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Nonperturbative many-body techniques applied to the Yang-Mills Hamiltonian in the Schwinger gauge

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Starting with the Hamiltonian formulation of pure Yang-Mills theory in the Schwinger gauge, nonperturbative many-body techniques (Bogoliubov theory) are applied to calculate the gluon spectrum and the vacuum energy expectation value. The Schwinger gauge offers the possiblity of implementing a lattice analogous regularization in a natural way. The main results are the appearance of a mass gap in the interacting theory and a satisfactory agreement of the behavior of the vacuum energy expectation value (as a function of the coupling constant) with lattice calculations.

I. INTRODUCTION

Strong efforts have been undertaken in the past few years to obtain an understanding of the structure of Yang-Mills theories in the nonperturbative regime, where characteristic phenomena of non-Abelian gauge theories should occur. Because of the enormous numerical expense of the most successful lattice Monte Carlo method (a formulation based on evaluating a Feynman path integral) one is looking for an alternative framework to get better insight into the structure of bound states, eigenfunctions, etc. The treatment of the field-theoretical Hamiltonian with nonperturbative techniques seems to be the obvious way. The reliable results of the lattice calculations can be used for testing these other approaches. New results of van Baal[1] stimulate the hope that principally it should be possible to get a genuine alternative in this framework.

It is the purpose of this paper to investigate the eigenvalue problem of the Yang-Mills Hamiltonian (with complete gauge fixing) by applying many-body methods well known, e.g., from the description of nuclear matter. Concerning the choice of gauge, it has been shown in a recent paper[2] that the Schwinger gauge $\mathbf{x}\cdot\mathbf{A}=0$ seems to be distinguished in calculating bound states, etc. Compared to Ref. 1, this formulation also allows a replacement of the torus by a spherical cavity as a more natural domain of the definition of the theory.

The various complete gauge fixings—Coulomb, axial, etc., as they are applied to Refs. 3–7—yield different formulations, but should in principle agree with the results for the observables because of the local gauge invariance of the theory. However, the choice of a suitable regularization scheme is expected to be decisive in order to obtain a scaling behavior of the observables which is a necessary condition for consistency.[2,8] In the continuum perturbation theory it is the dimensional regularization[9] and in the nonperturbative sector it is the lattice formulation[10,11] that satisfy the above criterion (though only in a scaling window) and conserve the so-called Ward identities, which are a direct consequence of gauge invariance.

One advantage of the Schwinger gauge now is the possibility of introducing a lattice analogous regularization. Details of the general structure of this formulation are discussed in Ref. 2.

Within this paper the mathematical structure and a numerical application of the simplest nonperturbative many-body technique, the Bogoliubov theory, are presented. As a result we find the appearance of an energy gap in the single-particle (SP) spectrum of the non-Abelian theory, a structure which is expected to occur for the spectrum of the Yang-Mills theory (finite glueball mass), see, e.g., Refs. 5 and 12. We also present results for the vacuum energy expectation value which turn out to be in satisfactory agreement with lattice results. These calculations are considered as a first step for the computation of glueballs.

The paper is organized in the following way. After canonical quantization of the pure Yang-Mills theory we specify the Hamiltonian in the Schwinger gauge taking the results of Ref. 2 (Sec. II). To calculate the expectation values of H with respect to a BCS ansatz for the vacuum it is advantageous to implement a Bogoliubov transformation (Sec. III). This yields a characteristic set of random-phase-approximation (RPA) equations for this transformation from a variational principle equivalent to a diagonalization of the SP normal-ordered Hamiltonian (Sec. IV). A lattice analogous regularization is introduced in Sec. V. With the matrices consisting of the eigenvectors of the solved RPA problem, we compute the vacuum energy expectation value (Sec. VI). Some conclusions are given in Sec. VII.

II. CANONICAL QUANTIZATON

With standard notation the pure Yang-Mills Langrangian is

$$\mathcal{L}_{\mathrm{YM}}=-\tfrac{1}{4}F^{\mu\nu,a}(x)F_{\mu\nu,a}(x) \tag{2.1}$$

with

$$F^{\mu\nu,a}(x)=\partial^{\mu}A^{\nu,a}(x)-\partial^{\nu}A^{\mu,a}(x)+gf^{abc}A^{\mu,b}(x)A^{\nu,c}(x) \tag{2.2}$$

and

$$F^{\mu\nu}=-gF^{\mu\nu,a}T_a,\quad A^{\mu}=-gA^{\mu,a}T_a\ ,$$

$T_a\in$ Lie algebra of SU(n),

$$[T_a,T_b]=if^{abc}T_c,\quad f^{abc}f^{dbc}=n\delta^{ad}\ . \tag{2.3}$$

$\mathcal{L}_{\rm YM}$ is invariant under the local SU(n) transformations

$$g(x)=\exp[-i\Theta^a(x)T_a],\quad \Theta^a\in\mathbb{R}\ , \tag{2.4}$$

$$A'_{\mu}(x)=g(x)(A_{\mu}-i\partial_{\mu})g^{-1}(x)\ . \tag{2.5}$$

The procedure for quantization is to impose gauge fixing via constraints.[13,14] The most convenient strategy is to impose the temporal gauge $A_0=0$ leaving open time-independent gauge transformations and the Gauss law on the states (gauge invariance of the states). This Gauss law is conveniently satisfied via complete gauge fixing (for details see Ref. 2).

Here we consider the "Schwinger" condition

$$\mathbf{x}\cdot\mathbf{A}(\mathbf{x})=0 \tag{2.6}$$

which yields the Hamiltonian[2]

$$H_{\rm Schwinger}=\tfrac{1}{2}\int d^3x\ \Pi_j^a\Pi_j^a+\tfrac{1}{2}\int d^3x\ \rho_a^{\dagger}\Gamma\rho_a$$
$$+\tfrac{1}{2}\int d^3x\ B_j^aB_j^a \tag{2.7}$$

with the color-charge density

$$\rho=-\nabla\cdot\mathbf{\Pi}+i[\mathbf{\Pi},\mathbf{A}]\ ,$$

and

$$\Gamma=-\partial^{-1}r^{-2}\partial^{-1}r^2,\quad \partial^{-1}=(\partial/\partial_r)^{-1},\quad r=|\mathbf{x}|\ . \tag{2.8}$$

To satisfy the Schwinger condition (2.6) we expand the fields and their conjugate momenta into the product of linear combinations of vector spherical harmonics[15] and nonspecified radial functions (see Ref. 2):

$$\mathbf{A}(\mathbf{x})=\sum\frac{1}{\sqrt{2\omega}}(a_{\alpha}+s_{\alpha}a_{\bar{\alpha}}^{\dagger})\mathbf{f}_{\alpha}(r,\Omega)\ , \tag{2.9}$$

$$\mathbf{\Pi}(\mathbf{x})=-i\sum\frac{\sqrt{\omega}}{\sqrt{2}}(a_{\alpha}-s_{\alpha}a_{\bar{\alpha}}^{\dagger})\mathbf{f}_{\alpha}(r,\Omega) \tag{2.10}$$

with

$$\mathbf{f}_{\alpha}(r,\Omega)=r^{-1}g_{nl\lambda}(r)\mathbf{Y}^{\lambda lm}(\Omega)T_a,\quad s_{\alpha}=(-1)^{m+\lambda}$$
$$\mathbf{Y}^{1lm}=\mathbf{Y}^{(l,1)lm}\ ,$$
$$\mathbf{Y}^{2lm}=i[l(l+1)]^{-1/2}\mathbf{e}_r\times\mathbf{L}Y^{lm}\ , \tag{2.11}$$
$$\mathbf{e}_r=|\mathbf{x}|^{-1}\mathbf{x}\Longrightarrow\mathbf{x}\cdot\mathbf{Y}^{\lambda lm}=0$$

and

$$g_{nl\lambda}(r=0)=0 \tag{2.12}$$

with $\alpha=(n,l,m,\lambda,a)$, $n\in\mathbb{N}$; $l\geq 1$; $-l\leq m\leq l$; $\lambda=1,2$; $a=1,\ldots,n^2-1$; $\bar{\alpha}=(n,l,-m,\lambda,a)$, for more details see Ref. 2. (ω is a fixed length scale which can be put equal to one at a later stage.)

The freedom of the choice of suitable radial functions is later used for defining a regularization scheme (see Sec. V). The operator ∂^{-1} in Γ is defined as in Ref. 2 to avoid singularities in the energy expectation value:

$$(\partial^{-1}g)(r=0)=0\Longrightarrow(\partial^{-1}g)(r)=\int_0^r dr'g(r')\ . \tag{2.13}$$

For the Fock-space operators we have the canonical commutation rules

$$[a_{\alpha},a_{\beta}^{\dagger}]=\delta_{\alpha\beta},\quad [a_{\alpha},a_{\beta}]=[a_{\alpha}^{\dagger},a_{\beta}^{\dagger}]=0 \tag{2.14}$$

and those of the fields contain now a "transversal" δ function:

$$[\Pi_j^a(\mathbf{x}),A_i^b(\mathbf{y})]|_{t=0}=-i\delta^{ab}\frac{\delta(r-r')}{r^2}\delta_{ij}^{\rm tr}(\Omega-\Omega')\ ,$$
$$\delta_{ij}^{\rm tr}(\Omega-\Omega')=\sum Y_i^{\lambda lm\,*}(\Omega)Y_j^{\lambda lm}(\Omega')\ . \tag{2.15}$$

III. THE BOGOLIUBOV TRANSFORMATION

To calculate the expectation values of the above Hamiltonian it is necessary to make an ansatz for the ground state, the interacting vacuum. From the many-body theory it is known[7,16] that if $|\psi\rangle$ is the ground state of a many-body system, there exists a representation of the form

$$|\psi\rangle=e^S|\phi\rangle \tag{3.1}$$

with S a function only of the creation operators and an uncorrelated ground state $|\phi\rangle$ (e.g., a Slater determinant in the case of fermions). In our case $|\phi\rangle\equiv|0\rangle$ is the Fock-space vacuum of the annihilation operators ($a_{\alpha}|0\rangle=0$) defined in the last section and

$$S=c_1^{\alpha}a_{\alpha}^{\dagger}+\tfrac{1}{2}c_2^{\alpha\beta}a_{\alpha}^{\dagger}a_{\beta}^{\dagger}+\frac{1}{3!}c_3^{\alpha\beta\gamma}a_{\alpha}^{\dagger}a_{\beta}^{\dagger}a_{\gamma}^{\dagger}+\cdots\ . \tag{3.2}$$

The c coefficients must be chosen such that $|\psi\rangle$ has the same trivial quantum numbers as the vacuum.

Within this paper we approximate S by the first two terms in (3.2). Extensions to higher orders with a suitable cluster expansion[17] will be devoted to future investigations. Although we cannot hope to comprehend the whole theory, the first interesting insights into characteristic processes should be expected.

Because of color symmetry, c_1 vanishes. So the ansatz for the vacuum

$$|\psi\rangle=\exp(\tfrac{1}{2}c_2^{\alpha\beta}a_{\alpha}^{\dagger}a_{\beta}^{\dagger})|0\rangle \tag{3.3}$$

is identical with the well-known BCS ansatz. Thereby, because of the vacuum symmetry, c_2 has to be of the form

$$c_2^{\alpha\beta}=(-1)^{m+\lambda}\delta_{ab}\delta_{\lambda\lambda'}\delta_{ll'}\delta_{m,-m'}c_2^{nn'l\lambda}\ . \tag{3.4}$$

The calculation of expectation values with this ansatz is facilitated essentially by the possibility of transforming to quasiparticle operator $(b,b^{\dagger})$ by a Bogoliubov transformation[16] defined by

$$b_{\alpha}^{\dagger}=\sum(U_{\beta\alpha}a_{\beta}^{\dagger}+V_{\beta\alpha}a_{\beta}),\quad b_{\alpha}=\sum(U_{\beta\alpha}a_{\beta}+V_{\beta\alpha}a_{\beta}^{\dagger}) \tag{3.5}$$

(we assume the matrices U,V to be real) such that one has

$$b_{\alpha}|\psi\rangle=0,\quad [b_{\alpha},b_{\beta}^{\dagger}]=\delta_{\alpha\beta},\ \text{etc.} \tag{3.6}$$

The condition that $|\psi\rangle$ should have trivial angular momentum, color, and parity leads in the framework of our spherical basis of SP states to the restrictions

$$U_{\alpha\beta}=\delta_{\lambda\lambda'}\delta_{ll'}\delta_{m,m'}\delta_{ab}U^{l\lambda}_{nn'}\ ,$$
$$V_{\alpha\beta}=\delta_{\lambda\lambda'}\delta_{ll'}\delta_{m,-m'}s_\alpha\delta_{ab}V^{l\lambda}_{nn'}\ . \tag{3.7}$$

Finally the commutation relations (6) determine a generalized orthogonality (for every fixed l,λ):

$$f\begin{bmatrix}U^t & V^t\\ V^t & U^t\end{bmatrix}^{l\lambda} f\begin{bmatrix}U & V\\ V & U\end{bmatrix}^{l\lambda}=1,\quad f=\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}$$
(U^t means U transpose) . (3.8)

In the case of fermions (anticommutation rules) f is equal to the identity. Therefore, the transformation is unitary for fermions and symplectic for bosons.

For the transformed fields we obtain ($\omega=1$)

$$\mathbf{A}(x)=\sum_{\alpha,n'}\frac{1}{\sqrt{2}}(b_\alpha+s_\alpha b^\dagger_{\bar\alpha})(U-V)^{l\lambda}_{n'n}\frac{g_{n'l\lambda}}{r}\mathbf{Y}^{\lambda lm}T_a\ ,$$
$$\mathbf{\Pi}(\mathbf{x})=-i\sum_{\alpha,n'}\frac{1}{\sqrt{2}}(b_\alpha-s_\alpha b^\dagger_{\bar\alpha})(U+V)^{l\lambda}_{n'n}\frac{g_{n'l\lambda}}{r}\mathbf{Y}^{\lambda lm}T_a\ . \tag{3.9}$$

This shows that on account of the ansatz (3.7) for U,V only the radial functions are transformed, while the spherical harmonics remain unchanged.

IV. THE RPA EQUATIONS

The Wick rule is the important tool for a convenient representation of operators composed of the fields specified in Sec. III. Here, normal ordering and contractions will be defined with respect to the operators $b,b^\dagger$ and the vacuum $|\psi\rangle$.

The standard structure of the Bogoliubov theory[16] is that a variation of the ground state (with respect to the parameters of the Bogoliubov transformation) is equivalent to a diagonalization of the single-particle part of the Hamiltonian yielding

$$H=\langle H\rangle+\tfrac{1}{2}\sum E_\alpha b^\dagger_\alpha b_\alpha+V \tag{4.1}$$

with the SP energies E_α and a higher-order term $V=:V:$ For a better understanding we specify the contributions of $H_{\text{Schwinger}}$:

$$H=H_\pi+H_2+V_2+H_3+V_3+V_4+H_4 \tag{4.2}$$

with

$$H_\pi=\tfrac{1}{2}\int d^3x\ \Pi^a_j\Pi^a_j\ ,$$
$$H_2=\tfrac{1}{2}\int d^3x(\nabla\times\mathbf{A})^a_j(\nabla\times\mathbf{A})^a_j\ ,$$
$$V_2=\tfrac{1}{2}\int d^3x(\nabla\cdot\mathbf{\Pi})^a_j\Gamma(\nabla\cdot\mathbf{\Pi})^a_j\ ,$$
$$V_3=-g\int d^3x(\nabla\cdot\mathbf{\Pi})^a\Gamma f^{abc}\Pi^b_jA^c_j\ ,$$
$$H_3=\frac{g}{2}\int d^3x(\nabla\times\mathbf{A})^a_jf^{abc}(\mathbf{A}^b\times\mathbf{A}^c)_j\ ,$$
$$V_4=\frac{g^2}{2}\int d^3x\ f^{abc}f^{ade}(\mathbf{\Pi}^b\mathbf{A}^c)\Gamma(\mathbf{\Pi}^d\mathbf{A}^e)\ ,$$
$$H_4=\frac{g^2}{8}\int d^3x\ f^{abc}f^{ade}(\mathbf{A}^b\times\mathbf{A}^c)_j(\mathbf{A}^d\times\mathbf{A}^e)_j\ . \tag{4.3}$$

Then $V=:V_3+H_3+V_4+H_4:$. The BCS ansatz for the vacuum yields vanishing expectation values for the three point terms H_3,V_3 (an extended ansatz for the ground state with nonvanishing expectation values of H_3 and V_3 is described in Refs. 7 and 18).

For technical reasons we represent the untransformed fields of Sec. II by the Bargmann-space variables related to the SP basis mentioned above:

$$\mathbf{A}=\sum q_\alpha\mathbf{f}_\alpha\ ,$$
$$\boldsymbol{\pi}=\sum p_\alpha\mathbf{f}_\alpha,\quad[q_\alpha,p_\beta]=i\delta_{\alpha\beta} \tag{4.4}$$

with ($\omega=1$)

$$q_\alpha=\frac{1}{\sqrt{2}}(a_\alpha+s_\alpha a^\dagger_{\bar\alpha})\ ,$$
$$p_\alpha=-i\frac{1}{\sqrt{2}}(s_\alpha a_{\bar\alpha}-a^\dagger_\alpha)\ . \tag{4.5}$$

Then we get the normal-ordered SP part of the Hamiltonian (4.1):

$$:H_{\text{SP}}:=:(q,p)\begin{bmatrix}A & 0\\ 0 & B\end{bmatrix}\begin{bmatrix}q\\ p\end{bmatrix}: \tag{4.6}$$

with

$$A^{\alpha\beta}=H_2^{\alpha\beta}+V_4^{\sigma\alpha\tau\beta}\langle p_\sigma p_\tau\rangle$$
$$+(H_4^{\sigma\alpha\tau\beta}+H_4^{\sigma\alpha\beta\tau}+H_4^{\alpha\sigma\beta\tau}+H_4^{\alpha\sigma\tau\beta})\langle q_\sigma q_\tau\rangle\ ,$$
$$B^{\alpha\beta}=H_\pi^{\alpha\beta}+V_2^{\alpha\beta}+V_4^{\alpha\sigma\beta\tau}\langle q_\sigma q_\tau\rangle\ . \tag{4.7}$$

Here $\langle\cdot\rangle=\langle\psi|\cdot|\psi\rangle$ and $H_2^{\alpha\beta}$, etc. means the Bargmann-space matrix element with respect to the SP basis. Mixed elements (in the p,q) are vanishing. [Appearing only through V_4 they do not contribute because of the symmetry of the matrix elements $V_4^{\alpha\sigma\beta\tau}$ and the canonical commutation rules of the Bargmann-space variables (4).]

With the Bogoliubov transformation for the $(q,p)\to(\bar q,\bar p)$,

$$q_{nlm\lambda a}=(U-V)^{l\lambda}_{nn'}\bar q_{n'lm\lambda a}\ ,$$
$$p_{nlm\lambda a}=(U+V)^{l\lambda}_{nn'}\bar p_{n'lm\lambda a} \tag{4.8}$$

[U,V real matrices, see Eq. (3.5)], we obtain the "elementary contractions"

$$\langle p_{nlm\lambda a}p_{n'l'm'\lambda'b}\rangle=\kappa_{nn'}(l,\lambda)s_{m\lambda}\delta_{ll'}\delta_{ab}\delta_{m-m'}\delta_{\lambda\lambda'}\ ,$$
$$\langle q_{nlm\lambda a}q_{n'l'm'\lambda'b}\rangle=\rho_{nn'}(l,\lambda)s_{m\lambda}\delta_{ll'}\delta_{ab}\delta_{m-m'}\delta_{\lambda\lambda'}\ ,$$
$$\langle q_\alpha p_\beta\rangle=-\langle p_\beta q_\alpha\rangle=i\tfrac{1}{2}\delta_{\alpha\beta} \tag{4.9}$$

with

$$\kappa_{nn'}(l,\lambda)=\tfrac{1}{2}(U+V)^{l\lambda}_{nk}(U^t+V^t)^{l\lambda}_{kn'}\ ,$$
$$\rho_{nn'}(l,\lambda)=\tfrac{1}{2}(U-V)^{l\lambda}_{nk}(U^t-V^t)^{l\lambda}_{kn'}\ . \tag{4.10}$$

Finally, by inserting (4.8) into (4.6), applying the generalized orthogonality [Eq. (3.8)], and going back to the Fock space, one sees that the Hamiltonian (4.6) is diagonalized if the following equation is satisfied:

$$\begin{pmatrix} h & \Delta \\ -\Delta & -h \end{pmatrix}\begin{pmatrix} U & V \\ V & U \end{pmatrix}=\begin{pmatrix} U & V \\ V & U \end{pmatrix}\begin{pmatrix} \mathbb{E} & 0 \\ 0 & -\mathbb{E} \end{pmatrix} \tag{4.11}$$

with the norm (see Sec. III)

$$fWfW^t=1, \quad W=\begin{pmatrix} U^t & V^t \\ V^t & W^t \end{pmatrix} .$$

Here $\mathbb{E}$ (the matrix consisting of the eigenvalues E_α) is diagonal and

$$h=A+B, \quad \Delta=B-A \ . \tag{4.12}$$

Since h,Δ depend on U,V this is a nonlinear eigenvalue equation with a non-Hermitian matrix related to the Hermitian matrices A,B. For fixed h,Δ it is a linear eigenvalue problem of the RPA type. The transformation W is symplectic, as seen from its symmetry with respect to the metric f. To every eigenvector $\binom{U}{V}_n^{l\lambda}$ with eigenvalue E_α there exists a corresponding eigenvector $\binom{V}{U}_n^{l\lambda}$ with eigenvalue $-E_\alpha$.

The next step is to compute the matrices A,B related to H_π,H_2,H_4,V_2,V_4. Because of the symmetries of U,V the A,B are of the form

$$\begin{aligned} A^{\alpha\beta}&=\delta_{ll'}\delta_{m,-m'}\delta_{\lambda\lambda'}s_\alpha\delta_{ab}\hat{A}_{nn'}(l,\lambda) \ , \\ B^{\alpha\beta}&=\delta_{ll'}\delta_{m,-m'}\delta_{\lambda\lambda'}s_\alpha\delta_{ab}\hat{B}_{nn'}(l,\lambda) \ . \end{aligned} \tag{4.13}$$

Therefore the RPA equation (4.11) can be divided into a system of coupled RPA equations consisting of eigenvalue equations for every (l,λ). Then the calculation is straightforward and just technical. The details of the different contributions of the Hamiltonian to A and B are specified in the Appendix. For general radial functions $g_{nl\lambda}$ we obtain five types of radial integrals (see the Appendix), which are reduced to three after introducing our special regularization.

V. REGULARIZATION

Up to this point all operators, especially the Hamiltonian, have to be understood as formal objects, because, as is well known, the representation of the Poincaré group resulting from the canonical quantization of a classical field theory with interaction is pathological. In order to give the generators a mathematical meaning, we have to find a consistent regularization scheme.

In gauge theories we have to respect gauge invariance, yielding, e.g., Ward-Slavnov-Taylor identities. In perturbation theory dimensional regularization[9] satisfies this criterion in the continuum theory and the lattice formulation in the nonperturbative sector in a natural and elegant way.[10,11]

Within the Hamiltonian formulation, a regularization may be defined by truncating the expansion [Eqs. (2.9) and (2.10)], of the field operators by taking into account only a finite number of terms. An essential problem is then the choice of the form of these modes. In the fixed-gauge investigations in Refs. 3–7, plain waves (in Ref. 3 the S^3 analog) have been used for the basis functions in the expansion [Eqs. (2.9) and (2.10)] and the truncation was given by a (sharp) momentum cutoff.

Especially the rather complicated structure of the Hamiltonian in the Coulomb gauge[4,17] related to the Gribov horizon[19] makes it technically very difficult to work with other single-particle functions.

A momentum cutoff, however, is very problematic, since it is well known from perturbation theory that in order to make it consistent with dimensional regularization, counterterms have to be introduced (new counterterms in each order of perturbation theory) which are difficult to overview. In fact, in all the above examples[3–7] the counterterms were neglected (except for normal ordering).

The great advantage of the Schwinger gauge is the rather simple structure of the Coulomb term [Eq. (2.7)] allowing general choices of radial functions without technical difficulties. As discussed in Ref. 2, the lattice regularization is related to an expansion of the field operators in terms of characteristic functions. Such functions can now also be introduced for the radial coordinate within our Schwinger gauge framework and we expect that because of the analogy to the lattice formulation this regularization has a better chance of being consistent with dimensional regularization without counterterms. A finite check of this consistency is only possible by looking at the scaling properties of reliably computed observables which goes beyond the scope of our paper. Thus our regularization is given by

$$g_{nl}(r)=\frac{1}{a}[\Theta(r-na)-\Theta(r-(n+1)a)]=g_n(r) \ , \tag{5.1}$$

where a is the "lattice parameter" and

$$n\in\mathbb{N}, \quad 1\le n\le N_{\max}, \quad r\in\mathbb{R}, \quad r\le(N_{\max}+1)a \ . \tag{5.2}$$

In that way we obtain a theory in a finite volume (as in a bag model or in Ref. 1) and with an ultraviolet momentum cutoff as in the lattice formulation $p\le\pi/a$, $l\le L_{\max}\approx\pi(N_{\max}+1)$.

The introduction of characteristic functions produces ambiguities in the definition of the regularized forms of the operators ∂^{-1}, ∂ (see Sec. II and the Appendix). The only matrix elements which we have to define are (see the Appendix)

$$\begin{aligned} a_{nk}&=\int \partial^{-1}g_n r^{-2}\partial^{-1}g_k dr \ , \\ b_{nk}&=\int \partial g_n \partial g_k dr \ . \end{aligned} \tag{5.3}$$

Within the presented calculation we have made the choice

$$\partial g_n\to a^{-1}(g_{n-1}-g_n), \quad n\ge 2$$

and

$$\partial^{-1}=\int dr\to a\sum_n \ . \tag{5.4}$$

We also have

$$\int g_n g_k dr=a^{-1}\delta_{nk} \ .$$

Putting the scale a equal to one ($\omega=1$ is also legitimate), we obtain, for the radial matrix elements of the Appendix

$$
\begin{aligned}
a_{nk} &= \sum_{j=M}^{N_{\max}+1} j^{-2}, \quad M=\max(n,k)\ , \\
b_{nk} &= (2\delta_{nk}-\delta_{n,k-1}-\delta_{n,k+1})\ , \\
c_{nk} &= \delta_{nk} n^{-2}\ , \\
d^{lj}_{nk} &= \delta_{lj} a_{nk}, \quad \text{where } g_n g_k = \delta_{nk} g_k\ , \\
e^{lj}_{nk} &= \delta_{lj}\delta_{nl} c_{nk}\ .
\end{aligned}
\tag{5.5}
$$

The choice in (5.4) is not unique; others are given in Ref. 2 (see Secs. VI and VII for discussion of this point). In order to test the quality of the definition, the analytically solvable Abelian case (see Ref. 2) was treated. Very accurate results were obtained already with $N_{\max}=25$. We used this value in the non-Abelian case yielding an acceptable CPU time as well as good precision.

VI. RESULTS

The single-particle spectrum and the vacuum energy expectation value calculated in this section do not have a direct physical meaning, but they yield the possibility of a first understanding of the complicated structure of the many-body problem and the techniques of the formulation describing it.

A. The non-Abelian single-particle spectrum

To calculate the non-Abelian spectrum of the Hamiltonian mentioned in Sec. IV we have to diagonalize the coupled nonlinear system of RPA equations (see Sec. IV) by iteration. With the former regularization and the results of the Appendix we obtain

$$
\begin{aligned}
\hat{A}_{nn'}(l,\lambda) &= \frac{b_{nn'}}{2}+\delta_{nn'}\delta_{\lambda 1}\frac{l(l+1)}{2n^2} \\
&\quad + N_c\frac{g^2}{16\pi}\sum \kappa_{nn'}(l',\lambda')(2l'+1)a_{nn'} \\
&\quad + N_c\frac{g^2}{16\pi}\sum \rho_{nn'}(l',\lambda')\frac{2l'+1}{n^2}\delta_{nn'}\ , \\
\hat{B}_{nn'}(l,\lambda) &= \frac{\delta_{nn'}}{2}+\delta_{\lambda 2}\frac{l(l+1)}{2}a_{nn'} \\
&\quad + N_c\frac{g^2}{16\pi}\sum \rho_{nn'}(l',\lambda')(2l'+1)a_{nn'}\ .
\end{aligned}
\tag{6.1}
$$

The summation is performed over l',λ' ($1\le l,l'\le l_{\max}$; $\lambda,\lambda'=1,2$; $1\le n,n'\le N_{\max}$) and N_c is related to the gauge group SU(N_c). The matrices ρ and κ (containing the U,V) are given in Sec. IV. The first sum in $\hat{A}$ and that in $\hat{B}$ results from V_4, the second one in $\hat{A}$ from H_4. With $\hat{A},\hat{B}$ we obtain h,Δ [see Eq. (4.12)] and the coupled system of RPA equations is solved through iteration (with suitable initial conditions, e.g., $U=\mathbb{1}$, $V=0$). The eigenvectors and eigenvalues converge very rapidly; for a good precision we need only 4–5 iterations. The resulting SP spectrum [$E_{nl}(N_{\max}+1)$ as a function of l,n fixed] for $N_c=3$ and radial quantum number $n=1$ is shown in Fig. 1.

The appearance of an energy gap for $g\neq 0$, as in the axial gauge[5] or lattice calculations[12] is striking. It is an open question whether this gap may be interpreted as the existence of massive gluons in the interacting theory since up to now no calculation has been done in the continuum limit and the nature of the longitudinal components of the gluon is unclear. The existence of the gap, however, is probably a necessary condition for having only massive (colorless) glueballs.

To look for something like scaling behavior let us understand the gap as nearly half the mass of a glueball, a physical object (as we hope). Then with the gap $\Delta\approx 600$ MeV (Ref. 12) for $g=1.0$ we get $a^{-1}=1.2$ GeV. In the range of $g=0.8\ldots 1.1$ we obtain a scaling behavior

$$
\Delta = c a^{-1}\exp\left[-\frac{1}{\beta_0 g^2}\right], \quad c=\text{const} \tag{6.2}
$$

with $\beta_0\approx 3.1$ (similar to the Coulomb[4] and axial[5] gauge) while the correct value is $\beta_0\approx 0.14$.

We hope to get in the future a better result by extending the vacuum ansatz (see e.g., Ref. 18) or/and by defining another prescription for the regularized operators ∂^{-1},∂: e.g.,

$$
\partial_\alpha = h_\alpha^{-1}(D_\alpha^{-1}-h'_\alpha) \quad \text{with } D_\alpha = h_\alpha^{-1}\partial^{-1}\ , \tag{6.3}
$$

h'_α the derivation of h_α, with a regular function h_α and a parameter α.

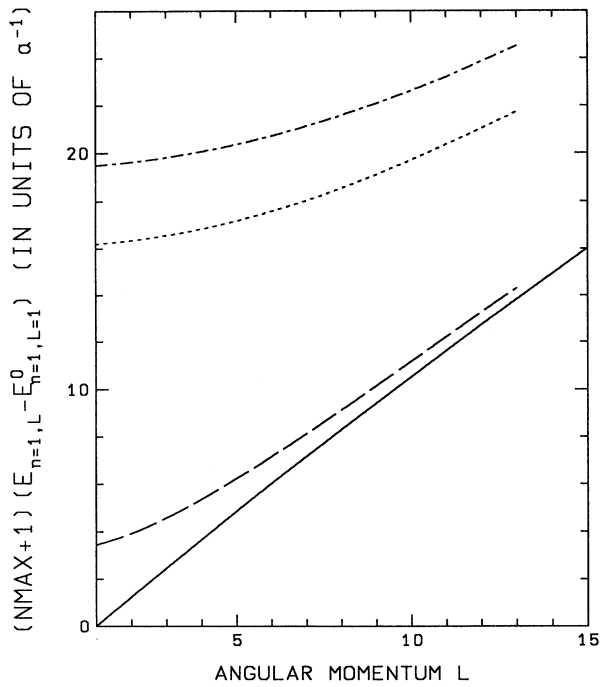

FIG. 1. The non-Abelian single-particle energies ($N_c=3$) for the lowest radial modes ($n=1$) as a function of the angular momentum L for $g=0.0$ (——), $g=0.1$ (— — —), $g=0.9$ (----), $g=1.2$ (—·—·-) relative to $E^0_{n=1,L=1}$, the energy of the lowest-lying state in the bag for $g=0$. The cutoff is given by $n_{\max}=25$.

B. The vacuum energy expectation value

From the calculation of the SP spectrum we have obtained the matrices U, V, which diagonalize the SP Hamiltonian approximatively. Therefore we can use these results to compute the vacuum expectation value of H and to compare it with lattice results from Chin, Long, and Robson.[20]

Only the complete contractions are to be determined:

$$\langle H\rangle=\langle H_\pi\rangle+\langle H_2\rangle+\langle V_2\rangle+\langle H_4\rangle+\langle V_4\rangle \ . \quad (6.4)$$

This calculation is analogous to the former one and yields the terms

$$\langle H_\pi\rangle=\tfrac{1}{2}D_c\sum(2l+1)\kappa_{nn'}(1,\lambda) \ ,$$

$$\langle H_2\rangle=\tfrac{1}{2}D_c\sum(2l+1)\rho_{nn'}(l,\lambda)\times\left[b_{nn'}+\delta_{nn'}\delta_{\lambda 1}\frac{l(l+1)}{n^2}\right] , \quad (6.5)$$

$$\langle V_2\rangle=\tfrac{1}{2}D_c\sum(2l+1)\kappa_{nn'}(l,\lambda)a_{nn'}l(l+1)$$

with D_c the dimension of the Lie algebra of SU(N_c).

Because of the antisymmetry of the structure constants only two types of contractions are nonvanishing in the four point terms, e.g., for H_4,

$$(\underline{A\overline{A}})(\underline{A\overline{A}}) \quad \text{and} \quad (\underline{A\overline{A}})(\underline{A\overline{A}}) \ , \quad (6.6)$$

yielding

$$\langle H_4\rangle=\frac{g^2}{16\pi}N_cD_c\sum(2l+1)(2l'+1)\times\rho_{nn}(l,\lambda)\rho_{nn}(l',\lambda')n^{-2} \ ,$$

$$\langle V_4\rangle=\frac{g^2}{16\pi}N_cD_c\sum(2l+1)(2l'+1)a_{nn'}\times[\rho_{nn'}(l',\lambda')\kappa_{nn'}(l,\lambda)-\delta_{\alpha\alpha'}] \ . \quad (6.7)$$

The sums include all appearing indices and both ρ,κ and a_{nk},b_{nk} are the same as in Secs. IV and V.

Figure 2 shows the expectation values in units of a^{-1} of the Abelian and non-Abelian contribution to $\langle H\rangle$. Chin, Long, and Robson[20] have calculated, in the lattice formulation,

$$e_{\rm KS}=\frac{a\langle H_{\rm KS}\rangle}{D_cN_p} \ , \quad (6.8)$$

with the Kogut-Susskind Hamiltonian[21] $h_{\rm KS}$ and $N_p=$ the number of the plaquettes.

To compare their results with ours a different definition of the coupling constant must be taken into account. Our formulation is related to that of Chin, Long, and Robson by the replacement

$$(A,\Pi)=>\left[gA,\frac{1}{g}\Pi\right] . \quad (6.9)$$

For comparison we also have to define an energy density per gluon degree of freedom:

$$e_S=\frac{a\langle H_{\rm Schwinger}\rangle}{D_cV} , \quad V=\frac{4\pi}{3}(N_{\rm max}+1)^3 . \quad (6.10)$$

In Fig. 3 the ratio $(e_S/\alpha)/e_{\rm KS}$ (with a constant α taking into account the various regularization schemes) is represented as a function of the coupling constant g. The value is nearly equal to one, which seems to be a good confirmation of our approach.

VII. SUMMARY AND REMARKS

Summarizing we may say that the application of nonperturbative many-body methods to the Yang-Mills Hamiltonian in the Schwinger gauge regularized by a lattice analogous scheme yields a technically transparent formulation to compute expectation values (spectra of bound states at a later stage) with a minimal numerical expense. The vacuum energy is described in a satisfactory way when compared to lattice results, and an effective mass for the gluon in the interacting theory can also be obtained, though with a scaling behavior not better than

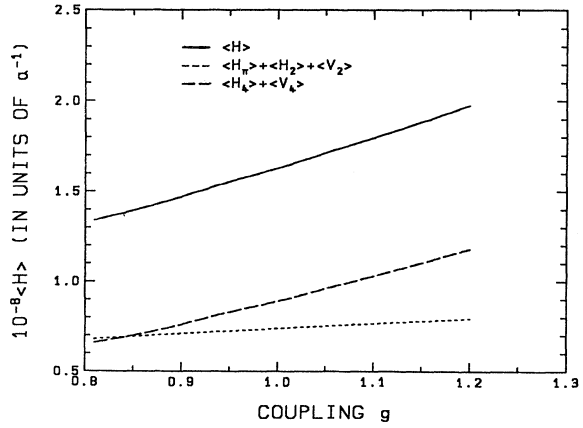

FIG. 2. The vacuum energy expectation values in units of a^{-1}. The cutoff is given by $N_{\rm max}=25$.

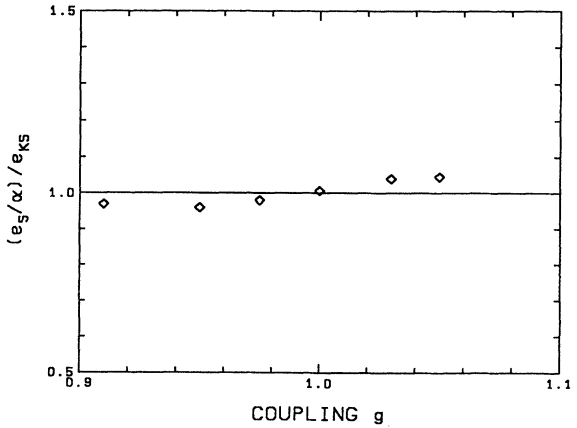

FIG. 3. Ratio of the vacuum energy densities. See text for details.

in other approaches. An extension of the vacuum ansatz to include three-point contributions and different choices of the regularization prescription for ∂^{-1} may perhaps improve this situation.[2]

Also it should be mentioned that a more detailed investigation of the boundary conditions of the fields might be necessary, motivated by the following remark: Generally every gauge fixing of the form $\Gamma(\mathbf{A})=0$ must be complete and unique (every orbit is cut from the hyperplane defined by the gauge condition only once). In the Coulomb gauge the uniqueness is not satisfied and therefore the Gribov horizon is introduced.

Here we have the problem that the gauge transformation to the Schwigner gauge,

$$g(\mathbf{x})=P\exp\left[i\partial^{-1}\frac{\mathbf{x}}{r}\cdot\mathbf{A}(\mathbf{x})\right]$$

(with the definition of ∂^{-1} given in Sec. II and Ref. 2), might not be consistent with respect to given boundary conditions of the gauge fields. This means that the Schwinger gauge condition eventually has to be extended. An example of this type has been discussed by Yabuki.[22]

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APPENDIX

Here we specify the results in the computation of the SP matrix elements of the Schwinger Hamiltonian. With the SP functions from Sec. II we obtain

$$\begin{aligned}
H_{\pi}^{\alpha\beta}&=\tfrac{1}{2}\mathrm{Tr}\int d^3x\ \mathbf{f}_\alpha\cdot\mathbf{f}_\beta=\tfrac{1}{2}s_\alpha\delta_{\bar{\alpha}\beta}\ ,\\
V_2^{\alpha\beta}&=\tfrac{1}{2}\mathrm{Tr}\int d^3x\ \nabla\cdot\mathbf{f}_\alpha\Gamma\nabla\cdot\mathbf{f}_\beta=\tfrac{1}{2}\delta_{\lambda 2}\delta_{ll'}\delta_{m\,-m'}\delta_{\lambda\lambda'}s_\alpha\delta_{ab}(l(l+1)a_{nl\lambda,n'l'\lambda'}\ ,\\
H_2^{\alpha\beta}&=\tfrac{1}{2}\mathrm{Tr}\int d^3x\ \nabla\times\mathbf{f}_\alpha\cdot\nabla\times\mathbf{f}_\beta=\delta_{ll'}\delta_{\lambda\lambda'}\delta_{m,\,-m'}s_\alpha\delta_{ab}(\tfrac{1}{2}b_{nl\lambda,n'l'\lambda'}+\tfrac{1}{2}\delta_{\lambda 1}l(l+1)c_{nl\lambda,n'l'\lambda'})
\end{aligned}\tag{A1}$$

with

$$\begin{aligned}
a_{nl\lambda,n'l'\lambda'}&=\int dr(\partial^{-1}g_{nl\lambda})r^{-2}(\partial^{-1}g_{n'l'\lambda'})\ ,\\
b_{nl\lambda,n'l'\lambda'}&=\int dr(\partial_r g_{nl\lambda})(\partial_r g_{n'l'\lambda'})\ ,\\
c_{nl\lambda,n'l'\lambda'}&=\int dr\, g_{nl\lambda}r^{-2}g_{n'l'\lambda'}\ .
\end{aligned}\tag{A2}$$

In the computation of the non-Abelian contributions we use the rotational invariance of a sum of a product of four spherical harmonics:

$$V_4^{\alpha\sigma\beta\tau}=-\frac{g^2}{2}\mathrm{Tr}\int d^3x[\mathbf{f}_\alpha,\mathbf{f}_\sigma]\Gamma[\mathbf{f}_\beta,\mathbf{f}_\tau]\ ;$$

therefore

$$V_4^{\alpha\sigma\beta\tau}\langle q_\sigma q_\tau\rangle=\hat{\delta}_{\alpha\bar{\beta}}N_c\frac{g^2}{16\pi}\sum_{\substack{n_1',n_2'\\ l_1',\lambda_1'}}\rho_{n_1'n_2'}(l_1',\lambda_1')(2l_1'+1)d^{n_1l_1\lambda_1,n_2l_2\lambda_2}_{n_1'l_1'\lambda_1',n_2'l_2'\lambda_2'}\quad(\text{with }\hat{\delta}_{\alpha\bar{\beta}}=\delta_{ll'}\delta_{\lambda\lambda'}\delta_{m,\,-m'}s_\alpha\delta_{ab})\ ,$$

$$V_4^{\sigma\alpha\tau\beta}\langle p_\sigma p_\tau\rangle=\hat{\delta}_{\alpha\bar{\beta}}N_c\frac{g^2}{16\pi}\sum_{\substack{n_1',n_2'\\ l_1',\lambda_1'}}\kappa_{n_1'n_2'}(l_1',\lambda_1')(2l_1'+1)d^{n_1l_1\lambda_1,n_2l_2\lambda_2}_{n_1'l_1'\lambda_1',n_2'l_2'\lambda_2'}\ ,$$

and

$$H_4^{\alpha\sigma\beta\tau}=-\frac{g^2}{2}\mathrm{Tr}\int d^3x\ \epsilon_{ijm}\epsilon_{ikl}[f_j^\alpha,f_m^\sigma][f_k^\beta,f_l^\tau]$$

$$\Longrightarrow(H_4^{\alpha\sigma\beta\tau}+H_4^{\alpha\sigma\tau\beta}+H_4^{\sigma\alpha\tau\beta}+H_4^{\sigma\alpha\beta\tau})\langle q_\sigma q_\tau\rangle=\hat{\delta}_{\alpha\bar{\beta}}N_c\frac{g^2}{16\pi}\sum_{\substack{n_1',n_2'\\ l_1',\lambda_1'}}\rho_{n_1'n_2'}(l_1',\lambda_1')(2l_1'+1)d^{n_1l_1\lambda_1,n_2l_2\lambda_2}_{n_1'l_1'\lambda_1',n_2'l_2'\lambda_2'}\tag{A3}$$

with

$$d_{\alpha\beta}^{\sigma\tau}=\int dr(\partial^{-1}g_\alpha g_\sigma)r^{-2}(\partial^{-1}g_\beta g_\tau),\quad e_{\alpha\beta}^{\sigma\tau}=\int dr\, g_\alpha g_\sigma r^{-2}g_\beta g_\tau\ . \tag{A4}$$

With the definition of the radial functions as "characteristic functions" we obtain the results for the radial matrix elements specified in Eq. (5.5).

Finally, the resulting matrices A,B are given in Eq. (6.1).

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[1]J. Koller and P. van Baal, Ann. Phys. (N.Y.) **174**, 299 (1987); P. van Baal and J. Koller, Nucl. Phys. **B302**, 1 (1988), and earlier references.

[2]D. Schütte, Phys. Rev. D **40**, 2090 (1989).

[3]R. E. Cutkosky and K. C. Wang, Phys. Rev. D **37**, 3024 (1988), and earlier references.

[4]P. Besting and D. Schütte, Phys. Rev. D **40**, 2692 (1989).

[5]A. Kovner and B. Rosenstein, Phys. Lett. B **177**, 71 (1986).

[6]S. Nojiri, Z. Phys. C **22**, 245 (1984).

[7]B. Faber, H. Nguyen-Quang, and D. Schütte, Phys. Rev. D **34**, 1157 (1986).

[8]P. Becher, M. Böhm, and H. Joos, *Eichtheorien* (Teubner, Stuttgart, 1981).

[9]J. Collins, *Renormalization* (Cambridge University Press, Cambridge, England, 1984).

[10]A. Hasenfratz and P. Hasenfratz, Phys. Lett. **93B**, 165 (1980).

[11]B. E. Baaquie, Phys. Rev. D **16**, 2612 (1977).

[12]J. E. Mandula and M. Ogilvie, in *Lattice Gauge Theory '86,* proceedings of the Conference, Upton, New York, 1986, edited by H. Satz, I. Harrity, and J. Potvin, NATO Advanced Study Institute, Series B: Physics, Vol. 159 (Plenum, New York, 1986).

[13]H. Cheng and E. C. Tsai, Chin J. Phys. **25**, 95 (1987).

[14]N. H. Christ and T. D. Lee, Phys. Rev. D **22**, 939 (1980).

[15]L. C. Biedenharn and J. D. Louck, *Angular Momentum in Quantum Physics* (Addison-Wesley, Reading, MA, 1981), p. 284.

[16]P. Ring and P. Schuck, *The Nuclear Many-Body Problem* (Springer, Berlin, 1980).

[17]D. Schütte, Phys. Rev. D **31**, 810 (1984).

[18]H. Nguyen-Quang, Ph.D. dissertation, Bonn, 1986.

[19]V. N. Gribov, Nucl. Phys. **B139**, 1 (1978).

[20]S. A. Chin, C. Long, and D. Robson, Center for Theoretical Physics Report No. CTP-TAMU 37 (unpublished).

[21]J. Kogut and L. Susskind, Phys. Rev. D **11**, 395 (1975).

[22]H. Yabuki, Phys. Lett. B **231**, 271 (1989).