PHYSICAL REVIEW D VOLUME 43, NUMBER 7 1 APRIL 1991

Generalized two-angle parametrization of the Cabibbo-Kobayashi-Maskawa matrix

H. G. Blundell and R. B. Mann
Guelph-Waterloo Program for Graduate Work in Physics, Department of Physics, University of Waterloo, Waterloo, Ontario, Canada N2L 3G1

U. Sarkar
Theory Group, Physical Research Laboratory, Navrangpura, Ahmedabad, India 380009
(Received 17 September 1990)

We demonstrate how to parametrize the Cabibbo-Kobayashi-Maskawa (CKM) matrix in terms of its eigenvalues and eigenvectors, generalizing a recent idea of Kielanowski's. In this version we are able to reproduce a symmetric CKM matrix with only two angles while predicting a range in the amount of *CP* violation. The relation between this parametrization and the standard one is studied. Some variations of this parametrization are worked out.

I. INTRODUCTION

In spite of the tremendous successes of the standard model of strong and electroweak interactions, we do not have any clue towards an understanding of its basic parameters. In the quark sector there are ten such arbitrary parameters: six quark masses, three mixing angles, and a *CP*-violating phase. It would be highly desirable to obtain a deeper knowledge of the origin of as many of these parameters as possible, and so a phenomenological study of this sector is highly advocated.

The three mixing angles $(\theta_1,\theta_2,\theta_3)$ and *CP*-violating phase δ are incorporated into the quark sector of the standard model via the Cabibbo-Kobayashi-Maskawa (CKM) matrix. Among the several conventional parametrizations of the CKM matrix that exist, we shall consider the one given in Ref. 1, which is

$$\tilde{V}=\begin{bmatrix} \tilde{c}_1 & \tilde{s}_1\tilde{c}_3 & \tilde{s}_1\tilde{s}_3 \\ -\tilde{s}_1\tilde{c}_2 & \tilde{c}_1\tilde{c}_2\tilde{c}_3-\tilde{s}_2\tilde{s}_3e^{i\delta} & \tilde{c}_1\tilde{c}_2\tilde{s}_3+\tilde{s}_2\tilde{c}_3e^{i\delta} \\ -\tilde{s}_1\tilde{s}_2 & \tilde{c}_1\tilde{s}_2\tilde{c}_3+\tilde{c}_2\tilde{s}_3e^{i\delta} & \tilde{c}_1\tilde{s}_2\tilde{s}_3-\tilde{c}_2\tilde{c}_3e^{i\delta} \end{bmatrix} \qquad (1)$$

where $\tilde{c}_i\equiv\cos(\theta_i)$ and $\tilde{s}_i\equiv\sin(\theta_i)$. By an appropriate redefinition of the quark field phases, each of the angles θ_i can be made to satisfy $0\le\theta_i\le(\pi/2)$.

The quark fields are each defined up to a phase, allowing us to rotate the phase δ without any loss of generality. Hence it becomes important to define a rephasing-invariant *CP*-violating parameter. This was done by considering a combination of elements of $\tilde{V}$, in which the number of incoming and outgoing up- and down-quark fields are the same, so that rephasing of these fields cannot alter the combination. Such a combination was given in Ref. 2:

$$J=\mathrm{Im}(t_{i\alpha j\beta})=\mathrm{Im}(\tilde{V}_{i\alpha}\tilde{V}_{j\beta}\tilde{V}^*_{i\beta}\tilde{V}^*_{j\alpha}) \ . \qquad (2)$$

Unitarity of $\tilde{V}$ then implies that for the three-generation case all possible $t_{i\alpha j\beta}$'s are related to each other. Consequently there is only one independent *CP*-violating quantity J. Thus if any element of $\tilde{V}$ is zero, J vanishes and there is no violation of *CP* in the quark sector. For example, if $\tilde{s}_3=0$ then $\tilde{V}_{13}$ vanishes, implying $J=\mathrm{Im}(t_{13j\beta})=0$. Similarly, for $\tilde{s}_2=0, \tilde{V}_{31}=0 \Longrightarrow J=\mathrm{Im}(t_{31j\beta})=0$. For $\tilde{s}_1=0$ the first generation completely decouples from the others and there is again no *CP* violation. Hence in the parametrization [Eq. (1)] the *CP*-violating phase can be completely absorbed into a redefinition of quark fields if any one of the mixing angles vanishes, leading to no *CP* violation in the quark sector.

Recently a new parametrization of the CKM matrix in terms of its eigenvalues and eigenvectors has been proposed by Kielanowski.[3] The rephasing freedom of the quark fields implies that two CKM matrices V and V' are physically equivalent provided

$$V=U_1U\tilde{V}U_1^\dagger \ , \qquad (3)$$

where $U=\mathrm{diag}(e^{i\phi_1},e^{i\phi_2},e^{i\phi_3})$ and $U_i=\mathrm{diag}(1,e^{i\psi_1},e^{i\psi_2})$. This freedom may be exploited to fix the eigenvalues of V such that $\mathrm{tr}V=0$ and $\det V=1$.[3] The eigenvectors may be constructed in terms of three angles $(\beta_1,\beta_2,\beta_3)$ and one phase α. In this parametrization the phase α drops out of the CKM matrix when $\beta_3=0$. However, unlike other parametrizations, the *CP*-violating quantity J does not vanish; rather it is given in terms of the other two angles β_1 and β_2. In this case the CKM matrix is symmetric and is completely determined by these two angles.

In this paper we generalize the latter parametrization so that choosing $\beta_3=0$ will yield a range of J values. In our general parametrization we have, in addition to the β_i and α, the variables ϕ, x, and γ, which can be continuously varied using the above rephasing freedom. In the observable results, these three new variables reduce to x and $\Gamma=\gamma-\phi/3$. For a particular choice of ϕ, x, and γ, the CKM matrix is defined by β_i and α. When $\beta_3=0$ the phase α again drops out, yielding a two-angle parametrization of V. J depends not only on β_1 and β_2 but also on the chosen values of the parameters. Different choices of these parameters for the same value of β_3 are not physically equivalent. We consider several variations of this generalized two-angle parametrization. In all cases we relate the two parameters β_1 and β_2 to the parameters of $\tilde{V}$ and give the relations which correspond to the condition $\beta_3=0$. Since for a given x and Γ (in this parametrization) we can eliminate two parameters (β_3 and α) by choosing $\beta_3=0$, in all cases we get two constraint equations in terms of the variables θ_i and δ of $\tilde{V}$. Hence by fixing two angles, $\tilde{V}$ and J are completely determined.

II. REPARAMETRIZATION OF THE CKM MATRIX

We will examine V, created from $\tilde{V}$ by rephasing as in Eq. (3). The eigenvalues of V satisfy its characteristic equation

$$\lambda^3-k_1\lambda^2+k_2\lambda-k_3=0 \tag{4}$$

where $k_1=\mathrm{tr}V, k_2=\frac{1}{2}[(\mathrm{tr}V)^2-\mathrm{tr}(V^2)]$, and $k_3=\det V$. The unitarity of V gives $k_2=(\mathrm{tr}V)^*\det V=k_1^*k_3$. The unitarity of U_1 means that it drops out of $\mathrm{tr}V$ and $\det V$, from Eq. (3). We then have

$$\mathrm{tr}V=\mathrm{tr}(U\tilde{V})=e^{i\phi_1}\tilde{V}_{11}+e^{i\phi_2}\tilde{V}_{22}+e^{i\phi_3}\tilde{V}_{33} \tag{5}$$

and

$$\det V=\det(U\tilde{V})=e^{i(\phi_1+\phi_2+\phi_3)}\det\tilde{V} \ . \tag{6}$$

Let $\mathrm{tr}V=xe^{i\gamma}$, a general complex number with real parameters. The unitarity of $\tilde{V}$ allows us to write $\det V=e^{i\phi}$, a phase. Equation (4) then becomes

$$\lambda^3-xe^{i\gamma}\lambda^2+xe^{-i\gamma}e^{i\phi}\lambda-e^{i\phi}=0 \ . \tag{7}$$

This can always be simplified by the transformations $\lambda=e^{i\phi/3}\,\Lambda$ and $\gamma=\Gamma+\phi/3$, to get

$$\Lambda^3-xe^{i\Gamma}\Lambda^2+xe^{-i\Gamma}\Lambda-1=0 \ . \tag{8}$$

Choosing $\mathrm{tr}V=0$ and $\det V=1$ (as in Ref. 3) gives $\lambda^3=1$ from Eq. (4). After solving for the eigenvalues λ_1, λ_2, and λ_3, the corresponding eigenvectors may be chosen as[3]

$$w_1=\begin{bmatrix} c_1 \\ s_1c_2 \\ s_1s_2 \end{bmatrix}, \quad w_2=\begin{bmatrix} -s_1c_3 \\ c_1c_2c_3-s_2s_3e^{i\alpha} \\ c_1s_2c_3+c_2s_3e^{i\alpha} \end{bmatrix}, \quad w_3=\begin{bmatrix} s_1s_3 \\ -c_1c_2s_3-s_2c_3e^{i\alpha} \\ -c_1s_2s_3+c_2c_3e^{i\alpha} \end{bmatrix} \tag{9}$$

where $c_i\equiv\cos(\beta_i)$ and $s_i\equiv\sin(\beta_i)$. The reparametrized CKM matrix may then be formed:

$$\begin{aligned} V&=\lambda_1w_1\otimes w_1^\dagger+\lambda_2w_2\otimes w_2^\dagger+\lambda_2w_3\otimes w_3^\dagger \\ &=W\hat{\Lambda}W^\dagger \end{aligned} \tag{10}$$

where W is the matrix of eigenvectors

$$W\equiv(w_1 \quad w_2 \quad w_3) \tag{11}$$

and $\hat{\Lambda}\equiv\mathrm{diag}(\lambda_1,\lambda_2,\lambda_3)$ is the diagonalized CKM matrix. Again by an appropriate redefinition of the quark field phases, each of the angles β_i can be made to satisfy $0\le\beta_i\le(\pi/2)$.

Then following this prescription for V, there is no need to make the above choices for $\det V$ and $\mathrm{tr}V$. The most general solution of Eq. (7) is easily obtained from Eq. (8) by applying the standard solution to a cubic; the resulting eigenvalues are

$$\begin{aligned} \lambda_1&=e^{i\phi/3}[e^{-2\pi i/3}(A+B)^{1/3}+e^{2\pi i/3}(A-B)^{1/3}+C] \ , \\ \lambda_2&=e^{i\phi/3}[e^{2\pi i/3}(A+B)^{1/3}+e^{-2\pi i/3}(A-B)^{1/3}+C] \ , \\ \lambda_3&=e^{i\phi/3}[(A+B)^{1/3}+(A-B)^{1/3}+C] \ , \end{aligned} \tag{12}$$

where

$$A=\frac{27-9x^2+2x^3e^{3i\Gamma}}{54} \ ,$$

$$B=\left[\frac{27-18x^2-x^4+8x^3\cos(3\Gamma)}{108}\right]^{1/2} \ ,$$

$$C=\tfrac{1}{3}xe^{i\Gamma} \ ,$$

and

$$\Gamma=\gamma-\phi/3 \ . \tag{13}$$

These eigenvalues are universal: they do not depend on any of the β_i or α, but only on the choices of ϕ, x, and γ. Note that the factor $e^{i\phi/3}$ is present in every eigenvalue, and hence in every element of the CKM matrix. This direct ϕ dependence will vanish in the observable results (the magnitudes of the CKM matrix elements, and J) leaving only x and Γ. It is thus convenient to consider $\det V=e^{i\phi}$ to be unconstrained, and to use $\mathrm{tr}V=xe^{i(\Gamma+\phi/3)}$ as a quantity with two arbitrary variables, x and Γ. We will follow this practice. The value of $\mathrm{tr}V=xe^{i(\Gamma+\phi/3)}$ is constrained by Eq. (5). Freedom to choose the ϕ_i allows Γ to be any angle and puts the following limits on x:

$$|x|\le|\tilde{V}_{11}|+|\tilde{V}_{22}|+|\tilde{V}_{33}| \ . \tag{14}$$

It is preferable to consider $\det V$ unconstrained, because if $\det V$ is set to a particular value, a constraint is placed upon the ϕ_i. This in turn restricts the range of x more than in Eq. (14). For example, choosing $\mathrm{tr}V$ to be real ($\gamma=0$) and $\det V$ to be one implies

$$\big||\tilde{V}_{22}|-|\tilde{V}_{33}|\big|\le|\tilde{V}_{11}-xe^{i\phi_1}|\le|\tilde{V}_{22}|+|\tilde{V}_{33}| \ , \tag{15}$$

which reduces to the limits found in Ref. 3 when $x=0$.

We pause here to comment on the nature of the choice of $\mathrm{tr}V$. Consider any two such choices, yielding eigenvalues $\{\lambda_1,\lambda_2,\lambda_3\}$, and $\{\lambda_1',\lambda_2',\lambda_3'\}$. If we use the eigenbasis [Eq. (9)] in each case then requiring the corresponding CKM matrices V and V' to be physically equivalent implies from Eq. (3) that

$$\mathrm{tr}\hat{\Lambda}'=\mathrm{tr}V'=\mathrm{tr}(W^\dagger VW\Lambda') \tag{16}$$

and so the J value and magnitudes of the elements of V'

will not be equivalent to those of V since $W^{\dagger}UW$ is not a diagonal phase matrix. Hence choosing $\beta_3=0$ for V is not equivalent to choosing $\beta_3=0$ for V'. Alternatively, if $\hat{\Lambda}'=U\hat{\Lambda}$ where U is a diagonal phase matrix, then

$$V'=WUW^{\dagger}V \tag{17}$$

and so V' and V are not physically equivalent. In this case requiring physical equivalence between V and V' entails the use of a different eigenbasis for each.

We close this section by examining the conditions necessary for the CKM matrix to be magnitude symmetric. Setting $|V_{12}|=|V_{21}|$ yields

$$0=\sin(\alpha)\sin(\beta_1)\sin(2\beta_1)\sin(2\beta_2)\sin(2\beta_3)[(\lambda_1-\lambda_2)^*\lambda_3+(\lambda_2-\lambda_3)^*\lambda_1+(\lambda_3-\lambda_1)^*\lambda_2] . \tag{18}$$

An identical condition follows from setting $|V_{23}|=|V_{32}|$ and also from setting $|V_{13}|=|V_{31}|$; hence Eq. (18) is necessary and sufficient for the CKM matrix to be magnitude symmetric. This condition is satisfied whenever any two eigenvalues are equal, or when any one of the β_i is a multiple of $(\pi/2)$, or when α is a multiple of π. By direct calculation using Eq. (10) it is straightforward (but somewhat tedious) to show that each of these choices yields a fully symmetric CKM matrix up to the phase equivalence (3). Hence any magnitude symmetric CKM matrix is phase equivalent to a fully symmetric CKM matrix, dependent upon only three parameters.

In the sequel we shall take $\beta_3=0$ to obtain a generalization of the two-angle CKM matrix of Ref. 3. Rather than working with the eigenvalues (12) we shall work with the parameters x and Γ; setting $x=0$ will then directly recover the results of Ref. 3.

III. EXAMPLES

We now go on to investigate how the J value and magnitudes of the CKM matrix depend upon the choice of $\mathrm{tr}V$. The most general expression one can write down is extremely complicated and will not be reproduced here. We shall proceed instead by considering various particular cases.

Consider first the choice $\mathrm{tr}V=xe^{i\phi/3}(\Gamma=0)$. Then the solutions of Eq. (7) are

$$\lambda_1=\begin{cases}\frac{1}{2}e^{i\phi/3}(x-1-i\sqrt{3+2x-x^2}), & -1\le x\le 3 ,\\ \frac{1}{2}e^{i\phi/3}(x-1-\sqrt{x^2-2x-3}), & x\le -1 \text{ or } x\ge 3 ,\end{cases}$$
$$\lambda_2=\begin{cases}\frac{1}{2}e^{i\phi/3}(x-1+i\sqrt{3+2x-x^2}), & -1\le x\le 3 ,\\ \frac{1}{2}e^{i\phi/3}(x-1+\sqrt{x^2-2x-3}), & x\le -1 \text{ or } x\ge 3 ,\end{cases} \tag{19}$$
$$\lambda_3=e^{i\phi/3} .$$

Note that, as discussed above, the factor $e^{i\phi/3}$ will vanish in the magnitudes of the CKM matrix elements, and in J, so the observable results are independent of $\det V$. If all the eigenvalues are real except for a common phase then the product $V_{11}V_{22}V_{12}^*V_{21}^*$ is real. Thus $J=0$ for $x\le -1$ or $x\ge 3$. Unitarity of V implies that the only relevant range of x is $-1\le x\le 3$; we shall consider the case in which x varies over this entire range, although it is possible to adjust ϕ so that $x>0$. We are interested in the case where $\beta_3=0$ and α drops out. Then the magnitudes of the CKM matrix elements are

$$\begin{aligned}
&|V_{11}|=\sqrt{1-\tfrac{1}{4}\sin^2(2\beta_1)(3+2x-x^2)} ,\\
&|V_{12}|=|V_{21}|=\tfrac{1}{2}\sin(2\beta_1)\cos(\beta_2)\sqrt{3+2x-x^2} ,\\
&|V_{13}|=|V_{31}|=\tfrac{1}{2}\sin(2\beta_1)\sin(\beta_2)\sqrt{3+2x-x^2} ,\\
&|V_{22}|=\sqrt{1-\tfrac{1}{4}\sin^2(2\beta_2)(3-x)-\tfrac{1}{4}\sin^2(2\beta_1)\cos^4(\beta_2)(3+2x-x^2)} ,\\
&|V_{23}|=|V_{32}|=\tfrac{1}{2}\sin(2\beta_2)\sqrt{(3-x)-\tfrac{1}{4}\sin^2(2\beta_1)(3+2x-x^2)} ,\\
&|V_{33}|=\sqrt{1-\tfrac{1}{4}\sin^2(2\beta_2)(3-x)-\tfrac{1}{4}\sin^2(2\beta_1)\sin^4(\beta_2)(3+2x-x^2)} ,
\end{aligned} \tag{20}$$

and $J=\mathrm{Im}(V_{11}V_{22}V_{12}^*V_{21}^*)$ is

$$J=\tfrac{1}{32}\cos(2\beta_1)\sin^2(2\beta_1)\sin^2(2\beta_2)(3+2x-x^2)^{3/2} . \tag{21}$$

If we ignore experimental constraints, the maximum range of J that we can get by varying β_1, β_2, and x is

$-(\sqrt{3}/18)\le J \le (\sqrt{3}/18)$.

To understand the implications of $\beta_3=0$ and any particular choice of x, we shall now relate the various angles and the phase of parametrization (1) to those of our new parametrization (10), modulo the redundant phase transformation (3). This is done by equating the modulus of four of the elements of the CKM matrix, for the two parametrizations (1) and (10). For $\beta_3=0$ the relations are given by

$$\theta_1=\arcsin[\tfrac{1}{2}\sin(2\beta_1)\sqrt{3+2x-x^2}],\quad \theta_2=\beta_2,\quad \theta_3=\beta_2,\quad \delta=\arctan\left[\frac{\cos(2\beta_1)\sqrt{3+2x-x^2}}{1-x}\right]. \tag{22}$$

The quadrant that δ lies in may be determined by equating J for the two parametrizations (1) and (10). As per the discussion at the end of Sec. II, for any value of x the CKM matrix is symmetric and $\theta_2=\theta_3$ for $\beta_3=0$. Different choices of x in this case imply different predictions of the *CP* violation. The fact that we are exploiting the rephasing freedom to get different choices of x, but then predicting different values for the rephasing-invariant parameter J, follows from the discussion at the end of Sec. II.

We use the experimental values of the magnitudes $\rho=|V_{13}/V_{23}|$ and J to set limits on β_1, β_2, and x. $J=(2.8-6.9)\times 10^{-5}$ is obtained from Refs. 3 and 4. We take the total spread of the average results in Ref. 5 to get $\rho^2=0.006-0.028$. The source of all magnitudes but $|V_{13}|$ is Ref. 6 (using the 2σ best-estimate values); $|V_{13}|$ is calculated from ρ and $|V_{23}|$. The experimental values of the magnitudes of the CKM matrix elements that we use are

$$\begin{bmatrix} 0.9748-0.9761 & 0.2173-0.2230 & 0.003-0.010 \\ 0.2169-0.2226 & 0.9734-0.9752 & 0.039-0.062 \\ 0.004-0.020 & 0.037-0.060 & 0.9980-0.9992 \end{bmatrix}. \tag{23}$$

First we solve for β_2 using the magnitudes of the first-row elements of the CKM matrix. We then solve for β_1 and x using the magnitudes of the first two rows, and our expressions for ρ^2 and J. We find that the ranges allowed by the data are

$$\beta_1=0.11-0.78,\quad \beta_2=0.013-0.046,\quad x=-0.9-1.3\,. \tag{24}$$

For these magnitudes, Eq. (14) becomes $|x|\le 2.9462-2.9505$.

As a second example, we choose the case where $\mathrm{tr}V=e^{i(\Gamma+\phi/3)}$ (a phase). Then the solutions of Eq. (7) are

$$\lambda_1=-ie^{i\phi/3}e^{-i\Gamma/2},\quad \lambda_2=ie^{i\phi/3}e^{-i\Gamma/2},\quad \lambda_3=e^{i\phi/3}e^{i\Gamma}\,. \tag{25}$$

As before, the factor $e^{i\phi/3}$ will vanish in the element magnitudes and in J. We are again interested in the case where $\beta_3=0$ and α drops out. Then the magnitudes of the CKM matrix elements are

$$\begin{aligned} &|V_{11}|=|\cos(2\beta_1)|,\quad |V_{12}|=|V_{21}|=\sin(2\beta_1)\cos(\beta_2),\quad |V_{13}|=|V_{31}|=\sin(2\beta_1)\sin(\beta_2)\,,\\ &|V_{22}|=\sqrt{1-\tfrac{1}{2}\sin^2(2\beta_2)-\sin^2(2\beta_1)\cos^4(\beta_2)+\tfrac{1}{2}\cos(2\beta_1)\sin^2(2\beta_2)\sin(\tfrac{3}{2}\Gamma)}\,,\\ &|V_{23}|=|V_{32}|=\tfrac{1}{2}\sin(2\beta_2)\sqrt{2-\sin^2(2\beta_1)-2\cos(2\beta_1)\sin(\tfrac{3}{2}\Gamma)}\,,\\ &|V_{33}|=\sqrt{1-\tfrac{1}{2}\sin^2(2\beta_2)-\sin^2(2\beta_1)\sin^4(\beta_2)+\tfrac{1}{2}\cos(2\beta_1)\sin^2(2\beta_2)\sin(\tfrac{3}{2}\Gamma)}\,, \end{aligned} \tag{26}$$

and $J=\mathrm{Im}(V_{11}V_{22}V_{12}^*V_{21}^*)$ is

$$J=\tfrac{1}{4}\cos(2\beta_1)\sin^2(2\beta_1)\sin^2(2\beta_2)\cos(\tfrac{3}{2}\Gamma)\,. \tag{27}$$

As above, if we ignore experimental constraints, the maximum range of J that we can get by varying β_1, β_2, and Γ is $-(\sqrt{3}/18)\le J\le(\sqrt{3}/18)$.

The old parameters from Eq. (1) can be expressed in terms of the new parameters, as described in the above section. For this special case, the old parameters are given in terms of the new parameters by

$$\begin{aligned} &\theta_1=\begin{cases} 2\beta_1, & 0\le\beta_1\le\dfrac{\pi}{4}\,,\\ \pi-2\beta_1, & \dfrac{\pi}{4}\le\beta_1\le\dfrac{\pi}{2}\,,\end{cases}\\ &\theta_2=\beta_2,\quad \theta_3=\beta_2\,,\\ &\delta=\begin{cases} \dfrac{3}{2}\Gamma+\dfrac{\pi}{2}, & 0\le\beta_1<\dfrac{\pi}{4}\,,\\ \dfrac{3}{2}\Gamma-\dfrac{\pi}{2}, & \dfrac{\pi}{4}<\beta_1\le\dfrac{\pi}{2}\,.\end{cases} \end{aligned} \tag{28}$$

Note that again, $\theta_2=\theta_3$ because the CKM matrix is symmetric for $\beta_3=0$, and the amount of *CP* violation can be varied over a range by choosing Γ.

Note that, by the argument at the end of Sec. II, each of the parametrizations (20) and (26) are equivalent up to transformations of the form (3).

IV. DISCUSSION

The proposed reparametrization (10) of the CKM matrix provides an interesting arena for testing new possible flavor symmetries.

In this work above, we have chosen to consider $\det V$ unconstrained and $\mathrm{tr}\,V=xe^{i(\Gamma+\phi/3)}$ to depend on the parameters x and Γ. Although the CKM matrix elements depend on ϕ, in the observable results it is only present within Γ and so can be ignored without loss of generality. Our special cases represent only two simplifications of the general eigenvalues (12).

Also in the above, we have followed Ref. 3 in setting $\beta_3=0$, for which the CKM matrix is symmetric. In this case α drops out of the matrix elements, and hence from their magnitudes and J as well. The same effect can be achieved by setting any one of the β_i to be 0 or $(\pi/2)$, by setting $\alpha=0$, or by setting any two of the eigenvalues equal to one another. However, setting $\beta_1=0$ puts some of the elements (and hence their magnitudes, and J) to zero. Symmetry of the CKM matrix and the matrix of magnitudes of its elements is related to the presence of α; in a symmetric CKM matrix α either vanishes or may be removed by a transformation of the form (3). Although setting $\beta_1=0$ would not be a good choice, the other possibilities could also have been used in our special cases.

We note that a CKM matrix with symmetric magnitudes is consistent with all present-day experimental data.[4] Arbitrarily choosing $\det V$ and $\mathrm{tr}\,V$ to have some definite values (as in Ref. 3) will yield a precise determination of β_1 and β_2, leading to an accurate prediction of ρ and J. Such predictions, however, are contingent upon this choice. We have shown that the present bounds on the various CKM angles in the new parametrization are in general rather poorly determined, even when β_3 is set to zero.

Note added. After completion of this work we received a paper by Branco and Parada[7] which reaches similar conclusions to ours concerning Kielanowski's parametrization and which also finds that ρ^2 and the *CP*-violating parameter are constrained for a symmetric CKM matrix.

ACKNOWLEDGMENTS

This work was supported by the Natural Sciences and Engineering Research Council of Canada.

[1] L.-L. Chau, Phys. Rep. **95**, 2 (1983).
[2] C. Jarlskog, Phys. Rev. D **36**, 2128 (1987); M. Gronau, A. Kfir, and R. Loewy, Phys. Rev. Lett. **56**, 1538 (1986); J. F. Nieves and P. B. Pal, Phys. Rev. D **36**, 315 (1987).
[3] P. Kielanowski, Phys. Rev. Lett. **63**, 2189 (1989).
[4] Particle Data Group, G. P. Yost *et al.*, Phys. Lett. B **204**, 1 (1988).
[5] R. Fulton *et al.*, Phys. Rev. Lett. **64**, 16 (1990).
[6] J. R. Cudell, F. Halzen, and S. Pakvasa, Phys. Rev. D **40**, 1562 (1989).
[7] G. C. Branco and P. Parada, CERN Report No. TH-5679/90 (unpublished).