Spectra of relic gravitons and the early history of the Hubble parameter

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The spectra of relic gravitational waves produced as a result of cosmological expansion of the generalized inflationary models are derived. It is shown how one can reconstruct the time dependence of the very early Hubble parameter and matter energy density from a measured frequency-dependent spectrum of relic gravitational waves.

I. INTRODUCTION

Relic gravitational waves are, quite possibly, the only source of information on the physical conditions in the very early Universe. The energy density and spectrum of relic gravitational waves depend on a specific rate of expansion of the Universe in its very distant past. The actual detection of relic gravitons, or even the experimental restriction of their possible spectral energy density, are capable of producing fairly meaningful conclusions about the parameters of the very early Universe (for a recent review of the subject, see Ref. 1). In general, the information stored in relic gravitons allows one to judge the overall evolution of the comological scale factor. However, in more specific models, which include very popular inflationary models of the Universe, this information gives us the direct data on the variability of the Hubble parameter. It is important to emphasize that the relic graviton spectrum measured "today" provides us with direct information on the Hubble parameter attributed to the very early Universe. Roughly speaking, the frequency dependence of the relic graviton energy density spectrum repeats exactly the time dependence of the very early Hubble parameter. This paper presents a more precise formulation of this statement and its proof.

II. BASIC EQUATIONS

We consider the simplest homogeneous isotropic models with the line element

$$ds^2=c^2dt^2-a^2(t)dl^2 \ . \tag{1}$$

Below, we will use the units $c=G=\hbar=1$; in these units the Planck density ρ_p is equal to one, $\rho_p=1$. The time derivative is denoted by an overdot, and a sign of the spatial curvature, $k=+1,0,-1$ corresponds to a closed, flat, or open space. We will also use the η-time variable $a(\eta)d\eta=dt$ and will denote the η-time derivative by a prime.

The gravity-wave perturbations $h_{ik}(\eta,\mathbf{x})$ superimposed on the background metric (1) obey the generalized wave equation. Each polarization component $h(\eta,\mathbf{x})$ can be presented as a sum (integral) over the independent mode functions $h_{\mathbf{n}}$ with the wave vector $\mathbf{n}=(n^1,n^2,n^3)$:

$$h(\eta,\mathbf{x})=\sum_{\mathbf{n}} h_{\mathbf{n}}, \quad h_{\mathbf{n}}=\frac{1}{a}\mu_n(\eta)U_{\mathbf{n}}(\mathbf{x}) \ .$$

The wave number n, $n^2=(n^1)^2+(n^2)^2+(n^3)^2$, is associated with the frequency ν (measured in Hz) according to the relation $\nu=n/2\pi a$. The time dependence of $h_{\mathbf{n}}$ is determined by $\mu_n(\eta)$ satisfying the equation[2]

$$\mu_n''+[n^2-V(\eta)]\mu_n=0 \ , \tag{2}$$

where $V(\eta)\equiv a''/a$ and, for simplicity, we have put $k=0$.

Equation (2) looks like the Schrödinger equation for a particle having the potential energy (potential) $V(\eta)$. For $n^2 \gg |a''/a|$ the general solution to Eq. (2) has the form

$$\mu_n=\frac{1}{\sqrt{2n}}(\alpha_n e^{-in\eta}+\beta_n e^{in\eta}) \ , \tag{3}$$

where α_n and β_n are arbitrary complex numbers. In the opposite limit, $n^2 \ll |a''/a|$, the general solution is

$$\mu_n=A_n a+B_n a\int^{\eta}\frac{d\eta}{a^2} \ . \tag{4}$$

It is known that a traveling gravitational wave passing through the barrier $V(\eta)$ will always be amplified.[2] In the quantum treatment of the problem, one says that the initial vacuum state of gravitons goes over into a final multiparticle quantum state—the particle creation takes place. The final quantum state belongs to a class of the so-called squeezed quantum states (for more details, see Ref. 3).

It is clear from Eq. (2) that the behavior of $\mu_n(\eta)$ is determined by the potential $V(\eta)\equiv a''/a$. Let us turn to the cosmological scale factor $a(t)$. The function $a(t)$ satisfies the Einstein equations

$$\frac{\ddot{a}}{a}=-\frac{4\pi}{3}(\rho+3p)\ , \tag{5}$$

$$H^2=\frac{8\pi}{3}\rho-\frac{k}{a^2}\ , \tag{6}$$

where the Hubble parameter $H(t)\equiv\dot{a}/a$. In the case of the equation of state $p=-\rho$, the solution to Eqs. (5) and (6) is the de Sitter metric. For $k=0$ it can be written in the form

$$ds^2=dt^2-a_0^2e^{2Ht}(dx^2+dy^2+dz^2)\ , \tag{7}$$

where $H=$const. It is known that the hypothesis according to which a sufficiently long period in the evolution of the early Universe has been described by Eq. (7) lies at the foundation of inflationary cosmological models which possess a number of attractive features.[4] Analogous advantages belong to the solutions which do not coincide with the de Sitter solution but for which the condition $\ddot{a}>0$ was satisfied for a sufficiently long interval of time (see, for example, Ref. 5). The condition $\ddot{a}>0$ can be called the condition of inflation. It is easy to see how this condition ensures the closeness of the present-day parameter Ω to unity. Indeed, Eq. (6) can be presented in the form

$$\Omega=1+\frac{k}{(Ha)^2}\ ,$$

where $\Omega=\rho/\rho_c$, $\rho_c=(8\pi/3)H^2$—the critical density. If $\ddot{a}>0$ at some stage of expansion, then the factor $\dot{a}\equiv Ha$ keeps growing. A sufficiently large growth of the factor Ha at this stage of evolution makes Ω so close to $\Omega=1$ that Ω remains in the necessary limits around 1 even long after the end of inflation and, in particular, during the present epoch. This solves the "flatness problem." In a similar way the "horizon problem" is solved. Indeed, the ratio l/R_H of the linear size of some region, $l\sim a$, to the Hubble distance R_H, $R_H\sim H^{-1}$, scales as $l/R_H\sim aH=\dot{a}$, that is, l/R_H increases at the $\ddot{a}>0$ stage. At the postinflationary stage the ratio l/R_H decreases. However, if the stage $\ddot{a}>0$ was sufficiently long, the entire Hubble volume of the present-day Universe could have developed from a single causally connected region.

To analyze the inflationary solutions it is convenient to use the function

$$\gamma(t)\equiv-\frac{\dot{H}}{H^2}\ . \tag{8}$$

Obviously, for the strictly de Sitter solution (7) one has $\gamma(t)=0$. If $a(t)$ is described by a power-law function $a(t)\sim t^P$, then $\gamma=1/P=$const and, in particular, $\gamma=2$ for $a\sim t^{1/2}$. For the arbitrary equation of state $p=q\rho$, where $q=$const (and assuming $k=0$), one obtains a simple relation $\gamma=\frac{3}{2}(q+1)=$const, that is, γ varies from 0 to 3 when q varies from -1 to $+1$. In general, since

$$\ddot{a}=(Ha)^{\cdot}=H(1-\gamma)Ha\ ,$$

the condition $\gamma<1$ is equivalent to the condition of inflation. We do not restrict $a(t)$ by any other conditions. (A family of exact solutions for Friedmann universes containing a scalar field with particular self-interaction potentials is found in Ref. 6.)

III. THE GRAVITON SPECTRA

The potential $V(\eta)$ can be rewritten as

$$V(\eta)=\frac{a''}{a}=(2-\gamma)(Ha)^2\ .$$

If the function $\gamma(t)$ changes slowly and the factor $(2-\gamma)$ is not very small, the potential $V(\eta)$ is approximately determined by the function $(Ha)^2$. For a wide class of problems the function $|V(\eta)|$ has a bell-like shape and goes to zero for $\eta\to\pm\infty$. The solution (3) is valid outside the potential barrier $|V(\eta)|$ while the solution (4) applies to the region where the function $|V(\eta)|$ dominates over n^2. The values of η where the regimes (3) and (4) interchange are determined by the condition

$$n^2=|V(\eta)|=|(2-\gamma)|(Ha)^2\ . \tag{9}$$

A wave with a given n enters the barrier region and leaves it in the "turning points" defined by Eq. (9). We will denote these points by η_i and η_f. The indices i and f will also be used to distinguish the values of the various functions at the turning points. We will match continuously at η_i and η_f the values of $\mu_n(\eta)$ and $\mu_n'(\eta)$ determined by the solutions (3) and (4).

To the left of the barrier, that is, for $\eta<\eta_i$, the solution is taken in the form $\mu_n=(2n)^{-1/2}e^{in\eta}$. To the right of the barrier, that is, for $\eta>\eta_f$, the solution has the form (3) where the numerical values of the coefficients α_n,β_n follow from the joining conditions. The actual values of α_n,β_n are

$$\alpha_n=\frac{1}{2in}e^{-in(\eta_i-\eta_f)}\left[\frac{a_i}{a_f}(H_ia_i+in)-\frac{a_f}{a_i}(H_fa_f-in)+a_ia_f(H_ia_i+in)(H_fa_f-in)J\right]\ , \tag{10}$$

$$\beta_n=\frac{1}{2in}e^{-in(\eta_i+\eta_f)}\left[\frac{a_f}{a_i}(H_fa_f+in)-\frac{a_i}{a_f}(H_ia_i+in)-a_ia_f(H_ia_i+in)(H_fa_f+in)J\right]\ , \tag{11}$$

where $J \equiv \int_{\eta_i}^{\eta_f} a^{-2} d\eta$.

In fact, we have calculated the coefficients of the Bogoliubov transformation which relates the in and out quantum states of gravitons. Let us recall that in the quantum theory the quantity $|\beta_n|^2$ defines the spectral energy density of gravitons created from the in vacuum state. The contemporary energy density of gravitons integrated over some frequency interval is given by the formula

$$\epsilon = \int \epsilon_\nu d\nu = \int \epsilon(\nu) \frac{d\nu}{\nu} ,$$

where $\epsilon(\nu) \equiv \epsilon_\nu \nu$. The spectral component $\epsilon(\nu)$ is given by

$$\epsilon(\nu) = \nu \frac{d\epsilon}{d\nu} = \frac{d\epsilon}{d \ln\nu} = 4\pi |\beta_v|^2 \nu^4 . \quad (12)$$

We would like to stress that Eq. (2) can be solved exactly in the case of the power-law scale factors $a(t)$ (see, for example, Refs. 2 and 7), but we are developing here a general, though approximate, method of calculation applicable to arbitrary functions $a(t)$.

Expressions (10) and (11) can be simplified if one transforms the integral J:

$$J = \int_{\eta_i}^{\eta_f} \frac{d\eta}{a^2} = -\frac{1}{3} \int_{\eta_i}^{\eta_f} \frac{da^{-3}}{H} = -\frac{1}{3} \left[\frac{1}{H_f a_f^3} - \frac{1}{H_i a_i^3} \right] + \frac{1}{3} \int_{\eta_i}^{\eta_f} \frac{dH^{-1}}{a^3} .$$

Further on,

$$\int_{\eta_i}^{\eta_f} \frac{dH^{-1}}{a^3} = \int_{\eta_i}^{\eta_f} \gamma \frac{d\eta}{a^2} = \langle \gamma \rangle \int_{\eta_i}^{\eta_f} \frac{d\eta}{a^2} = \langle \gamma \rangle J ,$$

where the number $\langle \gamma \rangle$ is defined as

$$\langle \gamma \rangle = \int_{\eta_i}^{\eta_f} \gamma a^{-2} d\eta \Big/ \int_{\eta_i}^{\eta_f} a^{-2} d\eta .$$

A reason for the notation $\langle \gamma \rangle$ is that this number presents the mean value of γ calculated with the weighting factor a^{-2} at the interval of time from η_i to η_f. Thus, one obtains

$$J = \frac{1}{3 - \langle \gamma \rangle} \left[\frac{1}{H_i a_i^3} - \frac{1}{H_f a_f^3} \right] . \quad (13)$$

We are mainly interested here in the inflationary expansions and hence the value of $\langle \gamma \rangle$ is of the order of 1. Therefore, the factor $3 - \langle \gamma \rangle$ in the formula (13) is also a number of the order of 1. Even in the most "dangerous" case $\gamma = 3$ expression (13) reduces to the uncertainty of the type $\frac{0}{0}$ which can be resolved by the direct computation of J to a finite number. The case $\gamma = 3$ corresponds to the scale factor $a(t)$ governed by the matter with the most "stiff" equation of state $p = \rho$. Graviton production in this case has been discussed elsewhere[7] and we leave this case aside.

By using Eq. (13) one can rewrite Eqs. (10) and (11):

$$\alpha_n = \frac{1}{2in} e^{-in(\eta_i - \eta_f)} \left\{ \left[H_i a_i \frac{a_i}{a_f} - H_f a_f \frac{a_f}{a_i} \right] \left[1 - \frac{1}{3 - \langle \gamma \rangle} \left[1 + \frac{in}{H_i a_i} \right] \left[1 - \frac{in}{H_f a_f} \right] \right] + in \left[\frac{a_i}{a_f} + \frac{a_f}{a_i} \right] \right\} , \quad (14)$$

$$\beta_n = \frac{1}{2in} e^{-in(\eta_i + \eta_f)} \left\{ \left[H_f a_f \frac{a_f}{a_i} - H_i a_i \frac{a_i}{a_f} \right] \left[1 - \frac{1}{3 - \langle \gamma \rangle} \left[1 + \frac{in}{H_i a_i} \right] \left[1 + \frac{in}{H_f a_f} \right] \right] + in \left[\frac{a_f}{a_i} - \frac{a_i}{a_f} \right] \right\} . \quad (15)$$

From Eq. (9) it follows that the quantities $(H_i a_i)^2$, $(H_f a_f)^2$ are numbers of the order of n^2, if $\gamma(\eta)$ is not too close to $\gamma = 2$ in the vicinity of the turning points η_i and η_f. In the opposite case, that is, if one of the conditions $|\gamma_i - 2| \ll 1$, $|\gamma_f - 2| \ll 1$, or both of them, are satisfied, one of the quantities $(H_i a_i)^2$, $(H_f a_f)^2$, or both of them, will be numbers much larger than n^2. In any case, the expressions in square brackets in Eqs. (14) and (15) are the complex numbers with the absolute values of the order of unity. We denote these numbers by C_1, C_2 and rewrite Eqs. (14) and (15) once more:

$$\alpha_n = \frac{1}{2in} e^{-in(\eta_i - \eta_f)} \left[\left[H_i a_i \frac{a_i}{a_f} - H_f a_f \frac{a_f}{a_i} \right] C_1 + in \left[\frac{a_i}{a_f} + \frac{a_f}{a_i} \right] \right] , \quad (16)$$

$$\beta_n = \frac{1}{2in} e^{-in(\eta_i + \eta_f)} \left[\left[H_f a_f \frac{a_f}{a_i} - H_i a_i \frac{a_i}{a_f} \right] C_2 + in \left[\frac{a_f}{a_i} - \frac{a_i}{a_f} \right] \right] . \quad (17)$$

The remarkable property of Eqs. (16) and (17) is that they contain only the initial (i) and the final (f) values of a and H. All the intermediate history of $a(t)$ between η_i and η_f is covered by the numerical factors C_1 and C_2 with the absolute values of the order of unity $|C_1|^2 \approx |C_2|^2 \approx 1$.

Equations (16) and (17) can be further simplified for particular numerical values of $H_i a_i$, $H_f a_f$, and a_f / a_i. For instance, in the case of $(H_i a_i)^2 \gg n^2$ and $(H_f a_f)^2 \gg n^2$ one obtains

$$\alpha_n \approx \frac{1}{2in} e^{-in(\eta_i-\eta_f)} \left[H_i a_i \frac{a_i}{a_f} - H_f a_f \frac{a_f}{a_i} \right] C_1 ,$$

$$\beta_n \approx \frac{1}{2in} e^{-in(\eta_i+\eta_f)} \left[H_f a_f \frac{a_f}{a_i} - H_i a_i \frac{a_i}{a_f} \right] C_2 .$$

On the other hand, if $(H_i a_i)^2 \approx n^2$, $(H_f a_f)^2 \approx n^2$, and $a_f/a_i \gg 1$ one derives

$$\alpha_n \approx \frac{1}{2in} e^{-in(\eta_i-\eta_f)} (in - C_1 H_f a_f) \frac{a_f}{a_i} ,$$

$$\beta_n \approx \frac{1}{2in} e^{-in(\eta_i+\eta_f)} (in + C_2 H_f a_f) \frac{a_f}{a_i} ,$$

which leads to the especially simple expressions for the absolute values of α_n, β_n. In the order of magnitude,

$$|\alpha_n|^2 \approx |\beta_n|^2 \approx \left[\frac{a_f}{a_i} \right]^2 \approx \left[\frac{H_i}{H_f} \right]^2 .$$

We will consider in more detail the case where the inflationary stage $(0<\gamma<1)$ ends with a rapid transition to the radiation-dominated stage $(\gamma \approx 2)$. The relevant barrier in Eq. (2) is described by the function $V(\eta)$ which first increases with η and then decreases sharply up to zero. Let us consider the waves interacting with this barrier. Their wave numbers obey the conditions $(H_i a_i)^2 \approx n^2$, $(H_f a_f)^2 \gg n^2$, and we assume also $a_f/a_i \gg 1$. For acceptable inflationary models these waves have a present-day frequencies in the interval 10^8 Hz $< \nu < 10^{-16}$ Hz.[1] From Eqs. (16) and (17) we obtain

$$\alpha_n = -\frac{1}{2in} e^{-in(\eta_i-\eta_f)} H_f a_f \frac{a_f}{a_i} C_1 ,$$

$$\beta_n \approx \frac{1}{2in} e^{-in(\eta_i+\eta_f)} H_f a_f \frac{a_f}{a_i} C_2 ,$$

and (assuming $|C_1|^2 = |C_2|^2 = 1$)

$$|\alpha_n|^2 \approx |\beta_n|^2 \approx \frac{1}{4n^2} (H_f a_f)^2 \left[\frac{a_f}{a_i} \right]^2 \approx \frac{1}{4n^4} H_i^2 H_f^2 a_f^4 . \tag{18}$$

By using $|\beta_n|^2$, in Eq. (12) we derive

$$\epsilon(\nu) \approx \frac{\pi}{(2\pi)^4} \frac{1}{a^4} H_i^2 H_f^2 a_f^4 . \tag{19}$$

For the barriers under consideration the values of a_f, H_f are the same for all n. The entire dependence on n (and, hence, on frequency ν) is contained in the factor H_i^2. In its turn, $H_i^2(n)$ is proportional to the matter energy density at the inflationary stage, namely, at those instants of time t_n when the waves with the corresponding wave numbers n were entering the under-barrier region $H_i^2(n) = (8\pi/3)\rho(t_n)$. It will now be shown that the other factors in Eq. (19) can be expressed in terms of quantities accessible for contemporary observations. The value of H_f^2 is proportional to the matter energy density at the end of inflation. Some part δ of this energy $(\delta \approx 1)$ has been transformed into the electromagnetic radiation. In the course of cosmological expansion the energy density of the radiation was decreasing in proportion to $(a_f/a)^4$. The present-day energy density ϵ_γ of the radiation (3-K microwave background) is related to H_f^2 according to

$$H_f^2 = \frac{8\pi}{3} \rho(t_f) = \frac{8\pi}{3} \delta^{-1} \left[\frac{a}{a_f} \right]^4 \epsilon_\gamma .$$

Using the expressions for H_i^2 and H_f^2, neglecting the numerical factors of order unity, and restoring ρ_p for the correct dimensionality, we obtain finally

$$\epsilon(\nu) = \epsilon_\gamma \frac{\rho(t_n)}{\rho_p} . \tag{20}$$

This very simple expression allows us to link the energy density $\epsilon(\nu)$ of relic gravitational waves measurable today with the time-dependent values of the matter density ρ and the Hubble parameter H attributed to the very early Universe. On can say, a little loosely, that the function $\epsilon(\nu)$ stores the information on the rate of expansion of the Universe in that distant past when the relic waves with frequencies ν first started to emerge from zero-point quantum fluctuations.

If the expansion law $a(t)$ were precisely known, Eq. (19) would give us a definite spectral dependence of $\epsilon(\nu)$. For example, for the power-law scale factors $a(t) \sim t^p$, one has $\gamma = p^{-1} =$const, $H(t) = pt^{-1}$. From the condition $H_i(t_n)a_i(t_n) = n$ we derive $n \sim t_n^{p-1}$, that is, $t_n \sim n^{1/p-1}$. Since $\rho(t_n) \sim H_i^2(t_n) \sim t_n^{-2} \sim n^{-2/p-1}$ one can obtain $\epsilon(\nu) \sim \nu^{-2/p-1} \sim \nu^{2\gamma/\gamma-1}$, that is, the power-law frequency dependence for $\epsilon(\nu)$. In the case of strictly de Sitter expansion ($\gamma=0$, $H_i=$const) one would obtain the "flat" Harrison-Zel'dovich spectrum $\epsilon(\nu)=$const, $\epsilon_\nu \sim \nu^{-1}$.[8] (We note in passing that a spectrum of the "flat" shape is generated not only by the de Sitter evolution governed by matter with the equation of state $p=-\rho$, but also by the "dustlike" evolution governed by matter with the equation of state $p=0$, as follows from the general theory of graviton creation.[7,9]) However, the law of expansion of the very early Universe is not known in advance. Of larger practical interest is the problem of reconstructing the $\rho(t)$ and $H(t)$ from the measured spectrum $\epsilon(\nu)$ [we assume, of course, that the $\epsilon(\nu)$ will be eventually known from the actual observations].

Equation (20) gives ρ and H as functions of frequency ν,

$$\rho(\nu) = \frac{\rho_p}{\epsilon_\gamma} \epsilon(\nu), \quad H^2(\nu) = \frac{8\pi}{3} \rho(\nu) ,$$

and, therefore, gives ρ and H as nonmanifest functions of time, since every ν corresponds to some t_n. The waves with larger frequencies ν describe the latter epochs of the evolution as the high-frequency modes interact with the inflationary barrier $V(\eta)$ at the comparatively latter instants of time (for the barriers of different shape it could be the other way around). One can also derive the manifest dependence of ρ and H on time t. To do this one should use the condition $H_i(t_n)a_i(t_n)=n$ and derive the relationship between ν and t. Let us show, for example, how to obtain a graph of the function $H(t)$ assuming that the graph of the function $H(\nu)$ is known from observations. First we differentiate the equation $Ha=n$ and get

$$\frac{dn}{dt}=\frac{d(Ha)}{dt}=(1-\gamma)H^2a=(1-\gamma)Hn \ ,$$

$$\frac{1}{n}\frac{dn}{dt}=\frac{1}{\nu}\frac{d\nu}{dt}=(1-\gamma)H \ ,$$

which yields to the equation

$$\frac{d}{dt}=[1-\gamma(\nu)]H(\nu)\frac{d}{d\ln\nu} \ . \quad (21)$$

The function $H(\nu)$ entering this equation is known. The function $\gamma(\nu)$ is also in effect known since it can be derived from $H(\nu)$ and $\epsilon(\nu)$ with the help of Eq. (8):

$$\gamma(\nu)=-\frac{dH}{d\nu}\frac{d\nu}{dt}H^{-2}(\nu)$$
$$=-\frac{1}{2}H^{-1}(\nu)\epsilon^{-1}(\nu)\frac{d\epsilon(\nu)}{d\nu}\frac{d\nu}{dt}$$
$$=-\frac{1}{2}(1-\gamma(\nu))\frac{d\ln\epsilon(\nu)}{d\ln\nu} \ ,$$

that is,

$$\gamma(\nu)=-\frac{[d\ln\epsilon(\nu)/d\ln\nu]}{2-[d\ln\epsilon(\nu)/d\ln\nu]} \ . \quad (22)$$

In other words, the relationship between dt and $d\ln\nu$ is known from observations. Thus, it follows from the equation

$$\frac{dH(t)}{dt}=[1-\gamma(\nu)]H(\nu)\frac{dH(\nu)}{d\ln\nu} \quad (23)$$

that the graph of the function $H(t)$ repeats the graph of the function $H(\nu)$ if one relates the scales t and $\ln\nu$ with the help of a changing (but known) factor $(1-\gamma)^{-1}H^{-1}$. Another useful form of Eq. (23) is

$$\frac{dH(t)}{dt}=\frac{1}{2}[1-\gamma(\nu)]\frac{8\pi\rho_P}{3\epsilon_\gamma}\frac{d\epsilon(\nu)}{d\ln\nu} \ .$$

We can conclude by saying that every variation of the function $\epsilon(\nu)$ finds its explanation in the very definite variations of the function $H(t)$ describing the very early Universe.

If the observed spectrum $\epsilon(\nu)$ happens to have a power-law dependence:

$$\epsilon(\nu)=K\nu^\beta, \quad H(\nu)=\left[\frac{8\pi}{3}\frac{\rho_P}{\epsilon_\gamma}K\right]^{1/2}\nu^{\beta/2} \ ,$$

$K=$const, the task of restoring the $H(t)$ is further simplified. Indeed, from Eq. (22) one finds $\gamma=\beta/\beta-2$ and from Eq. (21) one finds the relationship

$$\left[\frac{8\pi}{3}\frac{\rho_P}{\epsilon_\gamma}K\right]^{1/2}\nu^{\beta/2}=\frac{\beta-2}{\beta}\frac{1}{t-t_0} \ ,$$

where $t_0=$const. Hence one obtains

$$H(t)=\frac{\beta-2}{\beta}\frac{1}{t-t_0}=\frac{1}{\gamma}\frac{1}{t-t_0} \ ,$$

and $a(t)\sim(t-t_0)^p$. For $\epsilon(\nu)$ having a power-law dependence $\epsilon(\nu)=K\nu^\beta$, the graph of the function $\ln\epsilon$ in terms of the variable $\ln\nu$ presents a straight line whose inclination depends on β. From this standpoint, the zero inclination of the "flat" spectrum $\epsilon(\nu)=\text{const}(\beta=0)$ produced in the de Sitter inflationary model is just a manifestation of the fact that the Hubble parameter $H(t)$ does not depend on time in this case, $H(t)=$const. For other inflationary models, $0<\gamma<1$, one has $\beta<0$ and the spectral density $\epsilon(\nu)=K\nu^\beta$ decreases toward the higher frequencies (see also Ref. 10).

IV. A PARTICULAR INFLATIONARY MODEL

In all inflationary models considered above, i.e., for $0<\gamma<1$, the parameter $H(t)$ is a decreasing function of time. Hence, one can expect in advance that the spectral density $\epsilon(\nu)$ will decrease for higher frequencies. If the position of the low-frequency end of the spectrum is restricted by the experimental data on the 3-K microwave background anisotropy,[11,1] the predicted $\epsilon(\nu)$ at the high-frequency end of the spectrum will be somewhat lower than in the case of a strictly de Sitter inflation.

We will consider in some detail a particular inflationary model governed by a scalar field $\phi(t)$ with the mass m. The Lagrangian of the field has the form

$$L=-\tfrac{1}{2}(\phi_{,\alpha}\phi^{,\alpha}+m^2\phi^2) \ .$$

It is shown in Ref. 12 that almost all the phase trajectories of the model inevitably get to the regime of the inflationary expansion. (This result was extended[13] to other models, some of which were used for an analysis of the relic gravity-wave production.[14]) In this regime, the scale factor $a(t)$ behaves as

$$a(t)=a(t_c)\exp\left[-\frac{2\pi}{m_P^2}[\phi^2(t)-\phi_c^2]\right] \ ,$$

where t_c denotes the time of the ending of the inflationary expansion, ϕ_c is the value of the scalar field at $t=t_c$, $\phi_c\approx m_P$, and m_P is the Planck mass. The scalar field $\phi(t)$ changes as

$$\phi(t)=\phi_c+\frac{mm_P}{(12\pi)^{1/2}}(t_c-t) \ ,$$

where it is assumed that the time variable t grows from the large negative values toward $t=t_c$. Since

$$H^2(t)\approx\frac{4\pi m^2}{3m_P^2}\phi^2(t) \ ,$$

one can write

$$H^2(t)\approx H_c^2\left[1+\frac{2m^2}{3H_c}(t_c-t)\right] \ , \quad (24)$$

where $H_c^2=(4\pi m^2/3m_P^2)\phi_c^2$. It is also easy to find the $\gamma(t)$ at $t=t_c$: $\gamma(t_c)\approx m_P^2/4\pi\phi_c^2\approx 1/4\pi$, which we will use later on.

From the condition $H(t_n)a(t_n)=\dot{a}(t_n)=n$ we find

$$-H_c(t_c-t_n)=\ln\frac{n}{n_c}=\ln\frac{\nu}{\nu_c}\ , \tag{25}$$

where n_c is the wave number characteristic for the end of inflation, $n_c=a(t_c)H_c$, and ν_c is the corresponding present-day frequency ($\nu_c\approx 10^8$ Hz for reasonable inflationary parameters). By substituting Eq. (25) into Eq. (24) one can find

$$H^2(t_n)\approx H_c^2\left[1-\frac{2m^2}{3H_c^2}\ln\frac{\nu}{\nu_c}\right]\ . \tag{26}$$

As far as the $|\beta_n|^2$ is determined by Eq. (18)

$$|\beta_n|^2\approx\frac{1}{4n^4}H^2(t_n)H_f^2a_f^4\ ,$$

where $H_f^2a_f^4=$const, the problem of deriving the predicted spectrum $\epsilon(\nu)$ being solved by Eq. (26). In other words, the "flat" spectrum $\epsilon(\nu)=$const acquires a logarithmic correction:

$$\epsilon(\nu)\sim\left[1-\frac{2m^2}{3H_c^2}\ln\frac{\nu}{\nu_c}\right]\ . \tag{27}$$

At the high-frequency end of the spectrum, that is, for $\nu\approx\nu_c$, the predicted $\epsilon(\nu)$ is about 5–10 times smaller than the value of $\epsilon(\nu)$ at the low-frequency end of the spectrum, that is, for $\nu\approx 10^{-24}\nu_c$.

The appearance of a logarithmic, frequency-dependent correction in Eq. (27) looks very natural in view of the fact that the t scale corresponds to the $\ln\nu$ scale. Let us see this in more detail. One can expand the function $H^2(t)$ into a power series near $t=t_c$:

$$H^2(t)\approx H_c^2+\left.\frac{dH^2}{dt}\right|_{t=t_c}\Delta t\ .$$

Then, one can use Eqs. (8) and (21) to obtain the estimate

$$\frac{H^2(t)}{H_c^2}\approx 1-\frac{2\gamma(t_c)}{1-\gamma(t_c)}\Delta\ln\nu$$

or, even a simpler expression,

$$\frac{H^2(t)}{H_c^2}\approx 1-2\gamma(t_c)\ln\frac{\nu}{\nu_c} \tag{28}$$

valid if $\gamma(t_c)\ll 1$. After substituting $\gamma(t_c)\approx m_P^2/4\pi\phi_c^2$ into Eq. (28) we return to Eq. (26) derived in the concrete model under consideration.

To summarize, we have shown what kind of conclusions about the very early Universe can be drawn from the measured spectrum of relic gravitational waves. We hope that the actual detection of relic gravity-wave background will be achieved in the not too remote future.

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