Sphalerons and axion dynamics in high-temperature QCD

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We consider the effects of sphaleronlike transitions in the high-temperature QCD plasma. Because of the chiral anomaly, these imply rapid chirality change, and strongly influence the dynamics of any axionic fields that may couple to the topological winding number of QCD. We derive a coupled set of equations describing the evolution of the axion field and chiral fermion density at high temperatures. The unsuppressed sphaleronlike transitions result in the damping of coherent axionic oscillations. The implications of this phenomenon for axion fields in early Universe cosmology is discussed.

I. INTRODUCTION

Non-Abelian gauge theories have a nontrivial vacuum structure.[1] In addition to the usual perturbative vacuum configuration $A_i=T^a A_i^a=0$, there are an infinite number of pure gauge configurations,

$$A_i=\frac{i}{g}\Omega\partial_i\Omega^{-1}, \quad \Omega\in G \tag{1}$$

labeled by an integer Chern-Simons number

$$N_{\rm CS}=\frac{g^2}{8\pi^2}\int d^3\mathbf{x}\,\epsilon_{ijk}\,{\rm tr}\left[A_i\partial_j A_k-\frac{2ig}{3}A_i A_j A_k\right], \tag{2}$$

all with zero energy.

In order to change from a vacuum configuration with one integer value of $N_{\rm CS}$ to that with another integer value, it is necessary to pass through a nonvacuum, i.e., *finite-energy* field configurations: Fig. 1. In a spontaneously broken gauge theory, such as the Weinberg-Salam model, it is possible to find the magnitude of the potential barrier between adjacent vacua by purely *classical* methods. This is because an energy scale is inserted into the theory at the tree level by means of the vacuum expectation value of the Higgs field. The gauge bosons acquire masses, and the barrier height in question is given by the energy of a certain static solution of the coupled

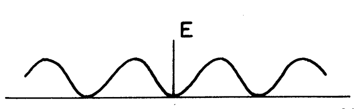

FIG. 1. The periodic vacuum structure of non-Abelian gauge theory in the absence of fermions.

Yang-Mills-Higgs classical field equations, called a sphaleron. In the Weinberg-Salam theory this energy barrier is of order M_W/α_W, or 10 TeV.[2]

Associated with the twisting of the gauge field from one vacuum state to another is the violation of chiral fermion number through the chiral anomaly and the Atiyah-Singer index theorem. For example, in a parity-violating theory such as the Weinberg-Salam model there is an anomaly in the lepton- and baryon-number currents;

$$\partial_\mu b^\mu=\partial_\mu l^\mu =\frac{n_f}{32\pi^2}[-2g^2{\rm tr}(F_{\mu\nu}\tilde{F}^{\mu\nu})+g'^2F'_{\mu\nu}\tilde{F}'^{\mu\nu}], \tag{3}$$

where $F_{\mu\nu}$ and $F'_{\mu\nu}$ are the field-strength tensors for the SU(2) and U(1) hypercharge gauge fields of the Weinberg-Salam theory, g and g' are the corresponding coupling constants, and n_f is the number of sequential generations (families) of quarks and leptons. Since $F_{\mu\nu}\tilde{F}^{\mu\nu}=\frac{1}{2}\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta}$ may be expressed as the total divergence of a four-vector whose time component is just that appearing in the definition of $N_{\rm CS}$, the nonconservation of B and L is related to the change in $N_{\rm CS}$ of the SU(2) gauge vacuum.

At zero temperature such transitions and concomitant B and L violations are very severely suppressed, since they require a quantum tunneling through the energy barrier between two degenerate vacua with different $N_{\rm CS}$. The semiclassical treatment of this tunneling problem by means of the Euclidean instanton solution indicates that the process is suppressed by a factor of $\exp(-4\pi\sin^2\theta_W/\alpha)\sim10^{-170}$, and hence is entirely negligible at zero temperature.[1]

At high temperatures ($T>M_W$), the situation is quite different. Because the energy barrier represented by the classical sphaleron solution is finite, the rate of classical real-time thermal transitions changing $N_{\rm CS}$ and therefore, the baryon and lepton number has no such exponential suppression in electroweak theory.[2–8] The rate of such B- and L-violating processes has been computed in the Weinberg-Salam theory by semiclassical methods for the temperature range $M_W(T) \ll T \ll M_W(T)/\alpha_W$.[3–5] At temperatures greater than $M_W(T)/\alpha_W$, the semiclassical analysis fails because perturbation theory around the zero-temperature ground state is unreliable. Equivalently, the breakdown of weak-coupling expansions for temperatures $T > M_W(T)/\alpha_W$ comes about because of infrared divergences in the effective three-dimensional gauge theory at high temperatures with massless gauge bosons.

The failure of the semiclassical approximation for the rate does not mean that the rate is small. On the contrary, it is possible to argue from general properties of scaling in the high-temperature phase that the rate of such transitions per unit volume is of order $\Gamma/V \sim \alpha_W^4 T^4$.[4,9] Study of two-dimensional models with features analogous to the (3+1)-dimensional electroweak theory leads to similar conclusions.[5,8,10]

A high rate of baryon- and lepton-number nonconservation at $T > M_W$ has a number of important cosmological implications. Any preexisting $(B+L)$ asymmetry would be eliminated by the time of the electroweak phase transition.[3,4,11] This excludes some grand unified models, such as the minimal SU(5) model for generating the observed baryon excess in the Universe. That the observed baryon excess might be *generated* at the electroweak phase transition is a tempting speculation, requiring further elaboration.[12]

The aim of the present paper is to extend the study of topological number change to finite-temperature QCD. Since there is neither symmetry breaking of the color group nor an energy scale in the bosonic sector at the tree level, QCD at any temperature is rather analogous to the electroweak theory for temperatures above the symmetry-restoration temperature. Like the electroweak theory above symmetry restoration, there is no finite-energy classical sphaleron solution in QCD and semiclassical weak-coupling methods are useless. However, drawing our lesson from the electroweak theory above the phase transition leads us to the conclusion that the transition rate per unit volume in $N_{\rm CS}$ for finite-temperature QCD must be at least of the order of $\alpha_s^4 T^4$, and quite unsuppressed.

In QCD the nontrivial vacuum structure leads to the introduction of the θ parameter, a nonzero value of which implies strong CP violation. To solve this strong CP problem a light pseudoscalar particle, the axion, was proposed.[13] Since the standard analysis of the generation of mass for the axion field relies on instanton tunneling transitions between neighboring vacuum states, it is interesting to ask whether the same analysis continues to be valid in light of the newer understanding of topology-changing transitions in non-Abelian gauge theories at finite temperature. In this paper we shall show that the original semiclassical analysis of the axion mass remains valid even at high temperatures, and in the process discover a new effect that has been overlooked in previous work, viz., the damping of coherent axionic oscillations in the thermal QCD plasma. This could be important for the dynamics of the axion field in the early Universe. The axion damping constant is not given by instanton methods, but is more closely related to the analogs of sphalerons in thermal QCD, in that unsuppressed, real-time processes are involved. The formalism we present is very general and should prove useful for discussing other processes as well, such as kinetic equilibration of fireballs in heavy-ion-collision experiments presently being planned.

II. SPHALERONLIKE CONFIGURATIONS IN QCD

Consider a one-parameter family of gauge fields beginning with

$$A_i = A_0 = 0 \ , \tag{4}$$

and ending with

$$A_0 = 0, \quad A_i = \frac{i}{g}\Omega\partial_i\Omega^{-1} \ , \tag{5}$$

where the gauge function Ω carries unit topological charge $N_{\rm CS}=1$. Because the Lagrangian of pure QCD is conformally invariant at the classical level, the maximal static classical energy of this family of configurations can be made arbitrarily small. A particular example of such a noncontractible loop in configuration space is the Euclidean instanton configuration with an arbitrary scale size λ, where Euclidean time τ may be regarded as the parameter along the loop:

$$A_\mu^a = \frac{1}{g}\frac{\eta^a_{\mu\nu}x^\nu}{\mathbf{x}^2+\tau^2+\lambda^2} \ . \tag{6}$$

If transformed to the $A_0=0$ gauge, this configuration will satisfy the boundary conditions of Eqs. (4) and (5). The maximal energy along the path of Eq. (6) is given by

$$E_{\rm max} = \frac{3\pi^2}{g_s^2}\frac{1}{\lambda} \ . \tag{7}$$

It corresponds to the configuration of Eq. (6) at $\tau=0$. It is clear from (7) that for infinitely large λ the barrier is absent.

Notice that this scaling argument relies on the scale invariance of the classical Lagrangian far into the infrared, precisely where the running coupling becomes strong and the quantum (scale-noninvariant) structure of the physical QCD vacuum becomes important. At finite temperatures the infrared behavior of correlation functions in the QCD plasma remains nonperturbative in character. Although the Debye screening length for electric correlation functions appears in one-loop perturbation theory, the magnetic screening length, presumed to be of order $(\alpha_s T)^{-1}$ cannot be calculated perturbatively.[14]

If we fix the scale size λ at a typical QCD length scale (such as $1/\alpha_s T$) by the method of constrained instantons,[15] then the field configuration (6) at $\tau=0$ in $A_0=0$ gauge has many properties in common with that of the sphaleron in electroweak theory. Like the classical sphaleron solution this configuration has $N_{CS}=\frac{1}{2}$, one negative mode, and lies atop of the potential energy barrier separating different gauge sectors. Moreover, there is a normalizable zero mode of Dirac operator in this background.[16] Hence, we can estimate the rate of topological transitions in QCD at high temperatures treating this configuration as "sphaleronlike." The calculation is essentially the same as in Ref. 4 where the estimate of the rate for $T>M_W(T)/\alpha_W$ was performed. The result of this estimate in the present case is

$$\Gamma_{\rm sph}=\left[\frac{\alpha_s}{4\pi}\right]^4 \mathcal{N}\mathcal{V}T^4\kappa\int\frac{d\lambda}{\lambda}e^{-3\pi/4\alpha_s\lambda T}\left[\frac{1}{\alpha_s\lambda T}\right]^{15/2} \tag{8}$$

$$\propto \alpha_s^4 T^4 , \tag{9}$$

where $\mathcal{N}\mathcal{V}$ is the normalization and volume factor for the translational and rotational zero modes. There is some indication from Ref. 4 that the proportionality constant in (9) is a number of order several thousand, because of the large factors involved in both the zero mode and fluctuating mode contributions. If this is the case, the rate of barrier hopping could well be as large or even larger than the rates of conventional perturbative processes in the hot QCD plasma, but we shall make no assumption of the size of this numerical constant in what follows.

Some progress has been made in understanding the nonperturbative dynamics of sphalerons using an effective Lagrangian which describes many of the features of confining gauge theories.[17] In this paper we shall not need any details of such sphaleronlike configurations in QCD, nor make use of any semiclassical expansion. We assume only that a crude estimate for the transition rate over the potential energy barrier separating vacua of different N_{CS}, such as (9) is valid.

III. AXION DYNAMICS IN THE HOT PLASMA

In this section we consider the response of a thermal system to the presence of an axion field which is varying in space and time on a scale large compared to $(\alpha_s^2 T)^{-1}$, a typical mean free path for particle scattering. The action is

$$S=S_0-\tfrac{1}{2}\int d^4x(\partial^\mu a)+\frac{1}{f}\int d^4x\, aq(x) , \tag{10}$$

where

$$q(x)\equiv\frac{g_s^2}{64\pi^2}\epsilon^{\mu\nu\alpha\beta}F^a_{\mu\nu}F^a_{\alpha\beta}=\frac{g_s^2}{16\pi^2}\text{tr}(F_{\mu\nu}\tilde{F}^{\mu\nu})$$

$$=\frac{\alpha_s}{2\pi}\mathbf{E}^a\cdot\mathbf{B}^a \tag{11}$$

is the density of Pontryagin number and S_0 is the QCD action in the absence of axions. The metric here is Lorentzian $(-+++)$ and the action S real. The axion field $a=f\Theta$, where Θ is a field periodic under $\Theta\to\Theta+2\pi$.

The Pontryagin density may be written as the total divergence of a topological current:

$$q(x)=\partial_\mu K^\mu , \tag{12}$$

where

$$K^\mu=\frac{g^2}{8\pi^2}\epsilon^{\mu\nu\alpha\beta}\text{tr}\left[A_\nu\partial_\alpha A_\beta-\frac{2ig}{3}A_\nu A_\alpha A_\beta\right] , \tag{13}$$

the time component of which is related to the Chern-Simons number:

$$N_{CS}=-\int d^3\mathbf{x}\, K^0 . \tag{14}$$

The minus sign enters here because of our convention that $\epsilon^{0ijk}=-\epsilon_{0ijk}=-\epsilon_{ijk}$.

The equation of motion for the thermal average of the axion field follows from the action (10):

$$\partial_\mu\partial^\mu\langle a(x)\rangle+\frac{1}{f}\langle q(x)\rangle=0 . \tag{15}$$

Here $\langle\mathcal{O}\rangle$ denotes the thermal average of the operator $\mathcal{O}$ in the presence of the axion background field:

$$\langle\mathcal{O}\rangle\equiv\frac{\text{Tr}\rho\mathcal{O}}{\text{Tr}\rho}\equiv\frac{1}{Z}\text{Tr}\rho\mathcal{O} , \tag{16}$$

where ρ is the density matrix, obeying

$$\frac{d\rho}{dt}=-i[H,\rho] \tag{17}$$

and H is the Hamiltonian corresponding to the action S. This is given by

$$H=\int d^3\mathbf{x}\left[\text{tr}\left[\mathbf{\Pi}+\frac{g^2}{8\pi^2 f}a\mathbf{B}\right]^2+\text{tr}(\mathbf{B})^2+\frac{1}{2}\Pi_a^2\right]$$

$$+H_{\text{fermion}} , \tag{18}$$

where

$$\mathbf{\Pi}=-\mathbf{E}-\frac{g^2}{8\pi^2 f}a\mathbf{B}=-\mathbf{E}-\alpha_s\frac{\Theta}{2\pi}\mathbf{B} \tag{19}$$

is the momentum conjugate to the gauge field $\mathbf{A}$ and Π_a is the momentum conjugate to the axion field. In the remainder of this section we shall ignore the fermionic part of the action, and discuss the influence of fermions on the axion dynamics in the next section.

Let us impose the initial conditions

$$a(t,\mathbf{x})\to 0,\quad t\to-\infty ,$$
$$\rho(t)\to\rho_0=\exp(-H_0/T), \tag{20}$$

and calculate the statistical averages in Eq. (15) to linear order in a. We find

$$\langle q(x)\rangle=\langle q(x)\rangle_0-\frac{i\alpha_s}{\pi f}\frac{1}{Z_0}\int_{-\infty}^{t}dt'\int d^3\mathbf{x}'\text{Tr}\{[\text{tr}(\mathbf{\Pi}\cdot\mathbf{B})a(t',\mathbf{x}'),\rho_0]q(x)\}+\mathcal{O}(a^2) , \tag{21}$$

where the zero subscript denotes the average with respect to ρ_0. Using Eqs. (11) and (19) we have

$$\langle q\rangle_0=-\frac{\alpha_s}{\pi}\langle \mathrm{tr}\mathbf{\Pi}\cdot\mathbf{B}\rangle_0-\frac{1}{2}\left[\frac{\alpha_s}{\pi}\right]^2 a\langle \mathrm{tr}(\mathbf{B})^2\rangle_0 , \tag{22}$$

where a will denote the expectation value of the axion field in all of the following.

The first term of Eq. (22) vanishes by the CP invariance of ρ_0. Therefore the linear response equation for the axion field in the QCD plasma is

$$0=-\partial_\mu\partial^\mu a+\frac{1}{2}\frac{1}{f^2}\left[\frac{\alpha_s}{\pi}\right]^2 a\langle \mathrm{tr}(\mathbf{B})^2\rangle_0-i\frac{1}{f^2}\left[\frac{\alpha_s}{\pi}\right]^2\int_{-\infty}^{t}dt'\int d^3\mathbf{x}'\langle[\mathrm{tr}(\mathbf{\Pi}\cdot\mathbf{B})(x),\mathrm{tr}(\mathbf{\Pi}\cdot\mathbf{B})(x')]\rangle_0 a(x') . \tag{23}$$

All averages are now with respect to the QCD thermal ensemble $\rho_0=\exp(-H_0/T)$ in which the axion field has been set to zero.

Let us introduce the retarded response function

$$\begin{aligned}G_R(t-t',\mathbf{x}-\mathbf{x}')&\equiv -i\left[\frac{\alpha_s}{\pi}\right]^2\theta(t-t')\langle[tr(\mathbf{\Pi}\cdot\mathbf{B})(t,\mathbf{x}),\mathrm{tr}(\mathbf{\Pi}\cdot\mathbf{B})(t',\mathbf{x}')]\rangle_0\\ &=\int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}\int\frac{d^3\mathbf{k}}{(2\pi)^3}e^{i\omega(t-t')}e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}\tilde{G}_R(\omega,\mathbf{k})\end{aligned} \tag{24}$$

whose Fourier transform $\tilde{G}_R$ is analytic in the upper half complex ω plane:

$$\tilde{G}_R(\omega,\mathbf{k})=\int_{-\infty}^{+\infty}d\omega'\frac{\rho(\omega',\mathbf{k})}{\omega-\omega'+i0} . \tag{25}$$

The spectral density ρ (not to be confused with the density matrix of which we have no further use) is determined by the matrix elements of the topological charge density:

$$\rho(\omega,\mathbf{k})=\frac{(2\pi)^3}{Z_0}\sum_{n,m}\left|\langle n|q(0)|m\rangle\right|^2 e^{-E_n/T}(1-e^{-\omega/T})\delta(\omega-E_m+E_n)\delta^3(\mathbf{k}-\mathbf{p}_m+\mathbf{p}_n) , \tag{26}$$

where the state $|n\rangle$ is an eigenstate of the full Hamiltonian.

If we focus now on spatially homogeneous axion fields, $a=a(t)$ that are slowly varying with time, and assume that there are no infrared divergences in QCD at finite temperature, then $\tilde{G}_R$ is analytic at $\omega=0$ and may be expanded in a Taylor series there:

$$\tilde{G}_R(\omega,\mathbf{k})=\tilde{G}_R(0,\mathbf{k})+\omega\frac{d}{d\omega}\tilde{G}_R(0,\mathbf{k})+\dots . \tag{27}$$

Substituting (27) into (23) and (24) yields the linear response equation for the axion field:

$$\frac{d^2a(t)}{dt^2}+\gamma\frac{da(t)}{dt}+M_a^2a(t)=0 , \tag{28}$$

where higher derivatives of a have been neglected. The axion mass M_a is given by

$$M_a^2=\frac{1}{2}\frac{1}{f^2}\left[\frac{\alpha_s}{\pi}\right]^2\langle \mathrm{tr}(\mathbf{B})^2\rangle_0+\frac{1}{f^2}\tilde{G}_R(0,\mathbf{0}) , \tag{29}$$

while the friction term

$$\gamma=\frac{i}{f^2}\frac{d}{d\omega}\tilde{G}_R(\omega,\mathbf{0})|_{\omega=0} \tag{30}$$

gives rise to the damping of the coherent axionic field oscillations.

Using the fact that $\rho(\omega,\mathbf{k})$ is an *odd* function of ω when $\mathbf{k}=0$, and therefore vanishes at $\omega=0$, we may write these two quantities in the form

$$M_a^2=\frac{1}{2}\frac{1}{f^2}\left[\frac{\alpha_s}{\pi}\right]^2\langle \mathrm{tr}(\mathbf{B})^2\rangle_0-\frac{1}{f^2}\int_{-\infty}^{\infty}d\omega\left[\frac{\rho(\omega,\mathbf{0})}{\omega}\right] , \tag{31}$$

and

$$\begin{aligned}\gamma&=-\frac{i}{f^2}\int_{-\infty}^{\infty}\frac{d\omega}{\omega-i0}\left[\frac{\rho(\omega,\mathbf{0})}{\omega}\right]\\ &=\frac{\pi}{f^2}\left[\frac{\rho(\omega,\mathbf{0})}{\omega}\right]\Bigg|_{\omega=0}\\ &=\frac{\pi}{f^2}\frac{d\rho}{d\omega}\Big|_{\omega=\mathbf{k}=0} .\end{aligned} \tag{32}$$

Despite appearances M_a^2 is positive and equal to the standard expression in terms of a Euclidean path integral

$$M_a^2=\frac{1}{f^2}\int_0^\beta d\tau\int d^3\mathbf{x}\frac{1}{Z_0}\int[\mathcal{D}A_\mu]e^{-S_E}q_E(0)q_E(\tau,\mathbf{x}) , \tag{33}$$

where q_E is the Euclidean continuation of (11). The expression (33) for the axion mass is derived in the Appendix. If we are in the weak-coupling regime, such as is obtained at extremely high temperatures, then (33) may be treated by semiclassical instanton methods, with the usual result, viz.,

$$M_a^2\propto f^{-2}e^{-2\pi/\alpha_s} . \tag{34}$$

This demonstrates that the leading contribution to the axion mass at finite temperature arises from quantum tunneling effects, just as at zero temperature, and receives *no* contributions from the classically allowed real-time sphaleronlike transitions.

This important result is a direct consequence of the fact that the classically allowed thermal transitions we are considering have *no* effect on the θ dependence of the free energy of pure QCD. The reason is that the θ parameter multiplies a total derivative in the action and does not enter the classical equations. It may be rotated away entirely and neglected in the classical theory applicable in the high-temperature limit. Therefore the free energy cannot receive any θ dependence from classically allowed thermal transitions, no matter how rapidly these occur in the hot QCD plasma. Since M_a^2 is proportional to the second derivative of the free energy with respect to θ, it too receives no contributions from the sphaleronlike transitions taking place in the hot QCD plasma. The result of Eq. (33) has been found by effective potential methods using instantons.[18] Our results confirm this analysis, and show that Euclidean effective potential techniques are not invalidated by real time topological transitions.

In contrast with M_a^2, the damping constant γ *cannot* be expressed simply in terms of Euclidean path integrals. Accordingly, it may and does receive contributions from classically allowed real time transitions, and is *not* instanton suppressed. This will be plain if we evaluate the last expression for γ, Eq. (32), by substituting the definition of the spectral density function (26). Proceeding in this manner we obtain

$$\gamma=\frac{8\pi^4}{f^2T}Z_0^{-1}\sum_{n,m}|\langle n|q(0)|m\rangle|^2e^{-E_n/T}\delta^3(\mathbf{p}_m-\mathbf{p}_n)\delta(E_m-E_n)\ . \tag{35}$$

This we recognize as the spectral decomposition for

$$\gamma=\frac{8\pi^4}{f^2T}\int_{-\infty}^{\infty}dt\int d^3\mathbf{x}\langle q(t,\mathbf{x})q(0)\rangle_0 \tag{36}$$

in the high-temperature limit. Since the matrix elements $\langle n|q(0)|m\rangle$ are nonzero in ordinary perturbation theory, there is no instanton suppression of the damping rate in the high-temperature plasma. This is in contrast with the zero temperature limit, where the spectral function and all of its derivatives vanish at $\omega=0$ because of the existence of a mass gap in the theory.

The representation for γ in Eq. (36) shows the connection between the axion damping and the real-time fluctuations of the topological charge,

$$Q(t)=\int_0^t dt\int d^3\mathbf{x}\,q(t,\mathbf{x})\ . \tag{37}$$

Indeed, in the absence of fermions we expect

$$\begin{aligned}\lim_{t\to\infty}\langle Q^2(t)\rangle&=\lim_{t\to\infty}\langle[N_{\rm CS}(t)-N_{\rm CS}(0)]^2\rangle\\&=2Vt\Gamma_{\rm sph}\ ,\end{aligned} \tag{38}$$

where the three-dimensional volume is denoted by V. This last relation is that of a random walk in the one-dimensional coordinate of field space parametrized by $N_{\rm CS}$. On the other hand, if we evaluate the average in Eq. (38) directly in terms of the density function of Eq. (26) we find

$$\begin{aligned}\langle Q^2(t)\rangle&=2V\int_0^{\infty}d\omega\,\rho(\omega,0)\frac{\sin^2(\omega t)}{\omega^2}\\&\to 2\pi Vt\rho(\bar{\omega},\bar{0})\ ,\end{aligned} \tag{39}$$

as $t\to\infty$. In order to arrive at this last form we have removed $\rho(\omega)$ from the integral and placed it with its value at some typical $\bar{\omega}\sim T$, by assuming that it is slowly varying compared to $\sin^2(\omega t)$ in the limit that t is very large. Furthermore, for long-time scales only the value of this function near $\omega=0$ is important, so that we may use the Taylor expansion

$$\rho(\bar{\omega},\mathbf{0})=\bar{\omega}\frac{d\rho}{d\omega}\bigg|_{\omega=\mathbf{k}=0}+\ldots\approx T\frac{d\rho}{d\omega}\bigg|_{\omega=\mathbf{k}=0}\ . \tag{40}$$

Comparing Eq. (32) or (36) with (38)–(40) we find

$$\gamma=\Gamma_{\rm sph}/f^2T\propto\alpha_s^4T^3/f^2\ , \tag{41}$$

which is valid for all temperatures high enough for the classical thermal activation over the barrier in Fig. 1 to dominate over quantum tunneling (instanton) processes. In QCD we expect the temperature at which this first occurs to be when the inverse magnetic coherence length α_sT becomes of order $N_{\rm QCD}$, or at temperatures of a few GeV. This analysis, and in particular, Eq. (38) neglects the presence of dynamical fermions, a shortcoming we address in the next section.

IV. AXION DYNAMICS IN THE PRESENCE OF FERMIONS

Fermions play a crucial role in the topology-changing transitions of the hot QCD plasma because of the chiral anomaly. Since QCD has only parity-conserving vector couplings, there is no fermion-number violation induced by these transitions. Only violations of chiral fermion number are generated. Let $Q_{\rm ch}$ be the total chiral charge of all quarks with mass $m\ll T$. For massless quarks the chiral anomaly informs us that

$$\left\langle\frac{dQ_{\rm ch}}{dt}\right\rangle=2n_fV\langle q\rangle\ ,\quad \Delta Q_{\rm ch}=-2n_fV\Delta N_{\rm CS}\ . \tag{42}$$

It should be remarked at this point that the minima of Fig. 1 are forced to be degenerate, since all integer $N_{\rm CS}$ are equivalent to each other by a (topologically nontrivial) gauge transformation. Unlike the Chern-Simons number, $Q_{\rm ch}$ is *gauge invariant,* so that states of different chiral fermion number may (and do) have different energies. Thus we should not expect a periodic potential with strictly degenerate minima in terms of Q_{ch} when the fermionic contribution to the energy is included.

If we introduce a nonzero average value of chirality into the plasma by the device of adding a chemical potential to the Hamiltonian,

$$H \to H - \mu Q_{ch} \ , \tag{43}$$

then it becomes energetically favorable to create a net chiral fermion number in the plasma. Near $Q_{ch}=0$ we have a negative linear term in the energy superimposed on the periodic potential of Fig. 1, because of the chemical potential term in (43). This leads to a washboard potential, which implies that $Q_{ch}=0$ is not the ground state of the system. If we evaluate the average value of the chiral fermion number in the Fermi-Dirac distribution with the Hamiltonian modified by the replacement (43), we find, to linear order in μ,

$$\langle Q_{ch} \rangle = \mu \frac{VT^2}{3} n_f \ . \tag{44}$$

This implies that the decreasing washboard potential cannot continue indefinitely for arbitrarily large Q_{ch}. The reason that the potential must turn upward for large Q_{ch}, and therefore possess a minimum, consistent with Eq. (44) is Fermi-Dirac statistics. Even if the fermions are treated as massless, it costs energy to create a fermion-antifermion pair with net chirality, since the pair must be created in an unoccupied momentum state. Since the spacing between states (and hence this energy cost) goes to zero in the infinite-volume limit, the value of Q_{ch} at which the potential begins to turn upward is of order V, which is consistent with the volume factor in Eq. (44).

Thus the potential with fermions included looks qualitatively like that sketched in Fig. 2. Stable dynamic equilibrium is maintained by the larger population of states with $Q_{ch}>0$ diffusing to lower Q_{ch}, and compensating for the linear skewing of the Hamiltonian in (43). Hence there is detailed balance, and the net rate of change of chirality is zero in equilibrium, $\langle dQ_{ch}/dt \rangle = 0$, with the average value given by (44).

Suppose now that the external chemical potential μ is removed suddenly (at $t=0$). Then the minimum of Fig. 2 is shifted suddenly to $Q_{ch}=0$. Now the initial overpopulation of states with $Q_{ch}>0$ is not the equilibrium configuration, and there will be a net decrease of $\langle Q_{ch} \rangle$; i.e., the net chirality will relax to zero. We may calculate the rate of relaxation if we assume that (44) continues to hold for $t>0$ as well, effectively *defining* a $\mu(t)$ in terms of the decreasing chirality as a function of time. This is valid only if the relaxation is slow enough so that the system is close to equilibrium during the relaxation, with an *effective* time-dependent chemical potential. This adiabaticity assumption permits us to use detailed balance and equate $\langle dQ_{ch}/dt \rangle$ for $t>0$ to the *negative* of the transition rate to the right with the Hamiltonian of (43):

$$\langle q(x) \rangle = -\frac{1}{V}\frac{dN_{CS}}{dt} = -(\Gamma_+ - \Gamma_-) = -2n_f \mu \frac{\Gamma_{sph}}{T} \ , \tag{45}$$

to linear order in μ, since

$$\Gamma_\pm = \Gamma_{sph} \exp\left[\pm \frac{\mu n_f}{T}\right] . \tag{46}$$

Then we may eliminate μ from Eqs. (42), (44), and (45) to secure

$$\langle \dot{Q}_{ch} \rangle = -12 n_f \frac{\Gamma_{sph}}{T^3} \langle Q_{ch} \rangle \ , \tag{47}$$

which determines the decay time of the net fermion chirality in the thermal plasma.

In spite of the fact that the rate Γ_{sph} is proportional to α_s^4 while the rates of usual kinetic processes such as $qq \to gg$, etc., are of order $\Gamma_{kin} \sim \alpha_s^2$, Γ_{sph} could be bigger than Γ_{kin} if the numerical coefficient in (9) is sufficiently large. In that case chirality-changing reactions may be the fastest reactions in the plasma, and the adiabaticity assumption used in deriving (47) may not be reliable enough for quantitative analysis. Even in this case we would expect the rate of chirality-changing reactions to be roughly given by Eq. (47) on dimensional grounds. Whether such rapid chirality flipping may be responsible in whole or in part for the restoration of chiral symmetry in high-temperature QCD is an interesting issue in itself that merits a more careful study.

In deriving Eq. (47) we have ignored the axion field, as well as explicit chirality violation in the theory (such as fermion masses). If we compare Eq. (15), the equation of motion of the axion field, and the chiral anomaly equation (42), it is clear that it is inconsistent to put the chiral density to zero when the time derivative of the axion field is nonzero, $\dot{a} \neq 0$. Explicitly, rewrite the above equation for Q_{ch} in terms of the chiral chemical potential μ, keeping also the term in the energy proportional to $\dot{a}$:

$$\Delta E_{gauge} = -\frac{1}{f} \dot{a} \, \Delta N_{CS} \ . \tag{48}$$

Then the equation for μ becomes

$$\frac{d\mu}{dt} = -\frac{6\Gamma_{sph}}{T^3}\left[2\mu n_f + \frac{1}{f}\frac{da}{dt}\right] - \Gamma_{ch}\mu \ . \tag{49}$$

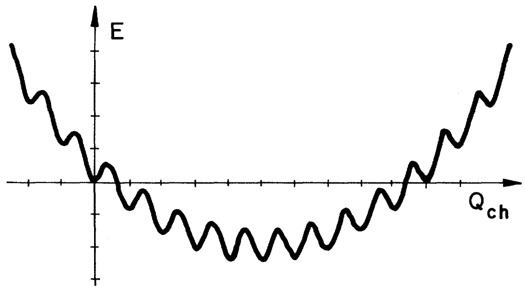

FIG. 2. The potential energy of a gauge field plus massless fermion system as a function of chiral charge. The potential is concave for large Q_{ch} in a finite volume, due to Fermi-Dirac statistics, as explained in the text.

The factor Γ_{sph} is the rate of sphaleron-induced transitions as before, Γ_{ch} is the rate of chirality-flipping transitions due to fermion masses and Higgs-boson exchanges, which were neglected previously. Γ_{ch} is determined by the cross sections of kinetic reactions such as $\bar{q}_L q_R \to gH$ where g is a gluon and H is a Higgs boson, as well as by the explicit violation due to a mass term.

The equation for the axion field in the presence of a nonzero chiral fermion density becomes

$$\ddot{a} + \gamma(\dot{a} + 2\mu f n_f) + M_a^2 a = 0 \ . \quad (50)$$

The second term which multiplies the coefficient γ above arises from Eqs. (41)–(47).

If the explicit chirality-breaking term can be neglected, we must have $2n_f\mu \to -(da/dt)1/f$. The chiral chemical potential therefore becomes proportional to the time derivative of the axion field. This may also be derived from the anomaly equation for chiral charge and the equation of motion of the axion field. In this case we arrive at an interesting situation. Suppose, that we started with some initial state with nonzero density of the chiral charge, corresponding to an initial μ_0 but zero $\dot{a}$. Then the large-time asymptotics of solution to Eqs. (49) and (50) is

$$\dot{a} \to -\frac{2n_f f_a \mu_0}{1+12 f_a^2 n_f / T^2} \ , \quad (51)$$

$$\mu \to -\dot{a}/2 f_a n_f \ , \quad (52)$$

provided that M_a is exponentially small. In other words, the axion field will rotate with constant speed around the point $a=0$.

In real QCD, $\Gamma_{ch} \neq 0$. This gives rise to a friction term for the axion equations of motion which is proportional to Γ_{ch} at asymptotically late times:

$$\gamma_{eff} = \gamma \left[\frac{\Gamma_{ch}}{2\gamma + 24\Gamma_{sph} n_f / T^3} \right] . \quad (53)$$

There is a complicated interplay between the axion field damping and the fermion chirality flipping in high-temperature QCD.

V. CONCLUSION

Sphaleronlike transitions at high-temperature QCD result in strong nonconservation of chirality. The rate of chirality-changing transitions could conceivably be competitive, or even larger than the rate of usual kinetic reactions. If this is the case, this effect can change the traditional treatment of thermalization processes in heavy-ion collisions. It may also be important for understanding the chiral phase transition of high-temperature QCD.

We have found that there is a damping of the axion oscillations in a hot plasma. In the expanding Universe, there is also a contribution due to the Hubble expansion. The sphaleron contribution, in order to be ignored, must satisfy

$$\gamma = \frac{\Gamma_{sph}}{f^2 T} \leq T^2 / M_{pl} \ . \quad (54)$$

If $f \sim 10^{12}$ GeV, as seems to be required for the invisible-axion hypothesis to be tenable at all,[19] we find that the sphaleron-induced damping of the axion field dominates over that due to the expansion of the Universe at all temperatures above a few hundred GeV. However, this extra damping coefficient does not affect the coherent axionic oscillations which do not begin until $H \sim M_a$, when the plasma has already cooled sufficiently for γ already to have turned off.[20]

It is clear that the sphaleron damping of axion oscillations is general in character. The results of this paper can easily be extended to any gauge theory with an axion-like field. One of the examples includes the pseudo-Goldstone-boson field of broken baryon number introduced in Ref. 21 for baryogenesis. If this field couples to the exactly conserved charge in electroweak theory $B-L$, then the effects discussed here can be ignored. However, if the field couples to the charge $\alpha B - \beta L$ with $\alpha \neq \beta$ then the results of Ref. 21 have to be reconsidered in light of the sphaleron-induced damping found in this paper.

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APPENDIX

In this appendix we derive Eq. (33), which proves that the axion mass can be represented simply in terms of a Euclidean path integral. We begin by considering the quantity

$$\partial_\mu \partial'_\nu \{ \theta(t-t')[K^\mu(x), K^\nu(x')] \} = \theta(t-t')[q(x), q(x')] + \delta(t-t')[K^0(x), q(x')] - \delta(t-t')[q(x), K^0(x')] + \frac{\partial}{\partial t'} \delta(t-t')[K^0(x), K^0(x')] \ . \quad (A1)$$

Since we eventually wish to integrate this expression with respect to $\mathbf{x}'$ and t', we may transfer the derivative on the δ function in the last term onto $K^0(x')$, by integrating by parts. Then the equal-time commutators may be evaluated by using Eq. (11) of the text, the expression for K^0,

$$K^0=-\frac{\alpha_s}{4\pi}\epsilon^{ijk}A_i^a\left[\partial_j A_k^a+\frac{g}{3}f^{abc}A_j^bA_\kappa^c\right], \quad (A2)$$

and the canonical commutation relation for the gauge fields,

$$[A_i^a(t,\mathbf{x}),\Pi_j^b(t,\mathbf{x}')]=i\delta^3(\mathbf{x}-\mathbf{x}')\delta^{ab}\delta_{ij} . \quad (A3)$$

The result of this calculation is

$$\partial_\mu\partial'_\nu\{\theta(t-t')[K^\mu(x),K^\nu(x')]\}$$
$$=\theta(t-t')[q(x),q(x')]$$
$$+i\delta(t-t')\left[\frac{a_s}{2\pi}\right]^2\delta^3(\mathbf{x}-\mathbf{x}')B_i^aB_i^a . \quad (A4)$$

This relationship between Wick ordering and Dyson ordering has been noted previously.[22] Integrating this result over $\mathbf{x}'$ and t' and comparing to Eq. (29) of the text yields

$$M_a^2=\frac{i}{f^2}\int_{-\infty}^{\infty}dt'\int d^3\mathbf{x}\,\partial_\mu\partial_\nu$$
$$\times\{\theta(t-t')\langle[K^\mu(x),K^\nu(x')]\rangle_0\} . \quad (A5)$$

We shall make use of this result momentarily.

Consider now a generic integral of the form

$$\int_{-\infty}^{\infty}dt'\theta(t-t')\langle[X(t),Y(t')]\rangle$$
$$=\int_{-\infty}^{t}dt'\langle X(t)Y(t')-Y(t')X(t)\rangle . \quad (A6)$$

By using the definition of the thermal average in terms of the density matrix equation (16) of the text, we may rewrite the second integral in the form

$$\int_{t+i\beta}^{-\infty+i\beta}dt'\langle X(t)Y(t')\rangle . \quad (A7)$$

Therefore the original integral, (A6) may be written as sum of contour integrals in the complex t' plane:

$$\int_{-\infty}^{\infty}dt'\theta(t-t')\langle[X(t),Y(t')]\rangle$$
$$=\int_{\mathcal{C}_1+\mathcal{C}_3}dt'\langle X(t)Y(t')\rangle$$
$$=-\int_{\mathcal{C}_2+\mathcal{C}_4}dt'\langle X(t)Y(t')\rangle , \quad (A8)$$

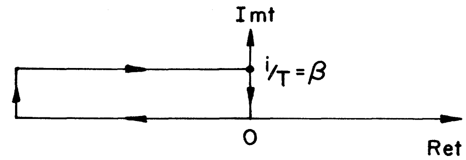

FIG. 3. The integration in the complex time plane for the time-ordered product of fields in Eq. (A9).

where the last equality follows provided the integrand is analytic in the interior of the rectangular region enclosed by the full contour $\mathcal{C}$, as illustrated in Fig. 3. If the integrand falls off at $t=-\infty$ the integral over the piece of the contour there, $\mathcal{C}_4$ may be neglected as well. Changing variables in the remaining integral yields

$$\int_{-\infty}^{\infty}dt'\theta(t-t')\langle[X(t),Y(t')]\rangle$$
$$=-i\int_0^\beta d\tau\langle X(t)Y(t+i\tau)\rangle$$
$$=-i\int_0^\beta d\tau\langle T_\tau\{X_E(0)Y_E(\tau)\}\rangle , \quad (A9)$$

where T_τ denotes Euclidean time ordering.

This time-ordered product is precisely what is obtained by means of Euclidean path integration. Applying this general result to the specific integral related to M_a^2 by Eq. (A5) gives

$$M_a^2=\frac{1}{f^2}\int_0^\beta d\tau\int d^3\mathbf{x}'\partial_\mu\partial_\nu\langle K^\mu(t,\mathbf{x})K^\nu(t+i\tau,\mathbf{x}')\rangle_0$$
$$=-\frac{1}{f^2}\int_0^\beta d\tau\int d^3\mathbf{x}'\partial_\mu\partial'_\nu\langle K^\mu(t,\mathbf{x})K^\nu(t',\mathbf{x}')\rangle_0|_{t'=t+i\tau}$$
$$=\frac{1}{f^2}\int_0^\beta d\tau\int d^3\mathbf{x}\frac{1}{Z_0}\int[\mathcal{D}A_\mu]e^{-S_E}q_E(0)q_E(\tau,\mathbf{x}) , \quad (A10)$$

since the Euclidean continuation of the Pontryagin density is pure imaginary, and the Euclidean path integral exactly reproduces the correct Euclidean time-ordered product given by the line above. This proves Eq. (33) of the text.

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