New look at supermassive cosmic strings

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Although it is well known that cosmic strings produced at grand-unified-theory (GUT) scales give rise to a conical spacetime, this picture must be revised for strings produced at much larger energy scales. An expression for the metric due to a supermassive cosmic string has recently been given by Laguna and Garfinkle. Here, we argue that their metric is not unique, and that a second solution exists which has different asymptotics. The existence of this new solution is verified numerically. Like the Laguna and Garfinkle metric, the solution we give is singular at finite distance from the core of the string. We further demonstrate that supermassive cosmic strings may also arise from symmetry breaking at GUT scales if the coupling between scalar and gauge fields is very weak. We argue that these low-energy supermassive strings are closely related to U(1) global strings, a result which is not surprising given their singular nature. By analogy with global strings, it is clear that the singularity of a low-energy supermassive string occurs at extremely large distances from the core of the string.

I. INTRODUCTION

One of the most interesting aspects of cosmic strings is the nature of their coupling to gravity. It was first demonstrated by Vilenkin[1] that associated to a static, infinite, straight cosmic string is an asymptotically conical metric. Although this result is now well established for a Nielsen-Olsen string arising from grand-unified-theory (GUT) scale symmetry breaking in the Abelian Higgs model, cosmic strings in different field-theory models may give rise to very different gravitational fields. Perhaps the best example of this is provided by the U(1) global string which has been shown to produce a singular spacetime.[2–4] It is also clear that if one looks at the Abelian Higgs model away from the standard GUT symmetry-breaking scale, then as the symmetry breaking energy scale is increased, the deficit angle of the string spacetime increases.[1] There is a critical symmetry breaking scale above which the deficit angle is greater than 2π, and for which one is forced to abandon the picture of a cone in favor of a new metric; cosmic strings of this type have been termed "supermassive." Supermassive cosmic strings have recently been studied by Laguna and Garfinkle,[5] and their conclusion is that such a string has a Kasner-type metric[6] outside the core. Although related to the regular Kasner asymptotics of Melvin's magnetic universe,[7] it is found that the supermassive solution is necessarily singular at finite distance from the core. However, at first sight, unlike global strings, supermassive cosmic strings seem to be allowed only at high-energy symmetry breaking.

In Sec. II we review the work of Laguna and Garfinkle and also the earlier work on supermassive cosmic strings of Gott.[8] In Sec. III we demonstrate that the solution given by Laguna and Garfinkle is not unique, in the sense that a second solution with different asymptotics exists for the same values of the parameters in the Abelian Higgs model. The solution is similar to that found by Gott in his approximate analysis, and is perhaps more natural, since the large energy at the core of the string is reflected in a collapse in the conical structure of the spacetime. In both Laguna and Garfinkle's and Gott's treatments it is implicitly assumed that supermassive strings arise only at large symmetry-breaking scales. In Sec. IV we study this question in more detail and demonstrate that supermassive strings with singular spacetimes can form in the Nielsen Olesen model at GUT scales. These strings, however, require an extremely small value of the constant α defined explicitly below, and as such are probably unphysical. In addition, by exploiting the relationship between supermassive strings and global cosmic strings, it is possible to show that any singularity arising from GUT-scale supermassive strings must occur at a distance from the core of the string which is many orders of magnitude greater than the present Hubble scale.

II. AN OVERVIEW OF RESULTS ON SUPERMASSIVE COSMIC STRINGS

The model for a cosmic string we shall consider in this paper is the Abelian Higgs model of Nielsen and Olesen[9] with Lagrangian

$$\mathcal{L} = -\tfrac{1}{2}\left(\mathcal{D}_\mu \Phi\right)\left(\mathcal{D}^\mu \Phi\right)^* - \lambda\left(\Phi^*\Phi - \eta^2\right)^2 - \frac{F_{\mu\nu}F^{\mu\nu}}{16\pi}, \tag{1}$$

where the gauge-covariant derivative is defined by $\mathcal{D}_\mu = \nabla_\mu + ieA_\mu$ and $F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu$ is the U(1) field strength. We have taken $\hbar = c = 1$. ∇_μ is a covariant derivative if we couple the system to gravity. Although the Lagrangian depends on three parameters η, λ, and

e, it is simple to show that in flat space solutions are determined by just one dimensionless parameter

$$\alpha = 4\pi e^2/\lambda$$

with η further determining the scale of the solution.[10]

An ansatz for a cosmic string which is straight and has cylindrical symmetry is given by

$$\Phi = Re^{i\theta}, \quad A = \frac{1}{e}(P-1)\,d\theta, \tag{2}$$

where R and P are functions of ρ, and ρ and θ are cylindrical polar coordinates. In flat space it is well known that this ansatz leads to the Nielsen Olesen solution, whose asymptotics are given by

$$R \sim \eta\left(1 - R_0 e^{-2\sqrt{2\lambda}\eta\rho}\right), \quad P \sim P_0 e^{-\sqrt{\alpha\lambda}\eta\rho}. \tag{3}$$

To calculate the metric associated with a U(1) string, it is necessary to make an ansatz for the metric with similar symmetries. In addition to being cylindrically symmetric, the Nielsen Olesen solution also has boost symmetry, and it follows that the most general metric for such a string takes the form

$$ds^2 = e^{A(\rho)}\left(-dt^2 + dz^2\right) + d\rho^2 + e^{D(\rho)}d\theta^2. \tag{4}$$

Inserting (1), (2), and (4) into the Einstein-Hilbert action yields the full set of nonlinear Einstein and field equations, which must be solved to obtain a consistent picture of a self-gravitating string. In curved space, the equations contain an additional dimensionless constant, $G\eta^2$, which may be thought of as determining the strength of the coupling of the string to gravity.[10]

The first study of the gravitational field of a cosmic string was performed by Vilenkin.[1] He used a distributional energy-momentum tensor with tension and mass per unit length $\pi\eta^2$, as a source term in the linearized Einstein equations, and thereby demonstrated that the spacetime of a cosmic string is conical, with a deficit angle of the order of $G\eta^2$. A slightly more physical approach was later adopted by Gott[8] to obtain a similar result. Gott took the energy momentum tensor to be constant within a small region corresponding to the core of the string, and zero outside this region. The more realistic nature of the source made explicit the fact that the metric of a cosmic string is only asymptotically conical, and that inside the core region, the metric is regular, the spacetime looking globally like a "snub-nosed" cone.

The conical picture was obtained in both cases for GUT scale cosmic strings, where the dimensionless parameter $G\eta^2$ is extremely small (approximately equal to 10^{-6}). As expected, this qualitative result is confirmed by a more detailed analysis taking into account the full system of Einstein and field equations.[10,11] If, however, $G\eta^2$ becomes comparable to unity, one must proceed more carefully. At these energy scales the deficit angle $\Delta\theta$ approaches 2π,[10] and clearly beyond the critical point where $\Delta\theta = 2\pi$, the conical picture is no longer valid. Moreover, it is no longer consistent either to use the weak-field approximation or to regard the string as a distributional source.

Gott's method for calculating the spacetime of a cosmic string differs from Vilenkin's in that by taking into account the finite width of the string, he is able to offer a possible metric for a supermassive string. His general method of splitting spacetime into a core region and an exterior region is equivalent to matching a two-dimensional flat space (corresponding to the exterior region) to part of a two-sphere (corresponding to the core region) to obtain the spatial part of the metric transverse to the axis of the string. For small $G\eta^2$, the spherical region is of very small solid angle, and provides the "snub-nose," while the flat space is just the conical exterior. For $G\eta^2 = 1/4\pi$, the energy density becomes sufficiently large for the spherical region to be a hemisphere, and hence the exterior region is cylindrical; this is the critical point at which the transition between conical and supermassive strings occurs in his model. For $G\eta^2$ greater than but close to $1/4\pi$, the solid angle subtended by the region of the sphere is greater than 2π, and hence the flat space matched onto the sphere must be like an inverted cone, which now has a singular point at finite distance from the core.[12] The three Gott metrics are illustrated in Fig. 1. Although the last solution provides a candidate for the metric of a supermassive string, it does not seem likely that the unwelcome singularity would be smoothed out in a more detailed treatment.

Independent of the results of Gott, the spacetime of supermassive cosmic strings has been examined in some detail by Laguna and Garfinkle.[5] Their approach was to consider carefully the possible asymptotics of the supermassive string metric. It is well known that if the energy momentum tensor of any cylindrical source with boost symmetry falls off sufficiently rapidly with distance from the core, the asymptotic form of the metric must be one of the two Levi-Civita metrics,[13]

$$ds^2 = -dt^2 + dz^2 + d\rho^2 + (a_1\rho + a_2)^2\,d\theta^2, \tag{5}$$

whence the cone, or

$$\begin{aligned} ds^2 = {} & (b_1\rho + b_2)^{4/3}(-dt^2 + dz^2) \\ & + d\rho^2 + (b_1\rho + b_2)^{-2/3}\,d\theta^2. \end{aligned} \tag{6}$$

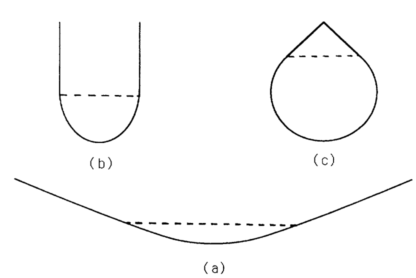

FIG. 1. An embedding diagram of the (r,θ) sections of the three Gott metrics. (a) $G\eta^2 < 1/4\pi$, (b) $G\eta^2 = 1/4\pi$, and (c) $1/2\pi > G\eta^2 > 1/4\pi$.

which is a Kasner metric.

In reality (5) and (6) are four different asymptotic metrics depending on whether a_1 and b_1 are positive or negative. For positive a_1, (5) is the standard cone, and for positive b_1, (6) is the asymptotic form of Melvin's magnetic universe. If a_1 or b_1 is negative, each of the metrics is asymptotically singular. In Ref. 5, Laguna and Garfinkle postulate that a supermassive cosmic string must correspond to a Kasner metric, (6). They show that if this is the case, then b_1 must be negative, implying that a supermassive string metric is singular. By solving the full field and Einstein equations numerically, they confirm that an asymptotic solution of this form may be matched onto a regular core. They also remark that the metric at the transition between conical and Kasner asymptotics is asymptotically cylindrical ($\mathbb{R}^3 \times S^1$ with $a_1 = 0$ or $b_1 = 0$), which, unlike their Kasner solution, agrees with Gott's prediction.

So far, we have not made reference to the considerable simplification to the problem of finding solutions to the full field and Einstein equations afforded by the analog of Bogomol'nyi's equations[14] in curved space.[15,16] The Bogomol'nyi equations apply when $\alpha = 8$, and reduce all differential equations to first order. In this special case, the deficit angle is given exactly by

$$\Delta\theta = 8\pi^2 G\eta^2 \tag{7}$$

so that the transition to supermassive strings indeed occurs exactly when $G\eta^2 = 1/4\pi$; for this value of the parameters, Linet[17] has demonstrated that the cylindrical solution referred to above may be written down in closed form. An immediate consequence of the Bogomol'nyi equations is that the metric coefficient A defined in equation (4) must vanish identically if $\alpha = 8$.[16,17] This is the case for the conical solution (when $\alpha = 8$), and for Linet's cylindrical solution. It is not, however, the case for Laguna and Garfinkle's metric, despite the fact that their solution includes the case $\alpha = 8$. We shall show in the next section that their solution is not unique, and that there is a second solution for a supermassive string which satisfies the Bogomol'nyi equations (with $A \equiv 0$) when $\alpha = 8$, and which is similar in character to that originally proposed by Gott.

III. A NEW SUPERMASSIVE COSMIC-STRING SOLUTION

Substituting the ansatz (2) into the Euler-Lagrange equations derived from the Einstein-Hilbert action, and redefining

$$X = \frac{R}{\eta}, \qquad r = \sqrt{\lambda}\eta\rho, \quad \text{and} \quad e^C = \lambda\eta^2 e^D$$

the Einstein equations and the scalar and gauge field equations are given by

$$\left(A'' + A'^2 + \frac{A'C'}{2}\right) = \frac{-16\pi G}{\lambda\eta^2}\left(T_t^t - \tfrac{1}{2}T\right), \tag{8}$$

$$\left(C'' + \frac{C'^2}{2} + A'C'\right) = \frac{-16\pi G}{\lambda\eta^2}\left(T_\theta^\theta - \tfrac{1}{2}T\right), \tag{9}$$

$$\left(2A'' + C'' + A'^2 + \frac{C'^2}{2}\right) = \frac{-16\pi G}{\lambda\eta^2}\left(T_r^r - \tfrac{1}{2}T\right), \tag{10}$$

$$X'' + X'\left(A' + \frac{C'}{2}\right) = X[P^2 e^{-C} + 4(X^2 - 1)], \tag{11}$$

$$P'' + P'\left(A' - \frac{C'}{2}\right) = \alpha P X^2, \tag{12}$$

where the components of the energy-momentum tensor are

$$\frac{2T_t^t}{\lambda\eta^2} = -\left(X'^2 + e^{-C}X^2P^2 + 2\left(X^2-1\right)^2 + \frac{P'^2 e^{-C}}{\alpha}\right),$$

$$\frac{2T_r^r}{\lambda\eta^2} = \left(X'^2 - e^{-C}X^2P^2 - 2\left(X^2-1\right)^2 + \frac{P'^2 e^{-C}}{\alpha}\right),$$

$$\frac{2T_\theta^\theta}{\lambda\eta^2} = \left(-X'^2 + e^{-C}X^2P^2 - 2\left(X^2-1\right)^2 + \frac{P'^2 e^{-C}}{\alpha}\right).$$

Equation (10) is purely a constraint equation that is automatically satisfied if we impose the boundary conditions

$$\begin{array}{llll} X \to 1, & P \to 0 & \text{as} & \rho \to \infty, \\ X \to 0, & P \to 1 & \text{as} & \rho \to 0, \end{array} \tag{13}$$

ensuring that we have a cosmic string configuration, and

$$A(0) = 0, \quad e^C \to r^2 \quad \text{as} \quad r \to 0, \tag{14}$$

ensuring that the metric is regular at the origin. We make no assumptions regarding the behavior of the metric at infinity, but in general X and P will behave asymptotically as in Eq. (3).

As mentioned above, the flat-space Bogomol'nyi bound may be generalized to curved space. The bound is saturated when $\alpha = 8$ by requiring that X and P satisfy

$$\frac{dX}{dr} = \mp X P e^{-C/2}, \tag{15}$$

$$\frac{dP}{dr} = \mp 4 e^{C/2}\left(X^2 - 1\right), \tag{16}$$

and that in addition

$$A' = 0$$

and

$$\frac{d}{dr}\left(e^{C/2}\right) = -4\pi G\eta^2 P\left(X^2 - 1\right) + 1 - 4\pi G\eta^2. \tag{17}$$

It follows directly from (17) that when $\alpha = 8$, and provided that $P\left(X^2 - 1\right)$ tends to 0 sufficiently fast and that $1 - 4\pi G\eta^2$ is positive, the deficit angle is exactly $\Delta\theta = 8\pi^2 G\eta^2$. When $1 - 4\pi G\eta^2$ is negative (or in other words when $\Delta\theta > 2\pi$), Eq. (17) implies that as $P \to 0$

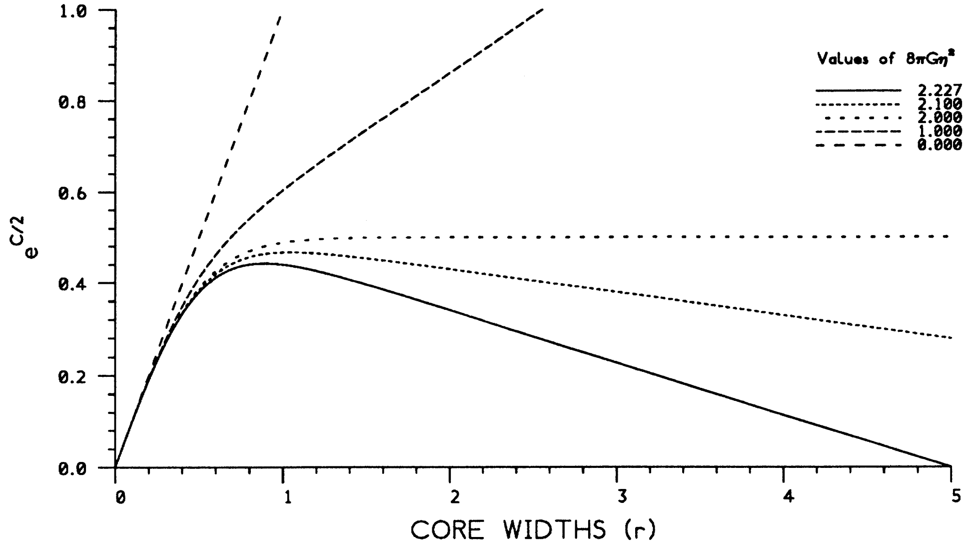

FIG. 2. The metric field $e^{C/2}$ as a function of r for several values of $G\eta^2$, with $\alpha = 8$. If $8\pi G\eta^2 > 2$, the function $e^{C/2}$ goes to zero at a singular point, which for $8\pi G\eta^2 = 2.227$ occurs at $r \sim 5$.

and $X \to 1$,

$$\left(e^{C/2}\right)' \to 1 - 4\pi G\eta^2 < 0.$$

The metric in this case is asymptotically just as in Eq. (5) with $a_1 < 0$. This gives a strong indication that at least in the case where $\alpha = 8$, there is a second solution that is again singular at finite r, but which is clearly not the Laguna and Garfinkle solution.

To verify the existence of this solution, we have solved the curved-space Bogomol'nyi equations numerically for $G\eta^2 < 1/4\pi$. The numerical calculations were performed using a polynomial approximation method called the tau method.[18,19] Details of this implementation of the tau method may be found in Ref. 20. Typically the solutions obtained yielded errors of order 10^{-5} when substituted back into the differential equations (15)–(17).

We found that the new solutions are almost exactly as predicted by Gott. The string fields X and P behave just as in the flat space solution; the behavior of $e^{C/2}$ is illustrated in Fig. 2. Note that we must constrain $G\eta^2$ to be small enough for the singularity to occur sufficiently far from the core that we can talk about a localized cosmic string. Beyond values of $G\eta^2$ where the singularity comes close to the core, it no longer makes sense to talk about string type solutions.

Having verified the existence of a solution to the Bogomol'nyi equations, it remains to see whether the new

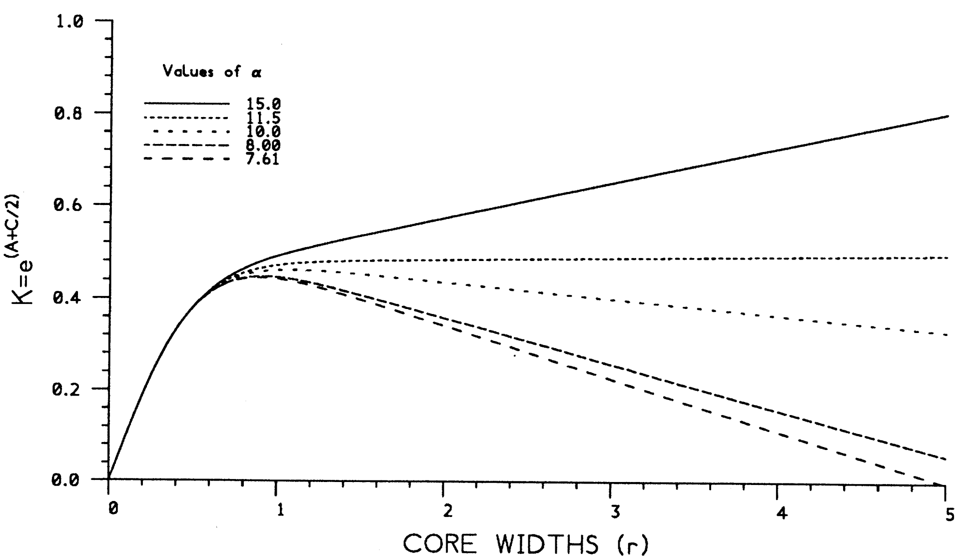

FIG. 3. The metric field $K = e^{(A+C/2)}$ as a function of r for several values of α, with $8\pi G\eta^2 = 2.2$. If $\alpha < \alpha_{\rm crit} \sim 11.5$, the string solution is supermassive and has a singularity at finite r.

solution exists away from $\alpha = 8$. This was indeed found to be the case, by solving the full system of Eqs. (8)–(12) numerically, provided that α is sufficiently close to 8 for the solution to remain within the supermassive region and for the solution to retain a distinct core region. In Fig. 3 we illustrate this with graphs of solutions for $K = e^{A+C/2}$ for various values of α, with $8\pi G\eta^2 = 2.2$. In these solutions e^A exhibited the asymptotics of (5), but without $A \equiv 0$. However, e^A was sufficiently close to 1 to regard its behavior as unimportant.

At first sight it would appear that this new inverted cone solution is the more natural given that it satisfies the Bogomol'nyi equations when $\alpha = 8$. However, the fact that both solutions are singular means that we cannot define an energy per unit length for either, and so it is difficult to justify such an argument. It is not clear whether a stability analysis of the Laguna and Garfinkle metric makes sense. A second factor in favor of the inverted cone solution is that the spacetime outside the core of the string becomes singular at a point of (nonregular) compactification. From an intuitive point of view, one would expect that an increase in the mass per unit length of a cosmic string would eventually result in the compactification of the sections transverse to the string. The Laguna and Garfinkle solution, on the other hand, has somewhat surprising asymptotics which do not lend easily to a physical interpretation. It is, however, the gauge string analog of the global U(1) string coupled to gravity as we shall describe below.

IV. GUT-SCALE SUPERMASSIVE STRINGS AND GLOBAL U(1) STRINGS

The phase space of static straight cosmic strings in the Abelian Higgs model coupled to gravity is parametrized by two dimensionless numbers, $G\eta^2$ and α defined above. It is normally assumed that cosmic strings were created by the breaking of a symmetry with α of order unity at GUT scales, corresponding to $G\eta^2 \sim 10^{-6}$, and for this choice of parameters the metric of the cosmic string is conical. It would therefore appear that supermassive strings cannot be produced at GUT scales.

The supermassive string solutions discussed above arise from the fact that as $G\eta^2$ increases, the deficit angle also increases. However, it is evident from the results given in Sec. III that the deficit angle also increases as α decreases. This phenomenon is reflected in the dependence on α of the critical value of $G\eta^2$ at which strings become supermassive. A graph of this dependence has been given by Laguna and Garfinkle in Ref. 5, and this shows that the critical value of η is a monotonically increasing function of α, at least for α and $G\eta^2$ both of order unity. Within this range it is clear that for a fixed symmetry-breaking scale, one should expect supermassive strings to form for any model with α smaller than a critical value of α depending on $G\eta^2$. In order to understand the role of α, and to determine whether this feature is still present at the much smaller $G\eta^2$ of GUT scales, it is necessary to examine the small α limit of the Abelian Higgs model in more detail.

Let us study the cosmic string solution in terms of the dimensionless unit of length specified by r. In these units, the width of the scalar field core is independent of both α and $G\eta^2$, whereas the width of the electromagnetic core is determined by α as may be seen directly from the asymptotics (3). $G\eta^2$ determines the strength of coupling to gravity of the energy contained within the two core regions. In the central core, where the derivative of $X(r)$ is of order 1, the energy density is of the same order of magntiude for all α. In the second, outer core (of width $r \sim 1/\sqrt{\alpha}$), $P \sim 1$, and $P'^2/\alpha \sim 0$. If one inserts these two values into the full field Eqs. (8)–(12), they become exactly the field equations for a U(1) global cosmic string coupled to gravity.[4] Thus, for small α, there is an outer core in which the gauge string solution behaves like a global string with X consequently falling off as $1/r^2$ in this region. As a result, the total mass per unit length of the cosmic string has an extra contribution from this outer core, which is given approximately in dimensionless units by

$$G\eta^2 \ln\left(1/\sqrt{\alpha}\right) . \tag{18}$$

When α is sufficiently small for this to become comparable to 2π, then a gauge cosmic string is supermassive rather than conical.

From a physical point of view, it is clear that this effect is almost certainly not relevant. In the first place, for the expression of Eq. (18) to be of order unity, we require that

$$\alpha \sim \exp\left(\frac{-1}{G\eta^2}\right) ,$$

which for GUT scales gives a value of α in the region of $e^{10^{-6}}$. This value is probably far too small to be of any interest. Secondly, either of the two supermassive metrics is approximately conical close to the core. The distance from the core at which the supermassive nature of the string starts to become important is given by $1/\alpha$ core widths which is clearly many orders of magnitude larger than the present horizon. This argument is exactly the one used to ignore the singularity in the metric of U(1) global strings.

Although the small-α limit does not appear to be directly relevant to cosmology, it does make clear the relationship between the gravitational fields of gauge cosmic strings and U(1) global cosmic strings. In the limit $\alpha \rightarrow 0$, we have seen that $P \rightarrow 1$, which reduces the field equations to those of a global string. The smaller α, the larger the region over which a Nielsen-Olesen string looks like a global string.

Looking at the Abelian Higgs model, the global U(1) limit occurs when the gauge and scalar fields decouple. This corresponds to $e = 0$, which is equivalent to $\alpha = 0$. We could have therefore expected the global string metric to be the $\alpha = 0$ limit of a supermassive string metric. A simple inspection shows that the global string metric has the same asymptotics (as discussed in Ref. 21) as the supermassive string metric of Laguna and Garfinkle. The reason why this, rather than the inverted cone solution is picked out may be easily understood. The term in the energy density proportional to

$$\frac{X^2P^2}{e^C}$$

is always finite in both supermassive string solutions. In the Laguna and Garfinkle solution, e^C tends to infinity. In the new solution, although e^C goes to zero at the singularity, so does P. However, if $P \equiv 1$, then only the Laguna and Garfinkle metric avoids infinite energy density at the singularity. Therefore a global string should correspond to the $\alpha \longrightarrow 0$ limit of their solution. The new solution found above apparently has no well-defined limit as $\alpha \longrightarrow 0$.

V. CONCLUSIONS

We have demonstrated the existence of a new supermassive cosmic string solution. In common with Laguna and Garfinkle's solution, the metric has a singularity, the existence of which may be easily interpreted in the new solution. The new metric takes the form of a teardrop as suggested by Gott. This type of metric has also been discussed in various other contexts (see for example the section on sigma model strings in Ref. 4). It has compact spatial sections which could provide a mechanism for dimensional reduction.

In addition we have shown that supermassive strings could be created at GUT scale phase transitions, although this requires fine tuning of the scales in the field-theory model, and would in addition not have any directly observable consequences. Finally, we have shown how gauge strings with small α are related to global cosmic strings.

The appearance of two solutions for the metric of a supermassive cosmic string suggests that the same phenomenon may occur for cosmic strings hitherto associated only with conical spacetimes. There is no mathematical reason to rule out the possibility of Melvin-type asymptotics for a cosmic string with a small mass per unit length. These would be of the form given by Eq. (6), with $b_1 > 0$, and consequently would have no singularity. It would be interesting if such solutions were found, as they would highlight the need for a dynamical mechanism favoring evolution to a conical solution after symmetry breaking.

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