Gravitation in 2+1 dimensions

N.J. Cornish and N.E. Frankel
School of Physics, University of Melbourne, Parkville, Victoria 3052, Australia

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We investigate gravitational field theories in 2+1-spacetime dimensions. The consequences of the lack of a Newtonian limit to general relativity are reviewed . Further insight into the implications of this fact is gained by considering a new, general class of exact hydrostatic solutions. We show that all self-gravitating polytropic structures have the same gravitational mass and produce matter-filled spaces of finite spatial volume. Other theories of gravitation are also considered and the behavior of one such theory with a Newtonian limit is studied. Cosmological solutions of these gravitational theories are also studied in detail.

I. INTRODUCTION

Recently there has been much attention given to studying gravitational theories in dimensions other than four. The reasons for this are many and varied; however, the principal motivation comes from string theory, grand unified theory, and quantum gravity. This paper shares none of those motivations; rather, this work was motivated by a study of degenerate fermion structures in d dimensions. The particular problems that arose for dimensions $d+1 \leq 3$ raised some interesting questions and produced some unusual results. The unique status of Einstein's field equations in two space and one time dimensions provides the principal reason to focus on 2+1 dimensions.

Because the Einstein and Riemann tensors are equivalent in 2+1 dimensions, spacetime is flat outside sources; there is no free gravitational field and no Newtonian limit. Deser, Jackiw, and 't Hooft[1] have studied the implications of this fact and have shown that for point sources gravity manifests itself as a global topology rather than local curvature with conserved quantities such as energy related to topological invariants. Deser *et al.* extended the static one- and two-body point-source solutions of Staruszkiewicz[2] to include N-body, static, point-source solutions, point particles with angular momentum and two-body solutions with orbital angular momentum. While these solutions provide insight into aspects of the theory that are especially important with a view toward quantization, they do not shed much light on the behavior of Einstein's field equations in the presence of extended sources. A collection of several exact solutions with an extended matter source was provided by Barrow, Burd, and Lancaster.[3] Because they only studied a few particular cases they did not attempt to draw any general conclusions about the nature of (2+1)-dimensional extended objects that obey Einstein's field equations. We shall present a general class of hydrostatic solutions all of which have the fascinating property that they produce spaces of finite spatial extent. There are an incredible diversity of matter distributions and compositions encompassed by this class of solutions. They range from constant to exponentially decaying density profiles, from the infinitely tenuous to the infinitely dense, from stiff to soft equations of state. All of this prompts us to conjecture that all hydrostatic structures in (2+1)-dimensional Einstein gravity produce matter-filled spaces with no matching to an external vacuum solution and thus represent static cosmologies. We also demonstrate in three particular cases that these solutions are stable.

In the light of all the unusual results cataloged above, and their dependence on the lack of a Newtonian limit to Einstein's equations, a variety of other field equations are considered to discover whether the lack of a Newtonian limit is peculiar to Einstein's equations. It was found that higher-derivative, higher-order, tensor equations also lack a Newtonian limit. The behavior of a scalar field equation which does have a Newtonian limit was studied and was found to yield some interesting cosmological and point source solutions.

II. EINSTEIN GRAVITY IN D+1 DIMENSIONS

In any dimension Einstein's equations are

$$G_{\mu\nu} = \kappa T_{\mu\nu} \Rightarrow R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{d-1}{d-2} G_d S_d T_{\mu\nu} \ , \tag{2.1}$$

where the constants have been chosen to agree with the corresponding Newtonian theory in the weak-field limit (c=1 throughout),

$$\nabla^2 \Phi \sim R_{tt} \sim G_d S_d T_{tt} \ , \tag{2.2}$$

where $S_d = 2\pi^{d/2}/\Gamma\left(\frac{d}{2}\right)$ is the solid angle, $G_d \equiv G_3 l^{d-3}$ and l is some fundamental length.

The singular nature of the coupling constant κ in $d = 1, 2$ spatial dimensions demands individual consideration. When $d = 1$, $G_{\mu\nu} = 0$ is an identity and Einstein gravity is devoid of content, this is reflected in Eq. (2.1) where $\kappa \to 0$ when $d \to 1$. The situation when $d = 2$ is not nearly as straightforward and provides the main reason for focusing on gravity in 2+1 dimensions. Forcing Einstein's equations to yield a Newtonian limit results in the coupling constant diverging when $d \to 2$. This is an unacceptable situation as all post-Newtonian terms are infinite and the theory as a whole is unworkable. A more satisfactory solution is to "renormalize" the gravitational constant in two spatial dimensions via $G_d/(d-2) \to G_d$ so that Einstein's equations in 2+1 dimensions read

$$R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R = 2\pi G_2 T_{\mu\nu} \quad , \tag{2.3}$$

which in the weak field, slow motion limit yields

$$\nabla^2 \Phi \sim R_{tt} = 0; \tag{2.4}$$

thus we have sacrificed the Newtonian limit so that the theory as a whole remains sensibly finite. Further insight can be gained by considering the Riemann tensor, which can be written as

$$\begin{aligned} R_{\lambda\mu\nu\kappa} = {} & \frac{G_d S_d}{d-2} \left[(g_{\lambda\nu} T_{\mu\kappa} - g_{\lambda\kappa} T_{\mu\nu} - g_{\mu\nu} T_{\lambda\kappa} + g_{\mu\kappa} T_{\lambda\nu}) \right] \\ & + \frac{G_d S_d}{d-2} \left(\frac{2T}{d} (g_{\mu\nu} g_{\lambda\kappa} - g_{\lambda\nu} g_{\mu\kappa}) \right) + C_{\lambda\mu\nu\kappa} \\ & \qquad (d > 2), \end{aligned} \tag{2.5}$$

$$\begin{aligned} R_{\lambda\mu\nu\kappa} = {} & 2\pi G_2 \left[(g_{\lambda\nu} T_{\mu\kappa} - g_{\lambda\kappa} T_{\mu\nu} - g_{\mu\nu} T_{\lambda\kappa} + g_{\mu\kappa} T_{\lambda\nu}) \right] \\ & + 2\pi G_2 \left[T \, (g_{\mu\nu} g_{\lambda\kappa} - g_{\lambda\nu} g_{\mu\kappa}) \right] + C_{\lambda\mu\nu\kappa} \\ & \qquad (d = 2). \end{aligned}$$

In general when $T_{\mu\nu} = 0$, $R_{\lambda\mu\nu\kappa} = C_{\lambda\mu\nu\kappa}$, so the curvature tensor depends solely on the Weyl tensor. This fact leads to the Weyl tensor being called the free gravitational field.[4] The Weyl tensor has $\frac{1}{12}(d+1)(d+2)(d+3)(d-2)$ linearly independent components which tells us that the free gravitational field vanishes in 2+1 dimensions and that $R_{\lambda\mu\nu\kappa} = 0$ outside of any source. In the the following section we shall see that the vanishing of curvature in empty space only allows global topological effects to influence geodesic motion.

III. STATIC SOLUTIONS TO EINSTEIN'S EQUATIONS IN 2+1 DIMENSIONS

A. Static equations

The most general circularly symmetric, static metric has a line element

$$ds^2 = e^{2\nu(r)} dt^2 - e^{2\eta(r)} dr^2 - r^2 d\theta^2 \quad . \tag{3.1}$$

Einstein's field equations in the presence of a perfect fluid,

$$R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R = 2\pi G_2 [(\rho + p) u_\mu u_\nu - p g_{\mu\nu}], \tag{3.2}$$

give

$$\frac{e^{-2\eta} \eta'}{r} = 2\pi G_2 \rho, \tag{3.3}$$

$$\frac{e^{-2\eta} \nu'}{r} = 2\pi G_2 p \tag{3.4}$$

$$\Rightarrow e^{-2\eta} = 1 - 2G_2 M(r), \tag{3.5}$$

$$M(r) = \int_0^r 2\pi \rho(x) x \, dx, \tag{3.6}$$

$$\nu' = \frac{2\pi G_2 p r}{1 - 2G_2 M(r)}, \tag{3.7}$$

where the constant in Eq. (3.5) has been chosen so that the origin is part of the spacetime. When combined with the radial component of $T^{\mu\nu}{}_{;\mu} = 0$, $T^{\mu r}{}_{;\mu} = p' + (p + \rho)\nu' = 0$, these equations describe hydrostatic equilibrium

$$p' = \frac{-2\pi G_2 p (p + \rho) r}{1 - 2G_2 M(r)} \quad . \tag{3.8}$$

Comparing this equation with its (3+1)-dimensional counterpart,

$$p' = -\left(\frac{4\pi G_3 p (p + \rho) r}{1 - 2G_3 M(r)/r} + \frac{G_3 M(r)(p + \rho)}{r^2 [1 - 2G_3 M(r)/r]} \right), \tag{3.9}$$

we see that the major difference is the lack of a Newtonian-like mass-mass $M(r)\rho$ piece in 2+1 dimensions.

In fact Eq. (3.8) demonstrates the essentially local nature of the theory because terms such as $p(p + \rho)$ depend on the pressure and density at a point, coordinate distance r from the center; in contrast a term such as $M(r)(p + \rho)$ depends on the total mass beneath coordinate radius r. The only "history" or nonlocal effect that remains in 2+1 dimensions is due to the denominator $[1 - 2G_2 M(r)]$. The results of the section on hydrostatic polytrope equilibrium show that this remaining nonlocal effect is directly responsible for the hydrostatic structures forming spaces of finite spatial volume.

B. Static sources

1. Static point source

For a point source $\rho = M\delta(\mathbf{r})$, $p = 0$, which yields ($G = G_2$ hereafter)

$$e^{-2\eta} = 1 - 2GM, \tag{3.10}$$

$$\nu' = 0 \tag{3.11}$$

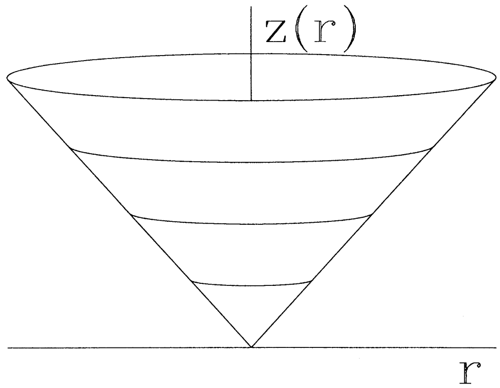

FIG. 1. Embedding diagram for case (a), given by Eq. (3.18).

and we are free to choose $\nu = 0$ as our scaling of time, producing the line element

$$ds^2 = dt^2 - \frac{1}{1-2GM}dr^2 - r^2 d\theta^2 \quad . \qquad (3.12)$$

To preserve the signature we require $M \leq \frac{1}{2G}$. A change of coordinates reveals that the space is flat except for a curvature singularity at the source ($r = 0$). Let

$$x = \left(\frac{1}{1-2GM}\right)^{1/2} r \Rightarrow 0 \leq x \leq \infty, \qquad (3.13)$$

$$\phi = (1-2GM)^{1/2}\,\theta$$

$$\Rightarrow 0 \leq \phi < 2\pi(1-2GM)^{1/2} \qquad (3.14)$$

$$\Rightarrow ds^2 = dt^2 - dx^2 - x^2 d\phi^2. \qquad (3.15)$$

The conical nature of the geometry is apparent from the restricted range of the angular variable which geometrically corresponds to removing a wedge of angle α and identifying the two cut edges, where

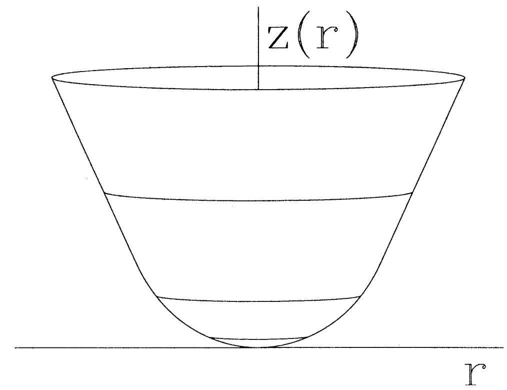

FIG. 3. Embedding diagram for case (c), given by Eq. (3.19).

$$\alpha = 2\pi[1-(1-2GM)^{1/2}]$$
$$= 2\pi GM + O((GM)^2) \quad . \qquad (3.16)$$

Thus we see matter producing global topological effects on the space. From the geodesic equation, $\ddot{x}^\mu + \Gamma^\mu_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta = 0$, we see that initially static bodies $\dot{x}^t = 1$, $\dot{x}^i = 0$ do not accelerate since $\Gamma^\mu_{tt} = (0, \nu' e^{2\nu-2\lambda}, 0) = \mathbf{0}$, reiterating the absence of a Newtonian limit.

2. Extended static sources

The above result has been generalized to N static point sources using elegant conformal transformation techniques.[1] Rather than repeating such a treatment we present the idealized case of evenly distributed dust confined within a radius R as an introduction to studying extended structures in hydrostatic equilibrium. A geometrical feel for the structure of such spacetimes is provided by embedding in a higher-dimensional space via the procedure

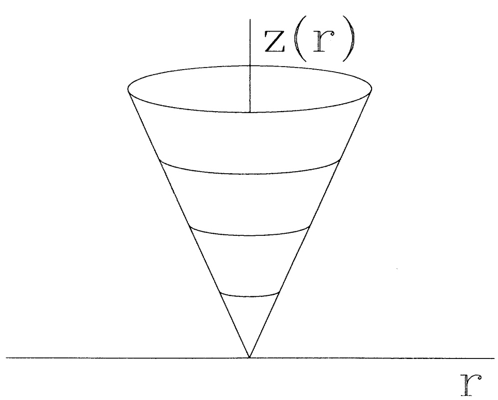

FIG. 2. Embedding diagram for case (b), given by Eq. (3.18).

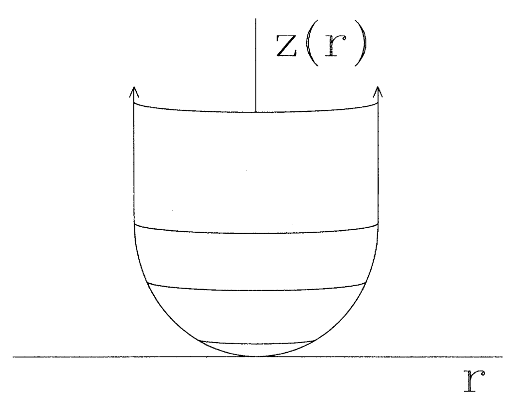

FIG. 4. Embedding diagram for case (d), given by Eq. (3.19).

$$dl^2 \rightarrow dl^2 = dz^2 + dr^2 + r^2 d\theta^2$$

$$\Rightarrow z(r) = \int_0^r \left(\frac{2GM\,(x)}{1-2GM\,(x)}\right)^{1/2} dx. \quad (3.17)$$

The embedding diagrams for (where H is the Heaviside function)

(a) $\rho = M\delta(\mathbf{r})$, $M = \frac{1}{4G}$,

(b) $\rho = M\delta(\mathbf{r})$, $M = \frac{2}{5G}$,

(c) $\rho = \frac{M}{\pi R^2} H(r-R)$, $M = \frac{2}{5G}$,

(d) $\rho = \frac{M}{\pi R^2} H(r-R)$, $M = \frac{1}{2G}$

are displayed in Figs. 1–4. The embedding function for cases (a), (b) is

$$z(r) = \left(\frac{2GM}{1-2GM}\right)^{1/2} r; \quad (3.18)$$

the embedding function for case (c) is

$$z(r) = \begin{cases} \frac{R}{(2GM)^{1/2}}[1-\sqrt{1-2GM(r/R)^2}], & 0 \le r \le R, \\ \left(\frac{2GM}{1-2GM}\right)^{1/2} r + \left(\frac{(1-2GM)^{1/2}-1}{(2GM)^{1/2}(1-2GM)^{1/2}}\right) R, & r > R. \end{cases} \quad (3.19)$$

while the embedding function for case (d) is

$$z(r) = \begin{cases} R\left(1-\sqrt{1-(r/R)^2}\right), & 0 \le r < R \\ R \rightarrow \infty, & r = R. \end{cases} \quad (3.20)$$

Note, the constant density configuration has a proper mass

$$M_0 = \int_0^{2\pi}\int_0^R \rho\left(g^{(2)}\right)^{1/2} d^2x$$
$$= \frac{1}{G}[1-(1-2GM)^{1/2}] \quad (3.21)$$

showing that the largest allowed proper mass for this configuration is $\frac{1}{G}$. The gravitational energy is

$$\Omega \equiv M - M_0$$
$$= \frac{1}{G}[GM - 1 + (1-2GM)^{1/2}]$$
$$= -\frac{1}{2G}[(GM)^2 + O((GM)^3)] \ . \quad (3.22)$$

The obvious question raised by these results is whether there exists an inequality analogous to Buchdahl's[5] in 3+1 dimensions which guarantees that objects in hydrostatic equilibrium preserve metric signature. In 2+1 dimensions such an inequality would have to guarantee $M(r) \le \frac{1}{2G}$. This inequality exists and will be discussed in the following section. Also to be discussed is the more surprising result that $M = \frac{1}{2G}$ for all polytropes and the implications this would appear to have for any physically reasonable structure. For this reason Fig. 4 deserves special attention since all the exact solutions we found have geometries topologically identical to this solution. This result forms the basis for the next section which is followed in turn by an investigation of the natural follow-up question: are these structures stable?

C. Hydrostatic equilibrium with relativistic polytropes

To solve the equations of hydrostatic equilibrium we require an equation of state $p(\rho)$. In keeping with astrophysical tradition we chose to study polytrope equations of state. Such a model has the advantage of exploring a wide range of constitutions, some of which can be shown to approximate particular physical systems. The equation of state is defined as

$$p = \kappa\, \rho^{1+1/n} \quad (3.23)$$

which can be parametrized as

$$p = \gamma\, \theta^{n+1}, \quad (3.24)$$

$$\rho = \lambda\, \theta^{n}, \quad (3.25)$$

where $\theta(0) = 1$, $\theta(R) = 0$ so

$\gamma =$ central pressure,

$\lambda =$ central energy density.

Some examples of physical systems here in 2+1 dimensions which obey such equations are

$n = 0$	constant energy density,
$n = 1$	nonrelativistic degenerate fermions,
$n = 2$	nonrelativistic matter (ρ dominated by rest mass) and radiation providing most of the pressure,
$n \rightarrow \infty$	pure radiation $\rho = 2p$, $\lambda = 2\gamma$,
$n \rightarrow \infty$	stiff equation of state $\rho = p, \lambda = \gamma$, speed of sound equal to speed of light.

For polytropes the equations describing hydrostatic equilibrium (3.6) and (3.8) become

$$M(r) = \int_0^r 2\pi r \lambda \theta^n dr, \tag{3.26}$$

$$n(\theta')^2 + \theta\theta'' = (\theta')^2 \left(\frac{2(n+1)\gamma\theta - \lambda}{\lambda + \gamma\theta}\right) + \frac{\theta\theta'}{r} . \tag{3.27}$$

After some rearrangement Eq. (3.27) becomes

$$(\ln\theta')' = (3n+4)\,[\ln(\lambda+\gamma\theta)]' - (n+1)\{\ln[\theta(\lambda+\gamma\theta)]\}' + (\ln r)' \tag{3.28}$$

$$\Rightarrow \quad \theta' = \frac{-2\pi G r(\lambda+\gamma\theta)^{2n+3}}{(n+1)(\lambda+\gamma)^{2n+2}\theta^{n+1}} \tag{3.29}$$

$$\Rightarrow \quad r^2 = \frac{(n+1)(\lambda+\gamma)^{2n+2}}{\pi G} \int_\theta^1 \frac{u^{n+1}du}{(\lambda+\gamma u)^{2n+3}}$$

$$= \frac{(n+1)(\lambda+\gamma)^{2n+2}}{\lambda^{n+1}\gamma^{n+2}\pi G} \sum_{k=0}^{n+1} \binom{n+1}{k} \frac{(-1)^k}{(k+n+1)} \times \left[\left(1+\frac{\gamma}{\lambda}\theta\right)^{-(k+n+1)} - \left(1+\frac{\gamma}{\lambda}\right)^{-(k+n+1)}\right] . \tag{3.30}$$

Equation (3.30) represents a complete analytic solution for the structure of all polytropes regardless of index, a situation unparalleled in 3+1 dimensions where exact solutions are only known for a few particular polytropes even when considered in the Newtonian approximation.

The most interesting result is obtained by substituting Eq. (3.29) into Eq. (3.26) to obtain

$$M = M(R) = \frac{\lambda(n+1)(\lambda+\gamma)^{2n+2}}{G} \int_0^1 \frac{\theta^{2n+1}}{(\lambda+\gamma\theta)^{2n+3}} d\theta$$

$$= \frac{(n+1)(\lambda+\gamma)^{2n+2}}{G} \int_0^{1/(\lambda+\gamma)} u^{2n+1} du$$

$$= \frac{1}{2G} \quad \forall\, n, \lambda, \gamma \; . \tag{3.31}$$

Thus all structures that obey a polytrope equation of state produce spaces of finite spatial volume with no external geometry; i.e., the "star" is the universe. Because polytrope equations of state encompass such a diverse range of physically reasonable equations of state, it is not unreasonable to conjecture that all perfect-fluid structures in (2+1)-dimensional Einstein gravity have a gravitational mass of $\frac{1}{2G}$ and thus represent compact spaces. We can at least be sure that $M \leq \frac{1}{2G}$ for all structures by employing the same arguments as Buchdahl for the $n = 0$, constant density polytrope; i.e., such structures have the largest allowed density gradient and an equation of state that bounds all physical systems in stiffness.

To complete the formal results for all polytropes we display exact expressions for the radius R , proper mass M_0 and gravitational energy Ω .

$$R^2(\lambda,\gamma,n) = \frac{1}{\pi G\lambda} \binom{2n+2}{n+1}^{-1} \sum_{k=0}^{n} \binom{2n+2}{k} \left(\frac{\gamma}{\lambda}\right)^{n-k} , \tag{3.32}$$

$$M_0(\lambda,\gamma,n) = +\frac{1}{G}\left[1 + \binom{2n+2}{n+1}^{-1} \sum_{k=1}^{n+1} \binom{2n+2}{n+1-k} \left(\frac{\gamma}{\lambda}\right)^k\right] , \tag{3.33}$$

$$\Omega(\lambda,\gamma,n) = -\frac{1}{G}\left[\frac{1}{2} + \binom{2n+2}{n+1}^{-1} \sum_{k=1}^{n+1} \binom{2n+2}{n+1-k} \left(\frac{\gamma}{\lambda}\right)^k\right] . \tag{3.34}$$

For cases where the polytropic index n is finite, these expressions are very useful. In particular, we find that the radius R is finite for all finite values of n and the parameter $\frac{\gamma}{\lambda}$. In the limit $n \to \infty$, asymptotic expansion of the integral in (3.30) for R [where $\theta(R) = 0$] is required. We find, most interestingly, that R is infinite for $\frac{\gamma}{\lambda} \geq 1$ and finite for $\frac{\gamma}{\lambda} < 1$ as can also be seen from the explicit examples, A and B, respectively, that are developed in Sec. III D. Since solutions with $\frac{\gamma}{\lambda} \geq 1$ have luminal or superluminal sound velocities they are physically irrelevant while the physically sensible cases with $\frac{\gamma}{\lambda} < 1$ produce geometries with finite spatial volume.

D. Stability of hydrostatic structures

The stability of these polytropic fluid spheres against infinitesimal, baryon-number-conserving, adiabatic, radial oscillations was studied using a method analogous to that developed by Chandrasekhar[6] in (3+1)-dimensional general relativity. Because an identical method was used only the resulting criteria for stability will be displayed here. The oscillation frequency ω is given as an extremal value with respect to the allowed trial functions ζ . Physically $\zeta(r)$ represents the Lagrangian displacement of a fluid element originally situated at a radius r in the equilibrium configuration:

$$\omega^2 = \text{extremum with respect to } \zeta \text{ of } \left(\frac{A+B+\Gamma C}{D}\right), \tag{3.35}$$

where

$$A = \int_0^R 4\pi G p(p+\rho)\zeta^2 e^{3\eta+\nu} r\, dr,$$

$$B = \int_0^R 2\pi G \rho p' \zeta^2 e^{3\eta+\nu} r^2 dr,$$

$$C = \int_0^R [(re^{-\nu}\zeta)\, ']^2 \frac{p}{r} e^{\eta+3\nu} dr,$$

$$D = \int_0^R (p+\rho)\zeta^2 e^{3\eta-\nu} r\, dr,$$

$$\Gamma = \text{ratio of specific heats}$$

$$= \frac{1}{p\left(\frac{\partial N}{\partial p}\right)}\left(N - (p+\rho)\frac{\partial N}{\partial \rho}\right)$$

(N = number density).

Γ is taken to be a constant throughout the structure, physically in 2+1 dimensions $\Gamma \geq \frac{3}{2}$ (in contrast with $\Gamma \geq \frac{4}{3}$ in 3+1 dimensions). The Lagrangian displacement ζ must satisfy the boundary conditions

$$\frac{\zeta}{r} \longrightarrow \text{finite limit as } r \longrightarrow 0, \tag{3.36}$$

$$\Delta p = -\Gamma p \frac{e^\nu}{r}\frac{\partial}{\partial r}(re^{-\nu}\zeta) \longrightarrow 0 \text{ as } r \longrightarrow R . \tag{3.37}$$

Also, to maintain adiabaticity (no shock fronts), we require $|\,\partial\zeta/\partial r\,| \leq 1$. Because the frequency is unaffected by a constant rescaling of ζ this condition becomes

$$\frac{\partial \zeta}{\partial r} \text{ should be finite for } 0 \leq r \leq R . \tag{3.38}$$

The absolute minimum of (3.37) is the squared frequency of the fundamental of mode pulsation. If it is negative the "star" is unstable, if it is positive the "star" is stable. Since D is positive definite the structures are stable if

$$A + B + \Gamma C > 0 \;\; \forall \text{ allowed } \zeta . \tag{3.39}$$

The stability condition can be expressed in terms of a critical ratio of specific heats

$$\Gamma_c = \frac{-(A+B)}{C} . \tag{3.40}$$

The structure is stable if $\Gamma > \Gamma_c$ .We also know that all physical ratios of specific heats exceed $\frac{3}{2}$ so there is no possibility for instability if $\Gamma_c < \frac{3}{2}$.

While it is not practical to test for stability by trying every one of the infinitely many types of allowed trial functions, a strong indication that the structure is stable can be gained by using some physical insight. First we need only consider the lowest radial mode of oscillation because if this mode is unstable then all other modes will be unstable; conversely, if this mode is stable we have shown that the structure is at least marginally stable. We can achieve this choice in practice by only considering trial functions without any "wiggles." Second, because this is a variational treatment a first-order difference between the true and trial Lagrangian displacements results in only a second-order difference in the frequency. The combination of these two factors has allowed this method to accurately distinguish between stable and unstable structures in 3+1 dimensions so we can apply it to 2+1 dimensions with some confidence.

Example A

Structures with a stiff equation of state, $p = \rho$, have $\Gamma = 2$ and density profiles given by [solution of (3.8)]

$$p = \rho = \gamma e^{-2\pi G \gamma r^2},$$

$$e^{2\eta} = e^{2\nu} = e^{2\pi G \gamma r^2} .$$

We choose a class of trial functions that satisfy the boundary conditions and are relatively "wiggle"-free (product of monotonic functions that have monotonic derivative) so as to provide the best approximation to the fundamental mode of oscillation:

$$\zeta \propto r^a e^{-2b\eta} = r^a e^{-2b\nu} .$$

The boundary conditions impose the restrictions $a \geq 1$, $b > 1$. The necessary integrals are straightforward and give

$$\Gamma_c(a,b) = \frac{2a(a+1-4b)}{(a+1)[(8a+4)b^2 - 4ab - 3a]} .$$

This expression is bounded above by 2; i.e., $\Gamma_c < 2$ as $(a \longrightarrow \infty\, ,\; b \longrightarrow 1)$. Because $\Gamma > \Gamma_c$ these structures are stable against perturbations of the type considered here.

Example B

For pure radiation or ultrarelativistic particles $p = \frac{1}{2}\rho$, $\Gamma = \frac{3}{2}$ and the structures are given by [solution of (3.8)]

$$p=\gamma\left[1-\left(\frac{r}{R}\right)^2\right]^3,$$

$$e^{2\eta}=\left[1-\left(\frac{r}{R}\right)^2\right]^{-4},$$

$$e^{2\nu}=\left[1-\left(\frac{r}{R}\right)^2\right]^2 .$$

For trial functions of the form

$$\zeta\propto r^a e^{b\nu}=r^a e^{-b\eta/2} ,$$

the boundary conditions impose the restrictions $a\geq 1$, $b>-2$ while the condition $|\partial\zeta/\partial r|\leq 1$ imposes the stricter condition $b>1$. We find

$$\Gamma_c(a,b)=\frac{12[(a+1)-b]}{2b^2+(a-3)b+9a+11} .$$

Since $\Gamma_c<\frac{6}{5},\Gamma>\Gamma_c$ and these structures are also stable.

Example C

For structures with ρ =const, $\Gamma\geq\frac{3}{2}$ and the structures are described by [solution to (3.8)]

$$p=\frac{\gamma\lambda[1-(\frac{r}{R})^2]^{1/2}}{\gamma+\lambda-\gamma[1-(\frac{r}{R})^2]^{1/2}},$$

$$e^{2\eta}=\left[1-\left(\frac{r}{R}\right)^2\right]^{-1},$$

$$e^{2\nu}=\left(1+\left(\frac{\gamma}{\lambda}\right)\left\{1-\left[1-\left(\frac{r}{R}\right)^2\right]^{1/2}\right\}\right)^2 .$$

For the choice of trial function

$$\zeta\propto r^a e^{-b\eta}e^{c\nu} ,$$

the boundary conditions impose the restrictions $a\geq 1$, $b>1$, and c is arbitrary. The condition $|\partial\zeta/\partial r|\leq 1$ imposes the stricter restriction $b\geq 2$. We find

$$\Gamma_c(a,b,c)=\frac{a(a-2b+3)}{(a+1)b(a+b-1)}+\left(\frac{\gamma}{\lambda}\right)O(a^{-\frac{1}{2}})$$

$$(\text{as } a\to\infty).$$

[Note, Γ_c is independent of c to first order in (γ/λ) as one would expect from the form of the metric function e^ν.] Since $\Gamma_c\leq\frac{1}{2}$, $\Gamma>\Gamma_c$ and these structures are also stable.

IV. COSMOLOGICAL SOLUTIONS IN (2+1)-DIMENSIONAL EINSTEIN GRAVITY

Because all the hydrostatic solutions proved to be cosmologies, it is interesting to consider how the more usual homogeneous, evolving cosmologies behave. It is also interesting to consider what might happen if any "galaxies" began to form. Assuming our conjecture, that all hydrostatic objects produce compact spaces, is correct then any condensations that form will pinch themselves off from the rest of the Universe. This situation is similar to the formation of primeval black holes in a (3+1)-dimensional Friedmann universe; the consequences would now, however, be even more drastic.

We first consider the equations governing homogeneous expansion before investigating how irregularities might grow. In terms of the (2+1)-dimensional Robertson-Walker line element,[7]

$$ds^2=dt^2-a^2(t)\left(\frac{dr^2}{1-kr^2}+r^2d\theta^2\right),\quad k=(-1,0,+1) . \tag{4.1}$$

Einstein's field equations become

$$\left(\frac{\dot a}{a}\right)^2+\frac{k}{a^2}=2\pi G\rho, \tag{4.2}$$

$$\frac{\ddot a}{a}=-2\pi Gp, \tag{4.3}$$

$$\frac{d}{dt}(\rho a^2)+p\frac{d}{dt}(a^2)=0 . \tag{4.4}$$

For a dust-filled universe, $p=0$ and

$$\rho a^2=\text{const}\equiv\rho_0a_0^2, \tag{4.5}$$

$$\dot a^2=2GM-k, \tag{4.6}$$

where $M=\pi\rho_0a_0^2$ and Eq. (4.6) requires

$$M\geq\frac{k}{2G}$$

then

$$a(t)=a_0\pm\sqrt{2GM-k}\,(t-t_0) .$$

We note that for the compact case, $k=+1$, with $M=\frac{1}{2G}$, $a(t)=a_0$ and the solution is identical to that of the $n=0$ polytrope in the limit $p\to 0$. For a radiation-dominated universe, $\rho=2p$ and

$$\rho a^3=\text{const}\equiv\rho_0a_0^3,$$

$$\dot a=\pm\left(\frac{2GM_0a_0}{a}-k\right)^{1/2},$$

$$\ddot a=-\frac{MGa_0}{a^2}$$

with expanding solutions

$$t=\begin{cases}\dfrac{2a^{3/2}}{3\sqrt{2GM_0a_0}} & \text{if } k=0,\\ 2GM_0a_0\left[\arcsin\left(\frac{a}{2GM_0a_0}\right)^{1/2}\left(\frac{a}{2GM_0a_0}\right)^{1/2}\left(1-\frac{a}{2GM_0a_0}\right)^{1/2}\right] & \text{if } k=+1,\\ 2GM_0a_0\left[\left(\frac{a}{2GM_0a_0}\right)^{1/2}\left(1+\frac{a}{2GM_0a_0}\right)^{1/2}-\operatorname{arcsinh}\left(\frac{a}{2GM_0a_0}\right)^{1/2}\right] & \text{if } k=-1.\end{cases} \tag{4.7}$$

For early times $a \propto t^{2/3}$ in comparison with $a \propto t^{1/2}$ in 3+1 dimensions. We see that this simple, noninflationary solution in 2+1 dimensions suffers the same horizon problem of its (3+1)-dimensional antecedent.

In an allied study, we considered the evolution of density and pressure fluctuations during the radiation-dominated epoch; following the treatment of Peebles,[8] it was found that the growing mode grew as $t^{1/3}$ which is similar to that in 3+1 dimensions ($t^{2/3}$). The similarity of the radiation-dominated solutions is not unexpected, for pure radiation $T^{\mu}{}_{\mu} = 0$ and

$$R_{\mu\nu} \propto T_{\mu\nu}$$

so there is no qualitative difference between the theories in $d = 2$ and 3 spatial dimensions for highly relativistic sources.

V. OTHER THEORIES OF GRAVITY IN 2+1 DIMENSIONS

A. Introduction

As the various results discussed so far have shown, Einstein gravity has some very peculiar features in 2+1 dimensions. It is pertinent to ask therefore whether alternative relativistic theories of gravity are able to restore mass-mass interactions. This surely is not an unreasonable demand to make of a theory of gravity.

One way of restoring propagating degrees of freedom is to go to higher-order, higher-derivative theories. Some examples of such theories have been considered by Barrow, Burd, and Lancaster[3] and Deser, Jackiw, and Templeton.[9] Barrow *et al.* considered gravitational actions of the form

$$\mathcal{S} = -\int \left(\alpha R_{\mu\nu}R^{\mu\nu} + \beta R^2 + R - \mathcal{L}_m\right)\sqrt{g}\, d^3x \ .$$

Analyzing the results of such a theory in the weak-field, slow-motion limit they showed that, in momentum space,

$$R_{tt}(\mathbf{p}) = \frac{2\pi G T_{tt}(\mathbf{p})\,[2p^2(2\beta+\alpha)]}{(1+\alpha p^2)[(3\alpha+8\beta)p^2-1]} \quad \propto p^2 T_{tt}$$

showing that this theory does not have Newton's equations as a nonrelativistic limit. The situation is similar for the topologically massive theory considered by Deser *et al.* The field equations for this theory

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \frac{1}{m}C_{\mu\nu} = \kappa T_{\mu\nu},$$

$$C^{\mu\nu} \equiv \frac{1}{\sqrt{g}}\epsilon^{\mu\alpha\beta}(R_{\alpha\beta} - \tfrac{1}{4}g_{\alpha\beta}R)^{;\nu}$$

have a slow-motion, weak-field limit given by

$$R_{tt} + \frac{1}{m}C_{tt} = 0$$

which has nontrivial dynamics but no Newtonian limit.

We are currently considering other classes of gravitational theory such as algebraically extended Hilbert theory and several bimetric theories in an attempt to find a theory that has a Newtonian limit in 2+1 dimensions while also being a viable alternative to Einstein gravity in 3+1 dimensions. Such a study has already been done in 1+1 dimensions[10] where it was found that algebraically extended Hilbert theory produced a field equation with nontrivial dynamics and a Newtonian limit. (Einstein gravity is devoid of content in 1+1 dimensions.) The field equation in this theory may be expressed as

$$R = -4G_1 T,$$

$$T^{\mu\nu}{}_{;\mu} = 0 \quad \text{(assumed)}.$$

It is not surprising that a successful gravitational theory in 1+1 dimensions has a scalar field equation since all two-dimensional spaces are conformally flat ($g_{\mu\nu} = \phi^2\eta_{\mu\nu}$); i.e., they possess a single scalar degree of freedom ϕ. Concomitantly the curvature tensor can be expressed purely in terms of the scalar curvature

$$R_{\mu\nu\lambda\kappa} = -\tfrac{1}{2}(g_{\mu\lambda}g_{\nu\kappa} - g_{\nu\lambda}g_{\mu\kappa})R \ .$$

Outside sources $T = 0$, $R = 0$ and space is flat as was the case with Einstein gravity in 2+1 dimensions. There are also no propagating degrees of freedom so why is there a Newtonian limit? The difference is that this theory results in a nontrivial, global, space*time* geometry whereas Einstein gravity in 2+1 dimensions only caused nontrivial, global, topological effects on the *spatial* geometry. This distinction is made more obvious if one considers the 1+1 dimensional line element in Euclidean form ($d\tau = i\,dt$) and transforms the spatial coordinate via $x \rightarrow r = \frac{1}{A}e^{-A|x|}$, then

$$ds^2 = dr^2 + A^2r^2d\tau^2,$$

where A is a function of the mass of the source situated at $x = 0$. The imaginary-time coordinate then has a period related to the strength of the source.[11] Because the time coordinate is affected, this conical geometry has a Newtonian limit while a completely analogous conical spatial geometry that did not involve the time coordinate in 2+1 dimensions has no Newtonian limit. It is important to note that the weak-field limit is achieved about $x = 0$ as one would expect in a dimension where the Newtonian potential grows linearly with distance.

It is interesting to consider how this $R = \kappa T$ field theory behaves in higher dimensions especially considering that it reduces to Newton's equations in any dimension. Einstein[12] actually tried this field equation in 1914 before discovering the correct tensor equations. Unfortunately this field equation contains insufficient information to fully determine an arbitrary assignment of metric functions; one has to choose a prior geometry that has only one scalar degree of freedom. A natural choice is $g_{\mu\nu} = \phi^2\eta_{\mu\nu}$, i.e., a conformally flat spacetime with $C_{\mu\nu\lambda\kappa} = 0$. Such a choice in 3+1 dimensions has a working Newtonian limit but fails to predict the deflection of light [obvious from the fact that

$ds^2 = \phi^2(dt^2 - dx^2 - x^2 d\Omega^2)$ has the same null geodesics as flat space modulo possible nontrivial homotopic considerations]. In addition the theory predicts Mercury to have a perihelion retreat rather than advance.[13] Such failings naturally resulted in this theory being rejected in 3+1 dimensions. It is a strange quirk therefore that this theory produces big-bang cosmologies with no horizon problem as the following solution demonstrates.

We generalize our choice of metric to those conformal to a spacetime with a maximally symmetric spatial subspace of the form

$$ds^2 = \phi^2 \left(d\tau^2 - \frac{\delta_{ij}\, dx^i \otimes dx^j}{(1 + kx^2/4)^2} \right), \tag{5.1}$$

which for homogeneous cosmologies can be transformed into the usual Robertson-Walker form, by a redefinition of the time coordinate,

$$ds^2 = dt^2 - a^2(t) \left(\frac{dr^2}{1 - kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta\, d\phi^2 \right) . \tag{5.2}$$

In terms of this line element, the field equation gives

$$\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} = 4\pi G_3(3p - \rho); \tag{5.3}$$

additionally $T^{\mu\nu}{}_{;\mu} = 0$ gives

$$\frac{d}{dt}(\rho a^3) + p \frac{d}{dt}(a^3) = 0 \ . \tag{5.4}$$

These equations only have a solution in both the radiation-dominated and matter-dominated epochs if $k = -1$. Then the solution that has a finite initial expansion rate in the radiation-dominated early stage is, with no arbitrary constants:

$$a(t) = t. \tag{5.5}$$

This solution then goes over to

$$\sqrt{a}\sqrt{a - 6G_3M} + 6G_3M \operatorname{arccosh}\left[\left(\frac{a}{6G_3M}\right)^{1/2}\right] = t + \text{const} \tag{5.6}$$

in the matter-dominated epoch. Conceivably the pressure would not be negligible until $a \gg 6G_3M$ at which time Eq. (5.6) becomes

$$a \sim t + \text{const.} \tag{5.7}$$

So, we have a complete cosmology that demands an open universe ($k = -1$) and never has a horizon problem.

While we are not suggesting that Einstein's equations should be amended to read

$$R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = 8\pi G_3 T_{\mu\nu} + \Lambda 24\pi G_3 g_{\mu\nu} T \ ,$$

where Λ is now vanishingly small, we cannot help feeling a sense of lamentation that nature does not comply with the $R = \kappa T$ theory as its cosmological predictions are very appealing in that they automatically admit solutions with no horizon problem without needing the scaffolding that supports the inflationary cosmologies currently in vogue.

B. R=κT theory in 2+1 dimensions

For the choice $g_{\mu\nu} = \phi^2 \eta_{\mu\nu}$

$$R = -4\phi^{-3}\Box\phi = 8\pi GT \tag{5.8}$$

$$\Rightarrow \Box\phi = -2\pi GT\phi^3 \ . \tag{5.9}$$

In a weak field $\phi \equiv 1 + \theta$ ($\theta \ll 1$, $T = \rho$), we have

$$\Box\theta = -2\pi G\rho \ , \tag{5.10}$$

which for static fields gives the Newtonian limit

$$\nabla^2\theta = 2\pi G\rho \ . \tag{5.11}$$

For a point source of mass M

$$\theta = GM \ln r \ . \tag{5.12}$$

For this potential the weak-field limit is about $r = 1$; the full line element becomes

$$ds^2 = (1 + GM \ln r)^2 (dt^2 - dr^2 - r^2 d\theta^2). \tag{5.13}$$

The metric is singular at $r = e^{-1/GM}$. This surface does not represent an event horizon however since radial photons can freely pass through and metric signature is preserved. One unusual feature of the circle at $r = e^{-1/GM}$ is that bound polar photon orbits can occur.

The equation for hydrostatic equilibrium with a perfect-fluid source is

$$\nabla^2 p - \frac{p'(\rho' + 2p')}{\rho + p} = 2\pi G(2p - \rho)(\rho + p) \times \exp\left(-2\int_{p_0}^{p} \frac{dp}{\rho + p}\right) . \tag{5.14}$$

Unfortunately we were unable to find any solutions to this equation except in the Newtonian limit. The philosophical desire for a static mass-mass interaction appears destined to lead to highly intractable equations. Thankfully the cosmological solutions proved to be more readily soluble.

If we generalize our choice of metric as we did in 3+1 dimensions to include line elements of the form

$$\begin{aligned} ds^2 &= \phi^2 \left(d\tau^2 - \frac{\delta_{ij}\, dx^i \otimes dx^j}{(1 + kx^2 i/4)^2} \right) \\ &= dt^2 - a^2(t) \left(\frac{dr^2}{1 - kr^2} + r^2 d\theta^2 \right) \end{aligned}$$

$$[\text{when } \phi \text{ purely } \phi(t)], \tag{5.15}$$

then the equations describing the evolution of the Universe are

$$2\frac{\ddot{a}}{a}+\left(\frac{\dot{a}}{a}\right)^2+\frac{k}{a^2}=4\pi G(2p-\rho), \tag{5.16}$$

$$\frac{d}{dt}(\rho a^2)+p\,\frac{d}{dt}(a^2)=0\ . \tag{5.17}$$

These equations have two distinct types of solution in both the radiation-dominated and matter-dominated epochs. One class demands $k=-1$ and are very similar to the solutions found in 3+1 scalar gravity. The other class has an identical form to those found in (2+1)-dimensional Einstein gravity during the radiation-dominated epoch.

Class 1:
k must $=-1$

(a) Radiation dominated:

$$a(t)=t. \tag{5.18}$$

(b) Matter dominated:

$$a(t)=\sqrt{1-4GM}(t-t_0)+a_0. \tag{5.19}$$

Class 2

(a) Radiation dominated:

$$t=\begin{cases} A\left[\arcsin\left(\frac{a}{A}\right)^{1/2}-\left(\frac{a}{A}\right)^{1/2}\left(1-\frac{a}{A}\right)^{1/2}\right] & \text{if } k=+1,\\ \frac{2}{3A^{1/2}}a^{3/2} \ \text{if } k=0, \\ A\left[\left(\frac{a}{A}\right)^{1/2}\left(1+\frac{a}{A}\right)^{1/2}-\text{arcsinh}\left(\frac{a}{A}\right)^{1/2}\right] & \text{if } k=-1,\end{cases} \tag{5.20}$$

where $A=$ arbitrary constant >0, (for $k=+1, A=a_{\max}$).

(b) Matter dominated:

$$t+\text{const}=\begin{cases} C\left[\arcsin\left(\frac{aB}{C}\right)^{1/2}-\left(\frac{aB}{C}\right)^{1/2}\left(1-\frac{aB}{C}\right)^{1/2}\right] & \text{if } B>0,\\ \frac{2}{3A^{1/2}}a^{3/2} \ \text{if } B=0, \\ C\left[\left(\frac{a|B|}{C}\right)^{1/2}\left(1+\frac{a|B|}{C}\right)^{1/2}-\text{arcsinh}\left(\frac{a|B|}{C}\right)^{1/2}\right] & \text{if } B<0,\end{cases} \tag{5.21}$$

where $B=4GM+k, C=$ arbitrary constant >0 (for $B>0, C/B=a_{\max}$) Note that there is no horizon problem for the class-1 solutions in analogy with the solutions to this theory in 3+1 dimensions. Further, the class-2 solutions are identical to those in Sec. IV for the Einstein theory during the radiation epoch. This is not fortuitous since Eqs. (4.2) and (4.3) add to give Eq. (5.16) during the radiation-dominated epoch, and Eqs. (4.4) and (5.17) are identical.

VI. CONCLUSION

After reviewing some of the unusual features of (2+1)-dimensional Einstein gravity in the presence of point sources, we extended our study to include matter distributions. We found that although such regions then possessed local curvature the lack of a Newtonian limit continued to have interesting ramifications.

Specifically we found that all relativistic polytropes in hydrostatic equilibrium produced compact spaces, an intriguing result that could well be symptomatic of all hydrostatic structures in (2+1)-dimensional Einstein gravity. Because all the "stars" turned out to be cosmologies we then considered the behavior of homogeneous, evolving cosmologies. It would be interesting to further study whether such cosmologies have a propensity to either subnucleate out static universes or to entirely transmute into closed, static cosmologies.

Finally we investigated other gravitational theories and found that the lack of a Newtonian limit was common amongst tensor theories. To gain some insight into the behavior of a relativistic theory in 2+1 dimensions that does have a Newtonian limit, we studied a theory that was developed recently for 1+1 dimensions. This theory has a history in 3+1 dimensions where it was considered by Einstein before being rejected on aesthetic and experimental grounds. The gross nonlinearity of this theory did not allow us to find any analytic relativistic hydrostatic solutions to compare with those that we found in (2+1)-dimensional Einstein theory. Cosmological solutions were found for this theory in both 3+1 and 2+1 dimensions, some of which had the attractive property of no horizon problem.

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