables), it turns out to be necessary to impose conditions on the fields as the foliating three-surfaces shrink to zero. An appropriate set of conditions may be found by insisting that the saddle points of the integrals over metrics and matter fields are *regular* solutions to the field equations on B^4, matching the prescribed data on the three-sphere boundary.[26]

We shall regard the scalar field $\phi(\mathbf{x},t)$ as a small perturbation on the gravitational field—it does not act as a source to the approximations in which we will be working. In the saddle-point approximation to the integral over metrics, (7.8) then leads to an expression of the form

$$\Psi_{NB}[h''_{ij}(\mathbf{x}),\phi''(\mathbf{x})] \approx \exp(-I_g[\bar{g}_{\mu\nu}]) \times \int \mathcal{D}\phi \exp(-I_m[\bar{g}_{\mu\nu},\phi]) , \quad (7.10)$$

where $\bar{g}_{\mu\nu}$ is the saddle-point metric, i.e., an extremum of the Einstein action (7.9). We are interested in the case in which the three-surface Σ is a three-sphere of radius a with the usual round metric $h_{ij}=a^2\Omega_{ij}$. Then, when $aH<1$, $\bar{g}_{\mu\nu}$ is real and is the metric (7.2) on the section of four-sphere closing off a three-sphere of radius a. When $aH>1$, $\bar{g}_{\mu\nu}$ is complex. It may be thought of as a section of de Sitter space described by the metric (7.1) matched at its minimum radius onto half a four-sphere with metric (7.2).

The form of Eq. (7.10) invites one to regard the quantities

$$\psi[a'',\phi''(\mathbf{x})] = \int \mathcal{D}\phi \exp(-I_m[\bar{g}_{\mu\nu},\phi]) \quad (7.11)$$

as matter wave functionals for a scalar field in a (possibly Euclidean) de Sitter space background. This is justified in that one may show that these wave functionals are solutions to the functional Schrödinger equation.[27] The no-boundary proposal implies that the integral over matter modes in (7.11) is over fields $\phi(\mathbf{x},\tau)$ on B^4 that match $\phi''(\mathbf{x})$ on the three-sphere boundary, and are such that the saddle point of the functional integral over ϕ corresponds to a *regular* solution to the scalar field equation on the background geometry.

Equation (7.11) is our candidate for a path integral generating de Sitter-invariant states. Indeed, by direct computation of the wave functionals, it may be shown that this state corresponds to the Euclidean vacuum, as defined above.[25] However, as noted above, a more heuristic argument for the de Sitter invariance of the no-boundary matter wave functionals has been given.[14] This argument shows that the de Sitter invariance is an inevitable consequence of the very geometrical nature of the no-boundary proposal, and is therefore true of all types of matter fields admitting de Sitter-invariant vacua.

Let us first recall the heuristic argument given in Ref. 14. Suppose one asks for the quantum state of the matter field on a three-sphere of radius $a<H^{-1}$. The no-boundary state is defined by a path integral of the form (7.11). One sums over all matter fields regular on the section of four-sphere interior to the three-sphere which match the prescribed data on the three-sphere boundary. The resulting state will depend on the geometry only through the radius of the three-sphere, and not on its intrinsic location or orientation on the four-sphere. One thus has the freedom to move the three-sphere around on the four-sphere without changing the quantum state—at each location one is summing over exactly the same field configurations to define it. These different locations are related to each other by the isometry group of the four-sphere, SO(5). It follows that the state is SO(5) invariant on the Euclidean section. On continuation back to the Lorentzian section, one thus finds that the state is invariant under SO(4,1), the de Sitter group; that is, the state is de Sitter invariant.

Although perhaps clear intuitively, there are at least two difficulties with this heuristic argument. First, in the usual functional Schrödinger quantization, strictly speaking one cannot talk about rotations of the three-sphere on the four-sphere, because the wave functionals depend only on the time coordinate label of the fixed foliation: they carry no information about the location or orientation of the three-surface on the four-sphere. Secondly, the key element of the above argument is that the fields on the cap of four-sphere enclosed by the three-sphere boundary "look the same" irrespective of the location of orientation of the three-sphere on the four-sphere. However, in concrete implementations of the path integral (7.11), one is obliged to take a particular foliation of the four-sphere. For example, one might take the leaves of the foliation to be surfaces of constant τ in the metric (7.2). This has the consequence that there is a preferred point on the four-sphere from which the foliation emerges at which it is necessary to impose regularity conditions. This means that the fields on the cap of the four-sphere enclosed by the three-sphere boundary will not in fact "look the same" for all possible locations and orientations of the three-sphere boundary.

Clearly the way around these difficulties is to introduce the embeddings as dynamical variables, as we have in the earlier parts of this paper. The first difficulty is comfortably handled because the embeddings *do* allow one to talk about motions or distortions of the three-surface. In particular, Eq. (4.33) captures in precise mathematical form the notion that a wave function is annihilated by the generator Q_A if it can be argued to be insensitive to the purely geometrical operation of moving the three-surface around on the four-sphere under the action of SO(5). With regard to the second difficulty, it is still true even when the embeddings are dynamical that one has to impose initial conditions at an "initial" point on the four-sphere from which the family of foliating surfaces emerges. However, in integrating over the embedding variables in the path integral, one integrates over all possible initial points. This means that there is no longer a preferred initial point, and the field configurations summed over on the cap of four-sphere will indeed look the same for all possible orientations of the bounding three-sphere.

The most convincing demonstration of the de Sitter invariance of (7.11), however, is to apply the results of Sec. V, and attempt to derive global constraints on the no-boundary wave function for the matter modes. We are therefore led to study the Euclidean version of the path-integral representation (6.1) of wave functionals of the parametrized field theory [the natural generalization of Eq. (7.11)], where the class of histories summed over is taken to be that specified by the no-boundary proposal of Hartle and Hawking. That is, we sum over histories (6.3) satisfying the final conditions (6.4) and satisfying such initial conditions as guarantee that the saddle points of the path integral correspond to *regular* solutions to the field equations for the scalar field and for the embeddings. Our object is to show that the conditions on the paths specified by this proposal imply that the de Sitter charges Q_A and/or G_A vanish initially, and thus, using the results of Sec. V, that the resulting wave functional is de Sitter invariant.

What, then, are the initial conditions on the histories summed over in (6.1) corresponding to the no-boundary amplitude?

Let us begin by considering the embedding variables. Strictly speaking, our analysis really only applies to four-manifolds of topology $\mathbb{R}\times\Sigma$. The case of the four-manifold B^4 considered here, therefore, must be obtained as a limiting case of manifolds of topology $\mathbb{R}\times\Sigma$, and one must expect to encounter some kind of singularities at the "beginning" of the foliation. It will then be necessary to impose suitable conditions on the embeddings to ensure regularity as the initial three-manifold goes to zero, and to ensure that the embeddings describe a "reasonable" foliation of the four-sphere. Heuristically, we envisage that the volume factor of the three-surface, $h^{1/2}$, must go to zero. In fact, because the charges Q_A and G_A are given by three-surface integrals [Eqs. (4.3) and (4.5)], this condition will guarantee that the charges vanish, provided that the integrands remain well behaved.

There are many ways in which the three-surface may shrink to zero. The plethora of ways may be characterized by the eigenvalues of h_{ij} at each point $\mathbf{x}$. For example, if just one of the eigenvalues goes to zero, than the three-surface volume goes to zero, but the three-surface just degenerates to a two-surface. On the other hand, if all the eigenvalues go to zero, than rather than degenerate to a lower-dimensional surface, the three-surface shrinks right down to a single point. Given that some kind of singular behavior is inevitable as the three-surface volume goes to zero, it seems desirable to restrict this singular behavior to just a single point, rather than to a one- or two-dimensional surface. We shall therefore demand that all the eigenvalues of h_{ij} go to zero. This is in fact completely equivalent to the condition that $h^{1/2}h^{ij}$ goes to zero (from which it follows that $h^{1/2}$ goes to zero). One could perhaps argue that this is in accord with the requirement that the embeddings describe a "reasonable" foliation of the four-sphere. We note that this condition is not obviously obligatory, on general grounds, and it would be interesting to investigate the consequences of choosing other initial conditions. However, it does seem to be the only possibility consistent with regularity of the constraint equations, as we shall see below.

It is difficult to see how the conditions $h^{1/2}h^{ij}=0$ can be enforced directly by imposing conditions on some combination of the canonical variables, ϕ, π, X^α, and P_α. Following the general approach of Ref. 26, we shall therefore demand only that the initial conditions $h^{1/2}h^{ij}=0$ be a *consequence* of the initial conditions we choose to impose, via the constraints or field equations.

Now consider the Euclidean constraint equations

$$P_\alpha=-\bar{n}_\alpha\mathcal{H}-X^i_{\alpha}\mathcal{H}_i \ . \tag{7.12}$$

Here, $\bar{n}_\alpha$ is the Euclidean normal, and is normalized according to $\bar{n}_\alpha\bar{n}^{\alpha}=1$. Recall that $\mathcal{H}$ and $\mathcal{H}_i$ are given by

$$\mathcal{H}=\tfrac{1}{2}h^{1/2}(h^{-1}\pi^2+h^{ij}\partial_i\phi\partial_j\phi+m^2\phi^2)\ , \tag{7.13}$$

$$\mathcal{H}_i=\partial_i\phi\pi\ . \tag{7.14}$$

In some (but not all) choices of gauge in the sum over histories, the integration over N^α will enforce the constraints as δ functions in the path integral. Let us ask what happens in the constraints as $h^{1/2}$ and $h^{1/2}h^{ij}$ go to zero, if we insist that all quantities remain regular. Clearly for the right-hand side to remain regular it is necessary that π goes to zero as $h^{1/2}$ goes to zero. Then we also deduce that $P_\alpha=0$.

But now let us turn the argument around. Let us insist that the initial conditions are

$$\pi(\mathbf{x})=0,\quad P_\alpha(\mathbf{x})=0\ . \tag{7.15}$$

Then the constraints imply that $h^{1/2}=0$, and $h^{ij}h^{1/2}=0$. Furthermore, the condition $\pi(\mathbf{x},0)=0$ will imply that the classical solutions to the scalar field equations are regular. We therefore conclude that the conditions (7.15) are the canonical initial data enforcing regularity of the classical solutions with the embeddings describing a reasonable foliation of the four-sphere. The conditions (7.15) are therefore reasonable candidates for the canonical initial data corresponding to the no-boundary amplitude.

Finally, it is readily seen that the conditions (7.15) imply that the charges G_A, Q_A vanish. We therefore deduce that the no-boundary wave functional for the scalar field is annihilated by the global operator constraints (4.27) and (4.28), and thus, that it is de Sitter invariant.

VIII. ANOMALIES

The treatment so far of quantum issues has been formal, in that we have not addressed the full field-theoretic aspects of the problem. We now discuss the important issue of anomalies in the algebra of the various sets of symmetry generators, both local and global.

The first point at which anomalies may occur is in the quantum version of the algebra of the diffeomorphism constraints, (2.26). On general grounds, one should in fact expect to find anomalies here, because the constraints $\Pi_\alpha(\mathbf{x})$ contain projections of the energy-momentum tensor of the scalar field, $T_{\alpha\beta}$, and it is known that anomalies generally arise in the commutators of $T_{\alpha\beta}$.[28]

To actually calculate the anomaly in the quantum version of the algebra (2.26) would be a very difficult task, and will not be attempted here. However, one can give general arguments about the form the anomaly must take. Moreover, the calculation has been carried out explicitly in the special case of a two-dimensional cylindrical spacetime,[29] where the problem essentially reduces to that of finding the anomaly in the Virasoro algebra, a problem which has a well-known solution.

On general grounds, a reasonable assumption (supported by the explicit two-dimensional example) is that the anomaly in the algebra of diffeomorphism constraints depends only on the embeddings, and has the form

$$[\hat{\Pi}_\alpha(\mathbf{x}),\hat{\Pi}_\beta(\mathbf{x}')]=F_{\alpha\beta}(\mathbf{x},\mathbf{x}';X^\alpha] \tag{8.1}$$

for some quantity $F_{\alpha\beta}(\mathbf{x},\mathbf{x}';X^\alpha]$, a function of the points $\mathbf{x},\mathbf{x}'$ and a functional of the embeddings X^α. Assuming that the Jacobi identity holds (it need not if there is a three-cocycle), one may deduce from the above that $F_{\alpha\beta}$ has vanishing exterior derivative, and thus that it is an exact curl,

$$F_{\alpha\beta}(\mathbf{x},\mathbf{x}';X]=\frac{\delta A_\alpha(\mathbf{x};X]}{\delta X^\beta(\mathbf{x}')}-\frac{\delta A_\beta(\mathbf{x}';X]}{\delta X^\alpha(\mathbf{x})} \tag{8.2}$$

for some "anomaly potential" $A_\alpha(\mathbf{x};X^\alpha]$. Because of the form of the anomaly, it may be removed by adding $-A_\alpha$ to the generators Π_α. One thus arrives at new quantum generators

$$\hat{\Pi}_\alpha^{\text{new}}=\hat{P}_\alpha+\hat{\mathcal{H}}_\alpha-A_\alpha \tag{8.3}$$

satisfying the correct algebra

$$[\hat{\Pi}_\alpha^{\text{new}}(\mathbf{x}),\hat{\Pi}_\beta^{\text{new}}(\mathbf{x}')]=0 \tag{8.4}$$

without anomaly. However, although finite in the two-dimensional example, it is quite possible that the anomalous term is infinite in the four-dimensional case, rendering the new generators (8.3) ill defined.[30]

The next point at which anomalies could conceivably arise is in the algebra of the global symmetry generators. Because of their particularly simple form, it seems reasonable to assume that the global symmetry generators G_A may be quantized without anomaly. For the generators Q_A, however, because they are constructed from the energy-momentum tensor, there is the possibility of anomalies. Such anomalies were not present in the two-dimensional example of Ref. 29, but there is no obvious reason why this should also be true of the four-dimensional case considered here.

Again one can apply general reasoning to find something out about anomalies in the algebra of the quantum generators $\hat{Q}_A$. It seems reasonable to assume that the quantum algebra has, at worst, the form

$$[\hat{Q}_A,\hat{Q}_B]=iK^C_{AB}\hat{Q}_C+F_{AB}\ , \tag{8.5}$$

where F_{AB} is a c number that does not expand on ϕ. This is consistent with the Jacobi identity (again assuming there is no three-cocycle). Suppose we shift the generator $\hat{Q}_A$ by a c-number term q_A, in Eq. (8.5). Then the anomalous term may be removed if we can find a q_A such that

$$iK^C_{AB}q_C+F_{AB}=0\ . \tag{8.6}$$

If the global symmetry group is semisimple [as is the de Sitter group, SO(4,1)], the Killing metric

$$g_{AB}=K^D_{AC}K^C_{BD} \tag{8.7}$$

is invertible. The indices on K^C_{AB} may be moved up and down using this metric and one may form the totally antisymmetric quantity

$$K_{ABC}=g_{CD}K^D_{AB}\ . \tag{8.8}$$

Equation (8.6) may then be solved for q_C, with the solution

$$q_C=-ig^{AD}g^{BE}K_{CAB}F_{DE}\ . \tag{8.9}$$

For semisimple groups, therefore, the anomaly may be removed [although a one-cocycle may appear in relations of the form (4.33)].

If the group is not semisimple (e.g., the Poincaré group), then the Killing metric is not invertible and the above argument does not go through. More explicitly, for generators $\hat{Q}_A$ in the invariant Abelian subgroup, the first term on the right-hand side of Eq. (8.5) is absent, and it is not possible to remove F_{AB} by shifting the generators. In this case, the algebra (8.5) would be anomaly-free only if F_{AB} happened to be zero.

Given these as-yet unresolved potential difficulties with the quantization of parametrized field theory, it is perhaps reasonable to ask to what extent their resolution might affect the main results of this paper. It may well be the case that the presence of anomalies rules out a consistent quantization of parametrized field theory. However, this would not necessarily mean that all is lost. In this paper we have been concerned with a completely general family of foliating surfaces; i.e., we imposed no restrictions on the embedding variables. For the purposes of discussing global symmetries, it would in fact have been sufficient to consider a very restricted family of foliating surfaces, related to each other by global symmetry transformations. Such a family of surfaces would be described not by the set of embeddings $X^\alpha(\mathbf{x})$, which effectively has $4\times\infty^3$ parameters, but by a *finite* set of parameters, corresponding to the parameters of the global symmetry group. As an example, in Minkowski space, one could obtain a Lorentz-covariant functional Schrödinger formalism by restricting to a family of embeddings of the form

$$X^\alpha(\mathbf{x},t)=A^\alpha(t)+B^\alpha_i(t)x^i \tag{8.10}$$

as suggested in Ref. 10. A theory of this type would have no local symmetries, just a set of global symmetries described by the generators Q_A and G_A, and there is a realistic possibility that it may be consistently quantized. In particular, it may be that the possibly infinite anomalous term A_α in Eq. (8.3) vanishes when smeared with Killing vectors in the construction of the quantum generators $\hat{Q}_A$ and $\hat{G}_A$.

A detailed treatment of anomalies in parametrized field theories will be the subject of future publications.[31]

IX. SUMMARY AND CONCLUSIONS

Two related issues motivated the work described in this paper.

(i) The derivation of the operator constraints of the Dirac quantization procedure from the path-integral representation of the wave function was considered in Ref. 13. However, a more detailed treatment of the case of constraints arising from global symmetries was called for. In particular, for the case of global *spacetime* symmetries, the operator constraints on the wave function cannot be derived from the path integral using the usual functional Schrödinger formalism. A covariant method, such as the one described in this paper, is needed.

(ii) An argument for the de Sitter invariance of the matter field quantum state in de Sitter space defined by the no-boundary proposal was given in Ref. 14. However, the rather heuristic nature of this argument called for a more formal approach. The introduction of the embeddings allows this heuristic argument to be formalized.

Given these motivations, we embarked, in this paper, on a study of global spacetime symmetries in the functional Schrödinger picture. We began, in Sec. II, by observing that the usual functional Schrödinger picture, in which one works with a fixed foliation of spacetime, does not permit a proper discussion of spacetime symmetries. This is because the field configurations on a fixed three-surface of a given foliation do not carry complete representations of spacetime symmetry groups. We then argued that a fuller appreciation of spacetime symmetries could be obtained by generalization to parametrized field theory, in which the spacetime coordinates describing the location of the three-surfaces are made dynamical. Section III covered some of the technicalities needed to handle the embeddings.

In Sec. IV we discussed global spacetime symmetries in parametrized field theory. We constructed two distinct sets of generators obeying the Lie algebra of the isometry group of the spacetime. We defined a globally invariant state to be one represented by a wave functional which is annihilated by the operator version of the generators. The utility of the embeddings in discussions of global spacetime symmetries is concisely summarized by Eq. (4.33). This equation shows the connection between the purely geometrical operation of the isometry group on the three-surface and the operation of the generator Q_A on the matter fields ϕ only.

With special emphasis on the case of global symmetries, in Sec. V we reviewed and elaborated on the results of Ref. 13, in which the derivation of operator constraints from the path-integral representation of the wave function was considered. The invariance of the class of histories summed over was identified as the source of the constraints.

In Sec. VI, we applied the results of Sec. V to the case of parametrized field theory. We derived the local constraints on the wave function (i.e., the generalized Schrödinger equation) from its path-integral representation. When the background spacetime possessed isometries, we showed that the wave functions were annihilated by the corresponding global generators provided that the class of histories summed over had vanishing initial charge. We emphasized the *necessity* of using the embeddings in the derivation of constraints corresponding to global spacetime symmetries.

In Sec. VII we reviewed quantum field theory in de Sitter space, and we reviewed a previously given heuristic argument for the de Sitter invariance of the wave function for the scalar field defined by the no-boundary proposal of quantum cosmology. We showed that the introduction of the embeddings allowed this argument to be made more mathematically precise. Applying the results of Secs. V and VI, we demonstrated explicitly that the wave function for the scalar field, as defined by the no-boundary path-integral proposal, is annihilated by the de Sitter generators. The issue of anomalies is an interesting and important one, and insufficient attention to it has been given in this paper. A brief discussion of anomalies was given in Sec. VIII. We also noted there that the possible difficulties with anomalies could be alleviated by considering very restricted classes of embeddings. A more detailed treatment of anomalies will be the topic of a future publication.[31]

Finally, we note that quantized parametrized field theory could be very useful in a number of other contexts. When doing quantum field theory in curved spacetime, it is often desirable to work with different (fixed) foliations of the spacetime. In de Sitter space, for example, one may work in $k=+1$ coordinates, in which the slices of the de Sitter hyperboloid are three-spheres, but one can also work in $k=0$ coordinates, in which one takes flat slices of the de Sitter hyperboloid. These different slicings are in fact related by de Sitter transformations. The formalism described in this paper may offer a means of relating the quantum theories (in the functional Schrödinger picture) using different slicings.

More generally, there are many situations in which there is considerable tension between covariant quantization methods and the functional Schrödinger approach, which is not manifestly covariant. This tension may be alleviated by quantized parametrized field theory. This has many of the benefits of the functional Schrödinger approach, such as its intuitive power and resemblance to simple quantum mechanics; but is also covariant, in that it involves all possible foliations of spacetime, rather than just one foliation. The possibilities this opens up will be pursued in future publications.

ACKNOWLEDGMENTS

I would like to express my gratitude to Gerald Dunne, Gary Gibbons, Jim Hartle, Karel Kuchař, Roman Jackiw, and Jorma Louko for useful conversations. This research was supported in part by funds provided by the U.S. Department of Energy (D.O.E.) under Contract No. DE-AC02-76ER03069.

APPENDIX: NORMALIZATION OF THE NORMAL AND CALCULATION OF THE JACOBIAN $\partial X/\partial x$

In this appendix we show how to calculate the normalization of the normal, (2.3), and calculate the Jacobian factor $\partial X/\partial x$ thus proving the relation (2.12).

We begin by defining the alternating symbols in n spacetime dimensions. They are defined by

$$\delta_{\alpha_1\alpha_2\cdots\alpha_n}=\delta^{\alpha_1\alpha_2\cdots\alpha_n}=\begin{cases}+1 & \text{if } \alpha_1\alpha_2\cdots\alpha_n \text{ is an even permutation of } 0,1,\ldots,(n-1),\\ -1 & \text{if } \alpha_1\alpha_2\cdots\alpha_n \text{ is an odd permutation of } 0,1,\ldots,(n-1),\\ 0 & \text{otherwise}.\end{cases} \tag{A1}$$

Using these symbols one can define the determinant of an $n \times n$ matrix A with elements $A_{\alpha\beta}$. It is

$$\det A = \frac{1}{n!} \delta^{\alpha_1\alpha_2 \cdots \alpha_n} \delta^{\beta_1\beta_2 \cdots \beta_n} A_{\alpha_1\beta_1} A_{\alpha_2\beta_2} \cdots A_{\alpha_n\beta_n} \ . \tag{A2}$$

Next we define the tensor density

$$\epsilon_{\alpha_1\alpha_2 \cdots \alpha_n} = (\epsilon g)^{1/2} \delta_{\alpha_1\alpha_2 \cdots \alpha_n} \ , \tag{A3}$$

where $g = \det(g_{\alpha\beta})$ and $\epsilon = \mathrm{sgn}(g)$. From (A3) it follows that

$$\epsilon^{\alpha_1\alpha_2 \cdots \alpha_n} = \epsilon(\epsilon g)^{-1/2} \delta^{\alpha_1\alpha_2 \cdots \alpha_n} \ . \tag{A4}$$

In particular, for the three- and four-dimensional cases of interest here, one has

$$\epsilon_{ijk} = h^{1/2} \delta_{ijk}, \quad \epsilon^{ijk} = h^{-1/2} \delta^{ijk} \ , \tag{A5}$$

$$\epsilon_{\alpha\beta\gamma\epsilon} = (-g)^{1/2} \delta_{\alpha\beta\gamma\epsilon}, \quad \epsilon^{\alpha\beta\gamma\epsilon} = -(-g)^{-1/2} \delta^{\alpha\beta\gamma\epsilon} \ . \tag{A6}$$

Using the above, it is possible to deduce the result of contracting together two alternating symbols. So for example, one of the results needed below is

$$\epsilon^{\alpha\mu\nu\rho} \epsilon_{\alpha\beta\gamma\epsilon} = -\delta^{\alpha\mu\nu\rho} \delta_{\alpha\beta\gamma\epsilon} = -6 \delta^{\mu}_{[\beta} \delta^{\mu}_{\gamma} \delta^{\rho}_{\epsilon]} \ , \tag{A7}$$

where $[\cdots]$ denotes antisymmetrization.

Now we may consider the normalization of the normal. Recall that the normal is

$$n_\alpha = k \epsilon_{\alpha\beta\gamma\sigma} X^\beta_i X^\gamma_j X^\sigma_k \epsilon^{ijk} \ , \tag{A8}$$

where k is a constant which we are to determine. From the normalization condition $n^\alpha n_\alpha = -1$, it follows that

$$k^2 \epsilon^{\alpha\mu\nu\rho} \epsilon_{\alpha\beta\gamma\sigma} X_{\mu l} X_{\nu m} X_{\rho n} X^\beta_i X^\gamma_j X^\sigma_k \epsilon^{lmn} \epsilon^{ijk} = -1 \ . \tag{A9}$$

Using (A7), and the fact that the three-metric is given by $h_{ij} = X^\alpha_i X_{\alpha j}$, Eq. (A9) becomes

$$-6k^2 h_{il} h_{jm} h_{kn} \epsilon^{ijk} \epsilon^{lmn} = -1 \ . \tag{A10}$$

From (A2) and (A5), it then readily follows that $k = \pm\frac{1}{6}$. We shall see below that the appropriate sign is to take $k = -\frac{1}{6}$.

Next we calculate the Jacobian arising in transforming from the spacetime coordinates X^α to the coordinates $x^\alpha = (t, x^i)$, and thus verify Eq. (2.12). We start with the relation

$$\det \left[\frac{\partial X^\alpha}{\partial x^\mu} \right] \delta_{\mu\nu\rho\sigma} = \frac{\partial X^\alpha}{\partial x^\mu} \frac{\partial X^\beta}{\partial x^\nu} \frac{\partial X^\gamma}{\partial x^\rho} \frac{\partial X^\epsilon}{\partial x^\sigma} \delta_{\alpha\beta\gamma\epsilon} \ , \tag{A11}$$

from which it follows that

$$\det \left[\frac{\partial X^\alpha}{\partial x^\mu} \right] \delta_{ijk} = \dot{X}^\alpha X^\beta_i X^\gamma_j X^\epsilon_k \delta_{\alpha\beta\gamma\epsilon} \ . \tag{A12}$$

Inserting the expression (2.8) for $\dot{X}^\alpha$, contracting with δ^{ijk}, and using (A5) and (A6), one thus obtains

$$\det \left[\frac{\partial X^\alpha}{\partial x^\mu} \right] = \tfrac{1}{6} N h^{1/2} (-g)^{-1/2} n^\alpha \epsilon_{\alpha\beta\gamma\epsilon} X^\beta_i X^\gamma_j X^\epsilon_k \epsilon^{ijk} \ . \tag{A13}$$

It is readily seen, however, that the last few terms are proportional to the normal, n_α, and using the normalization $n^\alpha n_\alpha = -1$, one obtains the desired result

$$\det \left[\frac{\partial X^\alpha}{\partial x^\mu} \right] = N h^{1/2} (-g)^{-1/2} \tag{A14}$$

proving (2.12), as intended.

Finally, note that if we had taken $k = +\frac{1}{6}$ in the expression for n_α, instead of $k = -\frac{1}{6}$, then we would have obtained a minus sign on the right-hand side of Eq. (A14). This would mean that the coordinates (t, x^i) would have the opposite orientation to the spacetime coordinates X^α. There is no obvious reason why one should not work with such coordinates, but it seems desirable to choose the coordinates (t, x^i) to have the same orientation as the coordinates X^α, and it is for this reason that we take $k = -\frac{1}{6}$.

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