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Equivalence between the covariant, Weyl, and Coulomb gauges in the functional Schrödinger picture

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Equivalence between the covariant, Weyl, and Coulomb gauges of scalar electrodynamics is shown by using the functional Schrödinger-picture formulation.

I. INTRODUCTION

The functional Schrödinger-picture formulation of quantum field theories has been found convenient for nonperturbative variational studies of various quantum field theories.[1] It has recently been found that the equivalence between the Weyl (temporal), Coulomb, and unitary gauges of the Abelian Higgs model can be shown almost trivially in the functional Schrödinger-picture formulation.[2]

In this Brief Report we will extend the method explored in Ref. 2 to show the equivalence between the covariant, Coulomb, and Weyl gauges of the Abelian Higgs model. In Sec. II the Schrödinger-picture formulation of scalar quantum electrodynamics in the covariant gauge is presented, and in Sec. III the equivalence between the covariant, Weyl, and Coulomb gauges is shown. In the last section we discuss some related problems.

II. THE ABELIAN HIGGS MODEL IN THE COVARIANT GAUGE

The Abelian Higgs model in the covariant gauge is described by the Lagrangian density

$$L=-\tfrac{1}{4}F^{\mu\nu}F_{\mu\nu}+\tfrac{1}{2}(D_{\mu}^{ab}\phi_b)(D_{ac}^{\mu}\phi_c)-V(\phi)-\frac{1}{2\alpha}(\partial_\mu A^\mu)^2-i(\partial_\mu\bar{\eta})(\partial^\mu\eta)\ , \tag{2.1}$$

where α is a gauge parameter, ϕ_a are real scalar fields, and η and $\bar{\eta}$ are real ghost fields. The potential V is a function of $(\phi_{\hat{1}}^2+\phi_{\hat{2}}^2)$, and the covariant derivative D_μ^{ab} is defined as

$$D_\mu^{ab}=\delta^{ab}\partial_\mu-e\epsilon^{ab}A_\mu,\quad a,b=\hat{1},\hat{2}\ , \tag{2.2}$$

where ϵ^{ab} is the usual antisymmetric real matrix. The corresponding Hamiltonian density is given by

$$\begin{aligned}\mathcal{H}=&\tfrac{1}{2}(\pi_k\pi_k+B_kB_k)+\tfrac{1}{2}P_aP_a+\tfrac{1}{2}(D_k^{ab}\phi_b)(D_k^{ac}\phi_c)\\&+V(\phi)-A^0(\partial_k\pi_k+e\epsilon^{ab}\phi_aP_b)-\pi^0\partial_kA_k\\&-\frac{\alpha}{2}(\pi_0)^2-i\pi_\eta\pi_{\bar{\eta}}-i(\partial_k\bar{\eta})(\partial_k\eta)\ ,\end{aligned} \tag{2.3}$$

where the conjugate momenta are defined as

$$\begin{aligned}&\pi_k=\partial_kA^0+\partial_0A_k,\quad \pi_0=-\frac{1}{\alpha}\partial_\mu A^\mu\ ,\\&P_a=\partial_0\phi_a-eA_0\epsilon_{ab}\phi_b\ ,\\&\pi_\eta=i\partial_0\bar{\eta},\quad \pi_{\bar{\eta}}=-i\partial_0\eta\ .\end{aligned} \tag{2.4}$$

We denote the spatial indices 1, 2, 3 by i, j, k and the scalar field components $\hat{1}$ and $\hat{2}$ by a, b, c.

In the Schrödinger picture, the wave functional of the physical system satisfies the Schrödinger equation

$$i\frac{d}{dt}\Psi(\phi_a,A^\mu,\eta,\bar{\eta};t)=H\Psi(\phi_a,A^\mu,\eta,\bar{\eta};t)\ , \tag{2.5}$$

where the Hamiltonian operator is given by

$$H=\int d^3x\,\mathcal{H}$$

and the conjugate momenta in the Hamiltonian are represented by functional derivatives with respect to the corresponding field variables in such a way that they are consistent with the equal-time (anti)commutation relations.[3] Since the Hamiltonian of the system is invariant under Becchi-Rouet-Stora (BRS) transformation, however, one must require that the physical states be invariant under the BRS transformation[4]

$$Q_B\Psi(\phi_a,A^\mu,\eta,\bar{\eta};t)=0\ , \tag{2.6}$$

where the BRS charge operator Q_B is given by

$$Q_B=\int d^3x(\eta\partial_k\pi_k+e\eta\epsilon^{ab}\phi_aP_b+i\pi_{\bar{\eta}}\pi^0)\ . \tag{2.7}$$

In order to find the physical wave functionals that satisfy the Schrödinger equation (2.5) and the constraint equation (2.6), it is convenient to use the polar coordinates for the scalar fields:

$$\phi_{\hat{1}}=\rho\cos\theta\ ,\qquad \phi_{\hat{2}}=\rho\sin\theta\ . \tag{2.8}$$

Then the BRS condition (2.6) becomes

$$i\int d^3x\left[\eta(\mathbf{x})\left[\partial_i\frac{\delta}{\delta A_i(\mathbf{x})}-e\frac{\delta}{\delta\theta(\mathbf{x})}\right]-\frac{\delta}{\delta\bar{\eta}(\mathbf{x})}\frac{\delta}{\delta A^0(\mathbf{x})}\right]\Psi=0\ . \tag{2.9}$$

By factorizing the ghost part of the wave functional as

$$\Psi(\phi_a;A^\mu,\eta,\bar{\eta};t)=\exp\left[\int d^3x\,d^3y\,\bar{\eta}(\mathbf{x})D(\mathbf{x}-\mathbf{y},t)\eta(\mathbf{y})\right]\Phi(\phi_a,A^\mu;t)\ , \tag{2.10}$$

the BRS condition can be expressed in terms of only the matter and gauge field variables:

$$\left[\partial_i\frac{\delta}{\delta A_i(\mathbf{x})}-e\frac{\delta}{\delta\theta(\mathbf{x})}-\int d^3y\,\tilde{D}(\mathbf{x}-\mathbf{y},t)\frac{\delta}{\delta A^0(\mathbf{y})}\right]\Phi(\phi_a,A^\mu;t)=0\ , \tag{2.11}$$

where $\tilde{D}(\mathbf{x}-\mathbf{y},t)=D(\mathbf{y}-\mathbf{x},t)$.

We can further simplify the constraint equation by splitting the vector potential to the transverse part $\mathbf{A}^T$ and the longitudinal part $\mathbf{A}^L$ as

$$\begin{aligned}A_i&=A_i^T+A_i^L\\&=\left[\delta_{ij}-\frac{\partial_i\partial_j}{\nabla^2}\right]A_j+\frac{\partial_i\partial_j}{\nabla^2}A_j\ .\end{aligned} \tag{2.12}$$

Then the wave functional Φ is expressed as a functional of gauge-independent variables (ρ, A^T) and gauge-dependent variables (θ, A^L, A^0), and the BRS condition (2.11) is expressed in terms of only the gauge-dependent variables. Equation (2.11) implies that there are only two independent variables among the three gauge-dependent ones. This fact can be seen more clearly by introducing the following transformation of variables:

$$\begin{aligned}r(\mathbf{x})&=\nabla\cdot\mathbf{A}(\mathbf{x})-\tilde{D}^{-1}(\mathbf{x}-\mathbf{x}',t)\nabla^2A^0(\mathbf{x}')\ ,\\s(\mathbf{x})&=\nabla\cdot\mathbf{A}(\mathbf{x})-\frac{1}{e}\nabla^2\theta(\mathbf{x})\ ,\\u(\mathbf{x})&=a\nabla\cdot\mathbf{A}(\mathbf{x})-\frac{b}{e}\nabla^2\theta(\mathbf{x})-c\tilde{D}^{-1}(\mathbf{x}-\mathbf{x}',t)\nabla^2A^0(\mathbf{x}')\ ,\end{aligned} \tag{2.13}$$

where a, b, and c are arbitrary constants with $a-b-c\neq0$, and the integration convention is used for the repeated arguments. Using the new variables r, s, and u, the BRS condition becomes

$$\frac{\delta}{\delta u(\mathbf{x})}\Phi(\phi_a,A^\mu;t)=0\ , \tag{2.14}$$

which implies that the wave functional is independent of the variable $u(\mathbf{x})$.

The problem of obtaining physical information from the covariant gauge Higgs model in the functional Schrödinger picture is now expressed as the problem of solving the functional Schrödinger equation (2.5) such that the wave functional satisfies the constraint equation (2.14). There exist many different ways of realizing this problem depending on the parameters a, b, and c in Eq. (2.13). As in the case of the Weyl gauge,[2] the different choice of the parameter corresponds to the different gauge-fixing conditions. We will show two interesting examples (the Weyl and Coulomb gauges) explicitly.

III. EQUIVALENCE BETWEEN THE COVARIANT, WEYL, AND COULOMB GAUGES

In the Weyl gauge it is well known that the gauge-fixing condition

$$A^0=0 \tag{3.1}$$

fixes only a part of the gauge degrees of freedom. The remaining spatial gauge degree of freedom is eliminated by the Gauss-law constraint

$$\partial_iE_i=ej^0\ , \tag{3.2}$$

which is realized as a constraint on the wave functional

$$\left[\partial_i\frac{\delta}{\delta A_i(\mathbf{x})}-e\frac{\delta}{\delta\theta(\mathbf{x})}\right]\Psi_w(\phi_a,A^i;t)=0 \tag{3.3}$$

in the functional Schrödinger picture.[2]

It is easy to show that, when $(a,b,c)=(0,0,1)$ in Eq. (2.13), the BRS condition (2.14) implies that the wave functional is independent of $A^0(\mathbf{x})$. Since the matrix elements of any quantum operators in the Schrödinger picture are defined by

$$\langle\Psi_1|O(\phi_a,A^\mu,\eta,\bar{\eta})|\Psi_2\rangle=\int D\phi_aDA^\mu D\eta\,D\bar{\eta}\,\delta(u)\Psi_1^*O\Psi_2\ , \tag{3.4}$$

the BRS condition (2.14) with the choice of $(a,b,c)=(0,0,1)$ is equivalent to the Weyl gauge condition (3.1). In Eq. (3.4) Ψ^* represents the dual state of Ψ. The Gauss-law constraint (3.3) can be obtained from one of the field equations in the covariant gauge:

$$\partial_i E_i = ej^0 = \partial_0 \pi^0 , \qquad (3.5)$$

which is expressed in the Heisenberg picture. In the Schrödinger picture the matrix elements of Eq. (3.5) are given by

$$\langle \Psi_1 | \partial_i E_i - ej^0 | \Psi_2 \rangle = -i\partial_0 \left\langle \Psi_1 \left| \frac{\delta}{\delta A^0(\mathbf{x})} \right| \Psi_2 \right\rangle , \qquad (3.6)$$

where Ψ_1 and Ψ_2 are the covariant gauge wave functionals that satisfy the constraint (2.14). The matrix elements of π^0 in the covariant gauge can be written as

$$\left\langle \Psi_1 \left| \frac{\delta}{\delta A^0} \right| \Psi_2 \right\rangle = \int D\phi_a DA^\mu D\eta D\bar{\eta}\, \delta(u) \Psi_1^* \left[\frac{\delta s}{\delta A^0} \frac{\delta}{\delta s} + \frac{\delta r}{\delta A^0} \frac{\delta}{\delta r} \right] \Psi_2$$

$$= \int D\phi_a DA^\mu D\eta D\bar{\eta}\, \delta(u) \Psi_1^* \frac{1}{a-b-c} \tilde{D}^{-1} \nabla^2 \left[c \frac{\delta}{\delta \partial_i A_i} + \frac{ce}{\nabla^2} \frac{\delta}{\delta \theta} + \frac{a-b}{\nabla^2} \tilde{D} \frac{\delta}{\delta A^0} \right] \Psi_2 . \qquad (3.7)$$

With the choice of $(a,b,c)=(0,0,1)$, therefore, one can easily show that the field equation (3.6) becomes the differential equation

$$\langle \Psi_1 | \partial_i E_i - ej^0 | \Psi_2 \rangle = -\partial_0 \tilde{D}^{-1} \langle \Psi_1 | \partial_i E_i - ej^0 | \Psi_2 \rangle , \qquad (3.8)$$

for the matrix elements of the generator of the spatial gauge transformations. The simplest solution of Eq. (3.8) is

$$\langle \Psi_1 | \partial_i E_i - ej^0 | \Psi_2 \rangle = 0 , \qquad (3.9)$$

which is just the Gauss-law constraint (3.3).

It can be easily shown that, when condition (3.9) is satisfied, the ghost field part of the Hamiltonian (2.3) decouples from the rest, and the Schrödinger equation (2.5) becomes that for the Weyl gauge. This implies that the covariant gauge formulation with the choice of $(a,b,c)=(0,0,1)$ becomes that of the Weyl gauge.

Similarly one can show the equivalence between the covariant and Coulomb gauges. With the choice of the parameters $(a,b,c)=(1,0,0)$ in Eq. (2.13), the constraint equation (2.14) becomes

$$\frac{\delta}{\delta \partial_i A_i} \Psi = 0 . \qquad (3.10)$$

This implies that the wave functionals are independent of the longitudinal part of A^μ. Together with the fact that the matrix elements of quantum operators are given by Eq. (3.4), Eq. (3.10) implies the Coulomb gauge condition

$$\nabla \cdot \mathbf{A} = 0 . \qquad (3.11)$$

Since the matrix elements of the operator $\delta/\delta\partial_i A_i$ in the covariant gauge are given by

$$\left\langle \Psi_1 \left| \frac{\delta}{\delta \partial_i A_i} \right| \Psi_2 \right\rangle = \int D\phi_a DA^\mu D\eta D\bar{\eta}\, \delta(u) \Psi_1^* \left[\frac{\delta s}{\delta \partial_i A_i} \frac{\delta}{\delta s} + \frac{\delta r}{\delta \partial_i A_i} \frac{\delta}{\delta r} \right] \Psi_2$$

$$= \int D\phi_a DA^\mu D\eta D\bar{\eta}\, \delta(u) \Psi_1^* \frac{1}{a-b-c} \left[(b+c) \frac{\delta}{\delta \partial_i A_i} + \frac{ae}{\nabla^2} \frac{\delta}{\delta \theta} + \frac{a}{\nabla^2} \tilde{D} \frac{\delta}{\delta A^0} \right] \Psi_2 , \qquad (3.12)$$

the matrix elements of $\partial_i E_i$ in Eq. (3.6) must be expressed as

$$\langle \Psi_1 | \partial_i E_i | \Psi_2 \rangle_{b=0=c} = i \left\langle \Psi_1 \left| e \frac{\delta}{\delta \theta} + \tilde{D} \frac{\delta}{\delta A^0} \right| \Psi_2 \right\rangle_{b=0=c} \qquad (3.13)$$

when $(a,b,c)=(1,0,0)$. Thus the field equation (3.6) becomes, in this case,

$$\tilde{D} \left\langle \Psi_1 \left| \frac{\delta}{\delta A^0} \right| \Psi_2 \right\rangle = -\partial_0 \left\langle \Psi_1 \left| \frac{\delta}{\delta A^0} \right| \Psi_2 \right\rangle . \qquad (3.14)$$

The simplest solution of the differential equation (3.14) is

$$\left\langle \Psi_1 \left| \frac{\delta}{\delta A^0} \right| \Psi_2 \right\rangle = 0 . \qquad (3.15)$$

Using the condition (3.15), one can easily show that the ghost part of the Hamiltonian (2.3) decouples from the rest, and thus the Schrödinger equation (2.5) becomes equivalent to the Coulomb gauge equation.[2] One can also show that the longitudinal part of the Hamiltonian (2.3), $\frac{1}{2}E_i^L E_i^L$, becomes the Coulomb interaction term because of relations (3.13) and (3.15).

IV. DISCUSSION

We have shown that the Schrödinger-picture formulation of covariant gauge scalar quantum electrodynamics can be realized in infinitely many ways parametrized by the parameters (a,b,c) in Eq. (2.13). Some special choices of these parameters correspond to the known gauge-fixing procedures, as shown in Sec. III. This procedure can easily be generalized to non-Abelian gauge theories. The freedom in the choice of the parameters (a,b,c) may help simplify some practical computations.

In showing the equivalence between the covariant, Weyl, and Coulomb gauges in Sec. III, we have chosen the simplest solutions for the differential equations (3.8) and (3.14). Other nonvanishing solutions of these equations correspond to adding gauge-dependent parts to the gauge-fixing constraint equations to make them Lorentz covariant. Thus the Schrödinger-picture formulation of gauge theories clearly shows the relations between different gauge-fixing conditions.

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