Phase transition in Reissner-Nordström black holes

Diego Pavón
Department of Physics, Faculty of Sciences, Autonomous University of Barcelona, 08193 Bellaterra, Barcelona, Spain
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It is argued that nonrotating electrically charged black holes present a nonequilibrium second-order phase transition in the limit $|Q| \to M$.

Soon after the discovery that quantum black holes were thermodynamic objects possessing a well-defined temperature T and entropy S, it was found that the heat capacity of rotating neutral black holes,

$$C_J = \frac{MTS^3}{\pi J^2 - T^2 S^3} \quad (\text{Kerr}) , \tag{1}$$

and nonrotating electrically charged black holes,

$$C_Q = \frac{4MTS^3}{\pi Q^4 - 4T^2 S^3} \quad (\text{Reissner-Nordström}) , \tag{2}$$

suffer an infinity discontinuity and change their sign at some given value of their parameters.[1] Effectively, at the "transition" point $J = (2\sqrt{3}-3)^{1/2} M^2$ and $|Q| = (\sqrt{3}/2) M$, the right-hand sides of Eqs. (1) and (2) diverge.

This result was interpreted by Davies[2] on a thermodynamic basis as a second-order phase transition. However, as pointed out by Sokolowski and Mazur[3] the nature of such "phase transitions" is very unclear. The event horizon does not lose its regularity and the internal state of the black hole remains unaffected, whence it is hard to see how these can be aptly named phase transitions.

However, one may still wonder whether the black-hole radiance is affected in some way by these "transitions," i.e., if at the "transition" points the second moments in the fluctuations of the rates of emission of energy and angular momentum or charge diverge or remain bounded. In a previous work[4] we used the Landau-Lifshitz hydrodynamic fluctuation theory[5] to calculate the second moments in the fluctuations of the fluxes of energy and angular momentum of a generic Kerr black hole. It turned out that nothing special happened at the "transition" point; however, we found that some second moments diverge in the limit $J \to M^2$, which corresponds to a maximally rotating black hole. This may be interpreted as a phase transition from an extreme ($J = M^2$) to nonextreme ($J < M^2$) Kerr black hole. Effectively, that transition is accompanied by a sudden change in the emission properties of the black hole. An extreme Kerr black hole is unable to radiate via spontaneous Hawking emission but only via superradiant scattering, whereas a nonextreme one gives off particles and radiation via both mechanisms. Furthermore, unlike ordinary Kerr black holes the inner and outer horizons become degenerate in the extreme limit.

The target of this work is to analyze the fluctuations in the emission rates of energy and electric charge of Reissner-Nordström black holes. It turns out, on the one hand, that some relevant correlations diverge in the extreme Reissner-Nordström limit ($|Q| \to M$), which can be seen as an indication that a true phase transition occurs at that point. On the other hand, no second moment diverges at the "transition" point $|Q| = (\sqrt{3}/2) M$. The latter outcome supports the view that no true physical transition can be ascribed to that point. These results fairly parallel those of Ref. 4. In what follows, for the sake of notational simplicity, we shall assume the charge Q to be positive; the general case is recovered by taking its absolute value.

To carry out our analysis we shall make use of the rates of emission of energy and charge by very massive and high electrically charged black holes. These, as calculated by Hiscock and Weems,[6] are

$$\dot{M} = -a\alpha\sigma_0 T^4 + \frac{Q}{r_+}\dot{Q} , \tag{3}$$

$$\dot{Q} \cong -\frac{e^4}{2\pi^3 \hbar m^2} \left[\frac{Q}{r_+} \right]^3 \exp\left[\frac{r_+^2}{QQ_0} \right] , \tag{4}$$

respectively. Here α is a fairly constant quantity that depends on, among other things, the number of massless neutrinos species. If the number is three, $\alpha \approx 2.0228$, whereas if it is zero, $\alpha \approx 0.267\,92$. σ_0 is the geometrical-optics cross section of the black hole and $a = \pi^2/15\hbar^3$. The quantity $r_+ = M + (M^2 - Q^2)^{1/2}$ stands for the radius of the outer event horizon, $Q_0 = \hbar e/\pi m^2$ with e and m the charge and mass of the electron, respectively. An upper dot means temporal derivative. The temperature is given by

$$T = \frac{\hbar}{2\pi} \frac{\Lambda}{r_+^2} , \tag{5}$$

with $\Lambda = (M^2 - Q^2)^{1/2}$. The first term on the right-hand side of Eq. (3) accounts for the emission of massless particles (neutrinos, photons, gravitons) while the second one accounts for the coupling between the emission rates of mass and electric charge.

These rates were derived under the following assumptions. (i) The rate of emission is so slow ($r_+^2 \gg QQ_0$, $T \ll 10^{-10}$ eV) that neither the mass nor charge of the hole vary appreciably on a geometrical time scale ($\simeq M$).

(ii) The black hole is in isolation so there is no surrounding medium accreting onto the black hole; otherwise the hole charge would neutralize quickly. From the astrophysical side this is not a realistic situation but it makes no difference for our purposes. (ii) The black-hole mass is so high ($M \gg Q_0$) that the emission rate of charge is overwhelmingly due to electron-positron pair creation and well described by the corresponding Schwinger formula.[7] It is obvious that these rates must experience spontaneous fluctuations around their quasi-steady average value—given by the right-hand sides of Eqs. (3) and (4), respectively. To calculate these fluctuations we shall resort to the Landau-Lifshitz theory,[5] which applies to situations such as this one, a slow varying process not far away from thermodynamic equilibrium. The latter means that the generation of entropy

$$\dot{S} = 2\pi \frac{r_+}{\Lambda}(r_+ \dot{M} - Q\dot{Q}) \tag{6}$$

is very small.

According to Landau and Lifshitz, if the flux $\dot{x}_i$ of a given thermodynamic quantity x_i, which is varying in a general dissipative process, is governed by

$$\dot{x}_i = -\sum_j \Gamma_{ij} X_j + \delta \dot{x}_i \tag{7}$$

and the entropy rate by

$$\dot{S} = \sum_i (\pm X_i \dot{x}_i) , \tag{8}$$

then the correlations in the fluctuations of the flux obey

$$\langle \delta \dot{x}_i \, \delta \dot{x}_j \rangle = k_B (\Gamma_{ij} + \Gamma_{ji}) \delta_{ij} . \tag{9}$$

Here and throughout, the angular brackets denote a statistical average with respect to the steady value $\langle \dot{x}_i \rangle$ whence $\langle \delta \dot{x}_i \rangle$ vanishes. The quantities Γ_{ij} and X_j indicate the phenomenological transport coefficients and the thermodynamic forces respectively. The Kronecker δ in Eq. (9) is to ensure that correlations between independent fluxes vanish.

To determine the second moments in the fluctuations in the fluxes of mass and electric charge we add a stochastic term $\delta \dot{M}$ and $\delta \dot{Q}$ to the right-hand sides of Eqs. (3) and (4), respectively, and combine the resulting relationships with Eq. (6). The standard procedure leads to

$$\langle \delta \dot{M} \, \delta \dot{M} \rangle = \frac{k_B}{\pi} \left[\alpha \sigma_0 T^4 + B \left[\frac{Q}{r_+} \right]^4 \exp\left[-\frac{r_+^2}{Q Q_0} \right] \right] \frac{\Lambda}{r_+^2} , \tag{10}$$

$$\langle \delta \dot{Q} \, \delta \dot{Q} \rangle = -\frac{k_B}{\pi} B \Lambda \frac{Q^2}{r_+^4} \exp\left[-\frac{r_+^2}{Q Q_0} \right] , \tag{11}$$

$$\langle \delta \dot{M} \, \delta \dot{Q} \rangle = \frac{Q}{r_+} \langle \delta \dot{Q} \, \delta \dot{Q} \rangle , \tag{12}$$

where $B \equiv e^4 / 2\pi^3 \hbar m^2$.

Note that at the "transition" point nothing special happens; that is to say, these fluctuations remain bounded. However they tend to vanish in the limit $Q \to M$. This is a very interesting property, since if it were not met Q could get bigger than M, causing the event horizon to disappear whence the singularity inside the black hole would reach a causal connection with external observers. Accordingly, the constraint $Q \le M$ can be viewed as a boundary condition. It is well known in hydrodynamic and electric systems that the autocorrelations vanish when a relevant parameter in the fluctuating system attains a boundary value.[8] An analogous behavior can be found in Kerr black holes.[4]

The second moments in the entropy and temperature rates are easily calculated from Eqs. (6) and (5). We get

$$\langle \delta \dot{S} \, \delta \dot{S} \rangle = \left[\frac{2\pi r_+}{\Lambda} \right]^2 (r_+^2 \langle \delta \dot{M} \, \delta \dot{M} \rangle + Q^2 \langle \delta \dot{Q} \, \delta \dot{Q} \rangle - 2 r_+ Q \langle \delta \dot{M} \, \delta \dot{Q} \rangle) , \tag{13}$$

$$\langle \delta \dot{T} \, \delta \dot{T} \rangle = \left[\frac{\hbar}{2\pi \Lambda r_+^3} \right]^2 ([M(M-\Lambda) - 2\Lambda^2]^2 \langle \delta \dot{M} \, \delta \dot{M} \rangle (\Lambda - M)^2 Q^2 \langle \delta \dot{Q} \, \delta \dot{Q} \rangle + 2[M(M-\Lambda) - 2\Lambda^2](\Lambda - M) Q \langle \delta \dot{M} \, \delta \dot{Q} \rangle) , \tag{14}$$

$$\langle \delta \dot{S} \, \delta \dot{T} \rangle = \frac{\hbar}{\Lambda^2 r_+^2} (r_+ [M(M-\Lambda) - 2\Lambda^2] \langle \delta \dot{M} \, \delta \dot{M} \rangle - (\Lambda - M) Q^2 \langle \delta \dot{Q} \, \delta \dot{Q} \rangle - [2M^2 + \Lambda^2 - M\Lambda] Q \langle \delta \dot{Q} \, \delta \dot{M} \rangle) . \tag{15}$$

Bearing in mind that the second moments given by Eqs. (10), (11), and (12) are proportional to Λ and that in the limit $Q \to M$ one has $\Lambda \to 0$ and $r_+ \to M$, one immediately realizes that in that limit the right-hand side of the last three equations diverge. From statistical mechanics it is well known that phase transitions are usually accompanied by the occurrence of divergences in some relevant second moments.[9] Accordingly, one may think that these divergences correspond to a nonequilibrium second-order phase transition from extreme to nonextreme Reissner-Nordström black hole. This conclusion seems reasonable since, on the one hand, no divergences are found in any of the second moments for $Q < M$, and on the other hand, extremely charged spherical black holes differ radically from nonextreme ones in their way of emission. While the former radiate only via superradiant scattering, as Hawking spontaneous emission is suppressed ($T=0$), the latter emit via both mechanisms.

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