BRST-invariant nonplanar primitive operators in the open-bosonic-string theory

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We investigate nonplanar operators in the open-bosonic-string theory by means of the covariant operator formalism based on the Becchi-Rouet-Stora-Tyutin (BRST) formulation. The detailed evaluations of two nonplanar primitive operators, that is, nonplanar self-energy and nonplanar two-loop tadpole, are presented. At the one-loop level, we show that the ghost contribution yields a right measure factor to the physical amplitude. For the nonplanar two-loop tadpole operator, we discuss the factorization of the nonplanar self-energy. In order to avoid multiple countings, the integration regions are suitably restricted by investigating the duality of the 5-Reggeon vertex and the periodicity of the nonplanar self-energy operator. We also argue that only one singularity associated with the nonplanar self-energy part occurs in the nonplanar two-loop tadpole.

I. INTRODUCTION AND PRELIMINARIES

In string theory, we know some different approaches for computing multiloop amplitudes which are distinguished by the choice of covering space of moduli. One of them is the Polyakov path-integral approach which is based on Teichmuller space.[1] In this covering space, the complex geometrical structure of string theory is well analyzed[2] and a modular-invariant definition of the amplitude is established. However an explicit expression for the amplitudes has not yet been obtained for more than four loops at present.[3]

On the other hand, in the covariant operator formalism based on the Becchi-Rouet-Stora-Tyutin (BRST) formulation,[4–14] the moduli space is parametrized by the Schottky parameters. Owing to this parametrization, it is possible to derive an explicit expression for the multiloop amplitude[15–17] and also to clarify the complex structure of the moduli space; the holomorphic factorization theorem is almost trivial.[6,18] Another advantage of the operator formalism is the factorizability, that is, all diagrams can be constructed from 3-Reggeon vertices and propagators. In addition, if one makes the formalism satisfy the duality property, it becomes possible to construct arbitrary multiloop operators from some primitive operators and trees. In the open-bosonic-string case, the primitive operators are known to be the following four operators:[19] planar tadpole, nonorientable tadpole, nonplanar self-energy, and nonplanar two-loop tadpole (Fig. 1). Along this line, in the previous papers,[9,10] we have proposed an operator formalism with operational duality and have computed all the one-loop operators according to this formalism. It is shown that the resultant amplitude has a correct measure owing to the ghost contributions. Furthermore, by making a multiloop operator from these primitive operators, one of the authors (H.K.) has shown that this program really works well at arbitrary order in the planar case.[11,14]

In the nonplanar case, however, any systematic investigation along this line has not yet been carried out; especially the nonplanar two-loop tadpole operator has not been calculated even in the old dual resonance model. In this paper, we carry out the systematic constructions of BRST-invariant nonplanar self-energy and nonplanar two-loop tadpole.

We first show the calculation of the nonplanar self-energy operator, as noted in Ref. 10. Although this operator as well as other one-loop primitive operators [Figs. 1(a)–1(c)] have been obtained by Gross and Schwarz within the framework of the old dual resonance model, they were subjected to the unphysical modes appearing in the loop diagrams.[20] We will show that the ghost sector correctly cancels these unphysical modes.

The nonplanar two-loop tadpole operator is another element for constructing the nonplanar multiloop amplitude. We will construct this operator from the reduced 5-Reggeon vertex by sewing two pairs of alternate legs through the propagators. The expression turns out to be similar to that of the planar two-loop tadpole operator[21] because of the coincidence of the Riemann doubles of both cases. The nonplanar two-loop tadpole has four real Schottky parameters. The integration regions with respect to these variables must be restricted to a certain region so as to evade the overcounting of the same configurations. We will carry this out by considering the duality of the 5-Reggeon vertex and the periodicity of the nonplanar self-energy part, which can be factorized in the nonplanar two-loop tadpole [Fig. 1(d)]. We next analyze the singularity which comes from the partition function. It will be shown that, in spite of the superficial coincidence between the planar and the nonplanar operators,

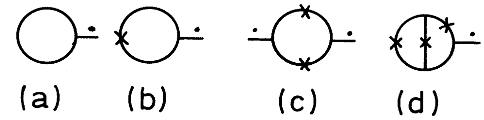

FIG. 1. The primitive operators in the open bosonic string: (a) planar tadpole, (b) nonorientable tadpole, (c) nonplanar self-energy, (d) nonplanar two-loop tadpole.

the singularity structures of both operators are quite different. We will argue that there is only one singular point in the nonplanar two-loop case. This is a crucial difference from the planar case, which has three independent singularities.[21]

Before going into the details of the evaluation of two primitive nonplanar operators, we here briefly review the basic ingredients of our calculations which are the following three BRST-invariant operators (A)–(C).

(A) *3-Reggeon vertex* (the CSV vertex):[4]

$$\langle V_{123}| = \langle 0a; q=3|\exp\left[\sum_{\substack{r\neq s\\ r=1}}^{3}\left[-\tfrac{1}{2}\sum_{n,m=0} a_n^r D_{nm}^{(0)}(U_r V_s) a_m^s - \sum_{n=2}\sum_{m=-1} c_n^r D_{nm}^{(1)}(U_r V_s) b_m^s\right]\right] \times \prod_{n=0,\pm 1}\left[\sum_{r=1}^{3}\sum_{m=0,\pm 1} D_{nm}^{(1)}(V_r) b_m^r\right]. \tag{1.1}$$

The $SL(2,R)$ elements U_r, V_r are Lovelace maps defined by[22]

$$V_r = \begin{bmatrix} \infty & 0 & 1 \\ Z_{r-1} & Z_r & Z_{r+1} \end{bmatrix}, \quad \Gamma = \begin{bmatrix} \infty & 0 & 1 \\ 0 & \infty & 1 \end{bmatrix}, \quad U_r = \Gamma V_r^{-1},$$

where Z_r is the Koba-Nielsen variable of the rth leg. The notation of the infinite-dimensional representations of $SL(2,R)$ $D_{nm}^{(0)}$ and $D_{nm}^{(1)}$ is defined in Appendix A. This vertex is BRST invariant in the sense that

$$\langle V_{123}|(Q_B^{(1)}+Q_B^{(2)}+Q_B^{(3)})=0 . \tag{1.2}$$

$Q_B^{(r)}$ is the BRST charge associated with the rth leg.

(B) *Propagators.* There are two types of propagators. One is the "twisted" propagator T, which is used for the orientable diagram, and the other is the "untwisted" propagator U for nonorientable diagrams.

The twisted propagator is given as[5]

$$T=(b_0-b_1)\int_0^1 \frac{dx}{x(1-x)}\mathcal{P}(x) , \tag{1.3a}$$

$$\mathcal{P}(x)=x^{L_0}\Omega S(x) , \tag{1.3b}$$

where Ω and $S(x)$ are the twist and gauge operators respectively, defined as

$$\Omega=(-)^{L_0}e^{-L_{-1}} , \tag{1.4}$$

$$S(x)=(1-x)^{L_0-L_1} . \tag{1.5}$$

Here $L_n=L_n^x+L_n^{\rm gh}$, $n=0,\pm 1$ are the generators of $SL(2,R)$. The relations

$$\mathcal{P}^\dagger(x)=\mathcal{P}(x) , \quad (b_0-b_1)\mathcal{P}(x)=\mathcal{P}(x)(b_0-b_{-1}) \tag{1.6}$$

guarantee the Hermiticity of T.

The untwisted propagator is obtained by replacing the integrand of the twisted propagator (1.3) as[10]

$$\mathcal{P}(x)\to\tilde{\Omega}^\dagger\mathcal{P}(x) , \tag{1.7}$$

where

$$\tilde{\Omega}=\Omega e^{-L_{-1}}, \quad \tilde{\Omega}^2=1 . \tag{1.8}$$

It is also Hermitian owing to the relations (1.6) with the replacement (1.7). Notice that the Hermiticity of both propagators guarantees the duality property of diagrams as discussed in Ref. 10. Both of them have the correct ghost number -1 and are BRST invariant after integration: $\{Q_B T\}=\{Q_B U\}=0$.

Another BRST-invariant scheme is possible using the propagator $b_0\int_0^1 dx\, x^{L_0-1}$ and the modified CSV vertex, one leg of which is twisted by Ω^T.[9] In this scheme however whether or not the duality can be maintained is uncertain at this stage.

(C) *Reflection operator.*[9] When one sews two legs with the propagator, one of them must be adjoined. The reflection operator plays this role. (In Ref. 14 an analogous operator is introduced as a sewing operator. There the geometrical meaning is also explained.) It is given as

$$|R_{23}\rangle=(\langle V_{123}|\Omega^{(2)\dagger}|0a;q=0\rangle_1)^\dagger = \prod_{n=0,\pm 1}(b_n^2-b_{-n}^3)\exp\left[-\sum_{n=1} a_{-n}^2 a_{-n}^3+\sum_{n=2}(c_{-n}^2 b_{-n}^3+c_{-n}^3 b_{-n}^2)\right]|0a;q=3\rangle_{23} . \tag{1.9}$$

This operator makes the bra state into a ket state in the BRST-invariant way, since it satisfies the BRST invariance

$$(Q_B^{(2)}+Q_B^{(3)})|R_{23}\rangle=0 .$$

The N-point extension $\langle V_{12\cdots N}|$ of the 3-Reggeon vertex $\langle V_{123}|$ is derived in the fourth paper of Ref. 4 according to the sewing rule of Lovelace.[22] The duality property is crucial in the construction of multiloop diagrams from primitive operators. We therefore introduce the "reduced N-Reggeon vertex" $\langle \tilde{V}_{12\cdots N}|$.[10] It is derived from $\langle V_{12\cdots N}|$ by putting the constraints $b_0 = b_1$ on all the external legs. These constraints guarantee the duality in the reduced vertex. The advantages of the reduced vertex are not only in its manifest duality but also in the simplification of practical calculations. It is given as follows.

$$\langle \tilde{V}_{12\cdots N}| = \langle V_{12\cdots N}| \,|_{b_0^r = b_1^r} ,$$
$$\langle \tilde{V}_{12\cdots N}| = \int \prod_{r=1}^{N} \frac{dZ_r \theta(Z_{r+1}-Z_r)}{(Z_{r+1}-Z_r)dV_{abc}} \langle \tilde{V}^{x}_{12\cdots N}| \otimes \langle \tilde{V}^{\mathrm{gh}}_{12\cdots N}| , \tag{1.10}$$

where dV_{abc} is the gauge volume of SL$(2,R)$ defined by

$$dV_{abc} = \frac{dZ_a dZ_b dZ_c}{(Z_a - Z_b)(Z_a - Z_c)(Z_b - Z_c)}$$

and the orbital sector and the ghost sector are defined by

$$\langle \tilde{V}^{x}_{12\cdots N}| = \langle 0_a | \exp\left[\sum_{r\neq s} \left\{ -\tfrac{1}{2} \sum_{n,m=0}^{\infty} a_n^r D_{nm}^{(0)}(U_r V_s) a_m^s \right\} \right] ,$$
$$\langle \tilde{V}^{\mathrm{gh}}_{12\cdots N}| = \langle q=3 | \exp\left[\sum_{r\neq s} \left\{ -\sum_{n=2} \sum_{m=-1} c_n^r D_{nm}^{(1)}(U_r V_s) b_m^s \right\} \right] \prod_{r=1}^{N} \frac{Z_{r+2}-Z_r}{Z_{r+1}-Z_r} [b_{-1}^r - b_0^r - D_{11}^{(1)}(U_r V_{r-1}) b_0^{r-1}] . \tag{1.11}$$

Although the constraints $b_0 = b_1$ break the manifest BRST invariance of $\langle \tilde{V}_{12\cdots N}|$, by attaching the propagators (1.3) or (1.7) or physical states on all the legs, we can recover the invariance. We can therefore keep the cancellation of the contribution of unphysical modes by that of the ghost modes in the following calculations.

In Sec. II, we give the calculation of the nonplanar self-energy. We then show that the ghost sector gives the right contribution to the measure factor in the on-shell physical amplitude. In Sec. III, we discuss nonplanar two-loop tadpole operator. This section consists of three subsections. In Sec. III A we construct this operator explicitly. In Sec. III B we investigate integration regions with respect to parameters which characterize the operator. Finally, Sec. III C is devoted to the discussion of singularity.

Some details of the notation and computations are given in the Appendixes. In particular, the notation throughout this paper is given in Appendix A. We give the details of the calculation of nonplanar two loop tadpole in Appendix B.

Throughout this paper we are always at the critical dimension $D = 26$ which comes from the nilpotency of the BRST charge.

II. NONPLANAR SELF-ENERGY OPERATOR

In this section, we briefly demonstrate the calculation of the nonplanar self-energy operator discussed in Ref. 10. This will become important for the discussion of the nonplanar two-loop tadpole operator in the next section. As illustrated in Fig. 2, it is constructed from the reduced 4-Reggeon vertex by sewing leg 1 to leg 3 with the twisted propagator:

$$ {}_{24}\langle \Sigma| = \int \frac{d^D k}{(2\pi)^b} \int_0^1 \frac{du}{u(1-u)} \frac{dv}{v(1-v)} \mathrm{Tr}^{13}[\langle \tilde{V}_{12E}|(b_0^E - b_1^E)\mathcal{P}^{(E)}(u) \langle \tilde{V}_{F34}| R_{EF}\rangle S^{(1)}(u) S^{(3)}(u)(b_0^1 - b_1^1)\mathcal{P}^{(1)}(v)|R_{11^+}\rangle]$$
$$= \int \frac{d^D k}{(2\pi)^b} \int_0^1 \frac{dv}{v(1-v)} \mathrm{Tr}^{13}[\langle \tilde{V}_{1234}|(b_0^1 - b_1^1)\mathcal{P}^{(1)}(v)|R_{11^+}\rangle] , \tag{2.1}$$

where Tr^{13} means the trace operation with respect to leg 1 and leg 3. The explicit expression of Tr^{ij} is given in Ref. 9. We introduce the gauge operators $S^{(1)}(u)$ and $S^{(3)}(u)$ according to the sewing rule of Lovelace[22] in order to make a 4-Reggeon vertex $\langle \tilde{V}_{1234}|$. After taking the trace, Eq. (2.1) becomes

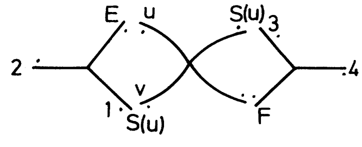

FIG. 2. The diagrammatic construction of the nonplanar self-energy.

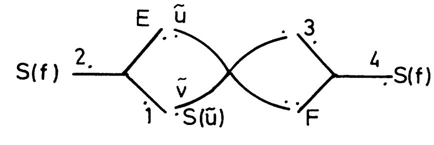

FIG. 3. The operator combination of the nonplanar self-energy in Ref. 20.

$$ {}_{24}\langle \Sigma| = \int \frac{d^b k}{(2\pi)^b} \int_0^1 \frac{du}{u(1-u)} \frac{dv}{v(1-v)} \, {}_{24}\langle \Sigma^x| \otimes {}_{24}\langle \Sigma^{\rm gh}| \ , \tag{2.2a} $$

where the orbital sector and the ghost sector are given by

$$ \begin{aligned} {}_{24}\langle \Sigma^x| = \prod_{n=1}^{\infty} (1-\omega^n)^{-D} \langle 0a| \exp \Bigg[& \tfrac{1}{2}k^2 \ln\omega - \sum_{r=2,4} [(a^r|U_r + (p_r|_0(U_r)][|\alpha) - |\beta)]k \\ & - \tfrac{1}{2} \sum_{r,s=2,4} \{[(a^r|U_r + (p_r|_0(U_r)](J-\delta_{rs})[V_s|a^s) + (V_s)_0|p_s)] \\ & + (1-\delta_{rs})(p_r|[{}_0(U_r)_0 + {}_0(V_s)_0]|p_s) + (1-\delta_{rs})(a^r|(U_r)_0|p_s)\} \Bigg] \ , \end{aligned} \tag{2.2b} $$

$$ \begin{aligned} {}_{24}\langle \Sigma^{\rm gh}| = \frac{1-v}{uv(1-u)} \prod_{n=2}^{\infty} (1-\omega^n)^2 \langle q=3| \exp \left[- \sum_{r,s=2,4} [c^r|U_r(J-\delta_{rs})V_s|b^s] + \text{cross terms} \right] \\ \times \prod_{r=2,4} \left[b^r_{-1} - b^r_0 + \frac{uv(1-u)}{1-v} b^r_0 \right] \ . \end{aligned} \tag{2.2c} $$

The notation of the exponents is given in Appendix A. The variables ω and α,β are the multiplier and the fixed points of the loop generator $\bar{P}$, respectively, where $\bar{P} = V_1 \mathcal{P}(v) U_3$ and $J = \sum_{l=-\infty}^{\infty} \bar{P}^{\,l}$ is the sum of all elements of the Schottky group generated by $\bar{P}$. The cross terms between the $n \geq 2$ and the $n = 0, \pm 1$ modes

$$ \sum_{n=2}^{\infty} \sum_{m=-1}^{1} c_n D^{(1)}_{nm} b_m $$

in the exponent of the ghost sector may be neglected because they have no contribution to the physical quantities. Hereafter we will always neglect the cross term.

The result of the orbital sector is the same as the one derived by Gross and Schwarz up to the following gauge operations (see also Refs. 23 and 24):

$$ V_3 \to \tilde{V}_3 = V_3 S^{-1}(u) \ , \tag{2.3} $$

$$ V_r \to \tilde{V}_r = V_r S(f), \quad r = 2,4 \ . \tag{2.4} $$

According to the gauge transformation (2,3), the propagator variables changes as

$$ \tilde{u} = u, \quad \tilde{v} = \frac{\tilde{u}(1-v)}{1-\tilde{u}v} \ . \tag{2.5} $$

These variables with tildes correspond to the variables used in Ref. 20. The gauge parameter f in Eq. (2.4) are defined by

$$ f = \frac{\omega\xi(1-\omega)}{1+\omega\xi} \tag{2.6} $$

with

$$ \tilde{u} = \frac{(1+\omega\xi)^2}{(1+\xi)(1+\omega^2\xi)}, \quad \tilde{v} = \frac{\omega(1+\xi)}{(1+\omega)(1+\omega\xi)} \ . \tag{2.7} $$

The above manipulations are depicted in Fig. 3. We will use this configuration for discussing the factorization of the nonplanar self-energy from the nonplanar two-loop tadpole in the next section.

Owing to these modification, the situation is very simplified; the new loop generator $\tilde{P} \equiv \tilde{V}_4^{-1} \bar{P} \tilde{V}_4 = (\tilde{V}_2^{-1} \bar{P} \tilde{V}_2)^{-1}$ becomes equivalent to the one that appeared in the planar one-loop tadpole[11,16] with the multiplier ω and two fixed points $\tilde{\alpha}, \tilde{\beta}$:

$$ \begin{aligned} & \hat{\tilde{P}}(z) = \omega z + 1 \ , \\ & \tilde{\alpha} \equiv \hat{\tilde{V}}_4^{-1}(\alpha) = \hat{\tilde{V}}_2^{-1}(\alpha) = \frac{1}{1-\omega} \ , \\ & \tilde{\beta} \equiv \hat{\tilde{V}}_4^{-1}(\beta) = \hat{\tilde{V}}_2^{-1}(\beta) = \infty \ . \end{aligned} \tag{2.8} $$

The multiplier are rewritten in terms of these new variables as

$$ \frac{\omega}{(1+\omega)^2} = \frac{\det\bar{P}}{(\mathrm{Tr}\bar{P})^2} = \frac{\det\tilde{P}}{(\mathrm{Tr}\tilde{P})^2} = \tilde{u}\,\tilde{v}(1-\tilde{v}) \ . \tag{2.9} $$

In the ghost part, the additional gauge operator $S(f)$ on leg 2 and leg 4 yield the factor $1/(1-f)^2$ in the $n = 0, \pm 1$ sector:

$$ \begin{aligned} & \frac{1}{(1-f)^2} \frac{1-\tilde{v}}{\tilde{u}\,\tilde{v}} \prod_{n=2}^{\infty} (1-\omega^n)^2 \langle q=3| \prod_{r=2,4} \left[b^r_{-1} - b^r_0 + \frac{\tilde{u}\,\tilde{v}}{1-\tilde{v}} (1-f)^2 b^r_0 \right] \\ & \qquad = \frac{1}{\omega} \frac{1}{(1-\omega)^2} \prod_{r=1}^{\infty} (1-\omega^n)^2 \langle q=3| \prod_{r=2,4} [b^r_{-1} - (1-\omega) b^r_0] \ . \end{aligned} \tag{2.10} $$

Because of the symmetries of the integrand, the integration regions with respect to $\tilde{u}$ and $\tilde{v}$ have to be restricted to

$$\tfrac{1}{2}\leq\tilde{u}\leq 1,\quad 0\leq\tilde{v}\leq\tfrac{1}{2}\ . \tag{2.11}$$

These restrictions are equal to $0\leq\omega\leq 1$, $\omega\leq\xi\leq 1$ in Ref. 20. This argument of restricting the integration regions will be used in the next section to discuss the nonplanar two-loop tadpole case.

Finally we discuss the BRST invariance of ${}_{24}\langle\Sigma|$. It is satisfied in the following form before neglecting the cross terms between $n\geq 2$ and $n=0,\pm 1$ in the ghost sector:

$${}_{24}\langle\Sigma|T^{(2)}T^{(4)}\{Q_B^{(2)}+Q_B^{(4)}\}=0\ . \tag{2.12}$$

Here we drop the surface term, which arises due to the restriction of the integration region, by suitably regularizing the divergence at the point $\omega=1$. The BRST invariance implies the decoupling of the unphysical modes from this operator. Indeed we can see the important role of the ghost sector in the calculation of the physical amplitude. The factor $1/(1-\omega)^2$ in (2.10) cancels the factor $(1-\omega)^2$ which comes from the Jacobian associated with the change of variables from x_i in the diagram of Fig. 4(a) to u_i of Fig. 4(b). Thus the amplitude has a correct measure.[10] Similar cancellations were seen in the planar and nonorientable cases.[9,10] Furthermore the partition function in the ghost sector cancels two powers in the orbital sector. In the nonplanar case, the two powers of the partition function of the ghost contribution are important to guarantee the absence of cut singularities in the limit $\omega\to 1$, so that only the physical poles appear in the amplitude. As these poles can be identified with the closed-string excitation modes, we can maintain the one-loop unitarity in the mixed system of the open and closed strings.

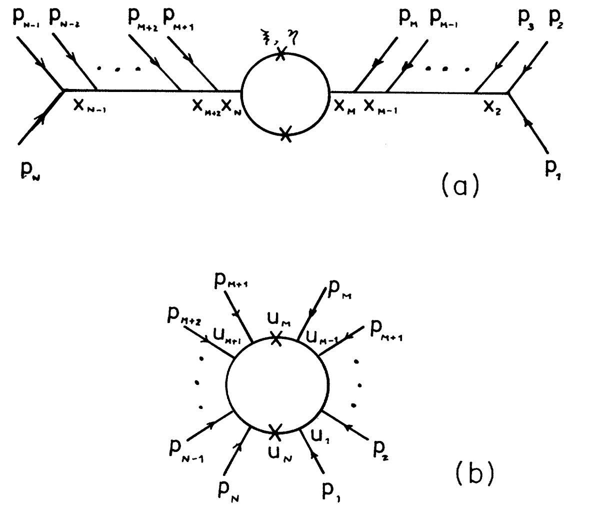

FIG. 4. The two different parametrizations of the N-point one-loop nonplanar amplitude. They are transformed with each other by a dual transformation.

III. NONPLANAR TWO-LOOP TADPOLE OPERATOR

Now let us construct the nonplanar two-loop tadpole operator shown in Fig. 1(d). This operator has not yet been obtained even in the old dual-resonance model. This operator is necessary for constructing an arbitrary multiloop amplitude. As a simple example, we can easily see that the nonplanar multiloop tadpole cannot be written by using only one-loop primitive operators, i.e., operators given in Figs. 1(a)–1(c).

A. Construction

Nonplanar two-loop tadpole operator $\langle\mathcal{T}^{(2)}|$ is built from the reduced 5-Reggeon vertex by sewing leg 1 to leg 4, and leg 2 to leg 5 with twisted propagators (see Fig. 5) and is given as

$$\begin{aligned}\langle\mathcal{T}^{(2)}|&=\frac{d^Dk_1}{(2\pi)^b}\int\frac{d^Dk_2}{(2\pi)^b}\int_0^1\frac{du}{u(1-u)}\int_0^1\frac{dv}{v(1-v)}\mathrm{Tr}^{14}\mathrm{Tr}^{25}[\langle\tilde{V}_{12345}|(b_0^1-b_1^1)\mathcal{P}^{(1)}(u)(b_0^2-b_1^2)\mathcal{P}^2(v)|R_{11^+}\rangle|R_{22^+}\rangle]\\&=\int\frac{d^Dk_1}{(2\pi)^b}\int\frac{d^Dk_2}{(2\pi)^b}\int_0^1\frac{du}{u(1-u)}\frac{dv}{v(1-v)}\frac{dx}{x(1-x)}\frac{dy}{y(1-y)}\langle\mathcal{T}^{(2)x}|\otimes\langle\mathcal{T}^{(2)\mathrm{gh}}|\ .\end{aligned} \tag{3.1}$$

The orbital sector is given by

$$\begin{aligned}\langle\mathcal{T}^{(2)x}|&=\mathrm{Tr}^{14}\mathrm{Tr}^{25}[\langle\tilde{V}^{\,x}_{12345}|\mathcal{P}^{(1)}(u)\mathcal{P}^{(2)}(v)|R_{11^+}\rangle|R_{22^+}\rangle]\\&=\det{}^{-D}(1-\bar{P}_1)\det{}^{-D}(1-\bar{P}_2)\det{}^{-D/2}[1-(J_1-1)(J_2-1)]\omega_1^{k_1^2/2}\omega_2^{k_2^2/2}\\&\quad\times{}_3\langle 0a|\exp\left\{\tfrac{1}{2}\sum_{i,j=1,2}\left[\left|\frac{1}{\beta_i}\right|-\left|\frac{1}{\alpha_i}\right|\right]Q_{ij}^{(1+2)}[|\alpha_j\rangle-|\beta_j\rangle]k_ik_j\right.\\&\quad\left.-\sum_{i=1,2}(a^3|\Gamma Q_i^{(1+2)}[|\alpha_i\rangle-|\beta_i\rangle]k_i-\tfrac{1}{2}(a^3|\Gamma(J^{(1+2)}-1)|a^3)\right\}\ .\end{aligned} \tag{3.2}$$

The details of this calculation are given in Appendix B. In this equation

$$\bar{P}_1 = V_3^{-1} V_1 \mathcal{P}^{(1)}(u) U_4 V_3, \quad \bar{P}_2 = V_3^{-1} V_2 \mathcal{P}^{(2)}(v) U_5 V_3$$

are generators of the Schottky groups $G^{(1)}$ and $G^{(2)}$, respectively. These generators correspond to the loops of Fig. 5. The variables ω_i, α_i, β_i $(i=1,2)$ are multiplier and fixed points of the generator $\bar{P}_i$, and $J_i = \sum_{l=-\infty}^{\infty} \bar{P}_i^l$ are the sum of all elements of the Schottky group $G^{(i)}$. In the exponent of Eq. (3.2), we use the notation[16]

$$J^{(1+2)} = J_2 \frac{1}{1-(J_1-1)(J_2-1)} J_1 = \sum_{\alpha \in G^{(1+2)}} \bar{P}_\alpha , \quad (3.3)$$

$$Q_i^{(1+2)} = J_j \frac{1}{1-(J_i-1)(J_j-1)} = \sum_{\alpha \in G^{(1+2)}}^{(i)} \bar{P}_\alpha \quad (i \neq j) , \quad (3.4)$$

$$Q_{ij}^{(1+2)} = \frac{1}{1-(J_j-1)(J_i-1)} = \sum_{\alpha \in G^{(1+2)}}^{(i,j)} \bar{P}_\alpha \quad (i \neq j) , \quad (3.5a)$$

$$Q_{ii}^{(1+2)} = (J_j-1) \frac{1}{1-(J_i-1)(J_j-1)} = \sum_{\alpha \in G^{(1+2)}}^{(i,i)} \bar{P}_\alpha \quad (i \neq j) . \quad (3.5b)$$

In each equation, $\bar{P}_\alpha$ are the elements of the Schottky group $G^{(1+2)}$; this group is generated by $\bar{P}_1$ and $\bar{P}_2$. The summation $\sum_\alpha^{(i)}$ is over all elements of $G^{(1+2)}$ which do not end by $\bar{P}_i$ or $\bar{P}_i^{-1}$, and $\sum_\alpha^{(i,j)}$ excludes the elements which begin with $\bar{P}_i$ or $\bar{P}_i^{-1}$ and end by $\bar{P}_j$ or $\bar{P}_j^{-1}$. When we denote the multiplier of $\bar{P}_\alpha$ by ω_α, the determinant part of Eq. (3.2) is rewritten as[16]

$$\det{}^{-D}(1-\bar{P}_1)\det{}^{-D}(1-\bar{P}_2)\det{}^{-D/2}[1-(J_1-1)(J_2-1)] = \prod_{\alpha \in G^{(1+2)}}' \prod_{n=1}^{\infty} (1-\omega_\alpha^n)^{-D} , \quad (3.6)$$

where $\prod_\alpha'$ denotes the product over all primitive elements of the Schottky group $G^{(1+2)}$; the elements cannot be written as powers of other elements.

Now we go ahead with the calculation of the ghost sector:

$$\langle \mathcal{T}^{(2)\mathrm{gh}} | = \mathrm{Tr}^{14}\mathrm{Tr}^{25}[\langle \tilde{V}_{12345}^{\mathrm{gh}} | (b_0^1 - b_1^1)\mathcal{P}^{(1)}(u)(b_0^2 - b_1^2)\mathcal{P}^{(2)}(v) | R_{11^+}\rangle | R_{22^+}\rangle] ,$$
$$= \frac{1}{\Delta} \prod_{\alpha \in G^{(1+2}}' \prod_{n=2}^{\infty} (1-\omega_\alpha^n)^2 \langle q=3 | \exp\{-[c^3 | \Gamma(J^{(1+2)}-1) | b^3]\} [b_{-1}^3 - (1-\Delta^2) b_0^3] \quad (3.7)$$

and

$$\Delta = \frac{xy(1-x)(1-y)uv}{(1-xy)(1-u)(1-v)} . \quad (3.8)$$

Note that the partition function in the ghost sector appears from the modes $n=2$. This situation is the same as that of the planar multiloop case.[5,11,13] The expression of the nonplanar two-loop tadpole operator is similar to that of the planar two-loop tadpole operator. The reason is that the period matrix, the first Abelian integral, and the prime form characterizing the loop amplitude are defined on the same Riemann double of the original open surface,[15] that is, the sphere with two handles. They are distinguished by the way of cutting the Riemann double to the two open surfaces, as depicted in Fig. 6. It amounts to the difference of arrangements of the isometric circles I_i and I_i^{-1} of the Schottky generators $\bar{P}_i$ $(i=1,2)$ on the real axis; in the planar case, the order is $I_1, I_1^{-1}, I_2, I_2^{-1}$,

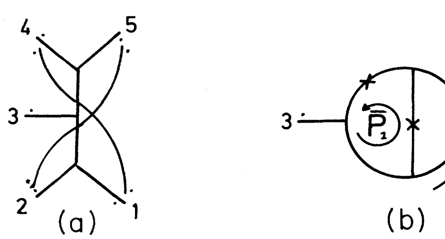

FIG. 5. The nonplanar two-loop tadpole operator: (a) the diagrammatic construction, (b) two generators.

whereas in the nonplanar case it is $I_1, I_2, I_1^{-1}, I_2^{-1}$.[15]

Finally we identify the coefficients of each term in the exponent with these mathematical quantities on the Riemann double mentioned above. This can be achieved in the same way as the planar case.[11,17]

(a) *The momentum linear term:*

$$\Gamma Q_i^{(1+2)}[|\alpha_i) - |\beta_i)] \equiv \sum_\alpha^{(i)} \sum_{n=1}^{\infty} D_{nm}^{(0)}(\Gamma P_\alpha) \frac{1}{\sqrt{m}} \times (\alpha^m - \beta^m) = \frac{\sqrt{n}}{n!} \left[\frac{\partial}{\partial z} \right]^n \phi_i(z) \Big|_{z=0} , \quad (3.9)$$

where $\phi_i(z)$ is the first Abelian integral associated with the ith loop, as shown in Fig. 7. They are given explicitly in the Schottky parametrization as

$$\phi_i(z) = \sum_\alpha^{(i)} \ln \left| \frac{z - \hat{\bar{P}}_\alpha(\beta_i)}{z - \hat{\bar{P}}_\alpha(\alpha_i)} \right| . \quad (3.10)$$

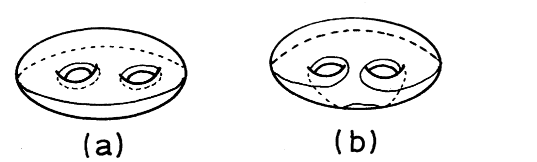

FIG. 6. The Riemann double of (a) the two-loop nonplanar diagram and (b) the two-loop planar diagram.

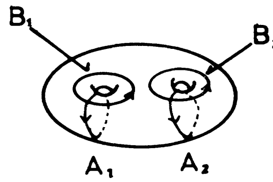

FIG. 7. The canonical cycles.

For the two canonical cycles A_i, B_i $(i=1,2)$ shown in Fig. 7, they are normalized as

$$\oint_{A_i} d\phi_j(z)=2\pi i\delta_{ij}, \quad \oint_{B_i} d\phi_j(z)=2\pi i\tau_{ij} , \tag{3.11}$$

where τ_{ij} is the period matrix defined by

$$\tau_{ij}=\frac{1}{2\pi i}\left\{\delta_{ij}\ln\omega_i + \sum_\alpha{}^{(i,j)}\ln\frac{[\alpha_i-\hat{\bar{P}}_\alpha(\alpha_j)][\beta_i-\hat{\bar{P}}_\alpha(\beta_j)]}{[\alpha_i-\hat{\bar{P}}_\alpha(\beta_j)][\beta_i-\hat{\bar{P}}_\alpha(\alpha_j)]}\right\} . \tag{3.12}$$

It is easy to check that τ_{ij} is connected with the coefficient of the momentum bilinear term.

(b) *The momentum bilinear term:*

$$\left[\left[\frac{1}{\beta_i}\right]-\left[\frac{1}{\alpha_i}\right]\right]Q_{ij}^{(1+2)}[\,|\alpha_j)-|\beta_j)]-\delta_{ij}\ln\omega_i =2\pi\,\mathrm{Im}\tau_{ij} . \tag{3.13}$$

(c) *The operator bilinear term.* The coefficient of the orbital sector is

$$[\Gamma(J^{(1+2)}-1]^x\equiv \sum_{\alpha\in G^{(1+2)}/\{I\}} D_{nm}^{(0)}(\Gamma\bar{P}_\alpha) =\frac{\sqrt{nm}}{n!m!}\left[\frac{\partial}{\partial z}\right]^n\left[\frac{\partial}{\partial y}\right]^m \times\ln\frac{E(z,y)}{z-y}\Bigg|_{z=y=0} , \tag{3.14}$$

where $E(x,y)$ is the prime form defined by

$$E(z,y)=(z-y)\prod_{\alpha\in G^{(1+2)}/\{1\}}{}'' \frac{[z-\hat{\bar{P}}_\alpha(y)][y-\hat{\bar{P}}_\alpha(z)]}{[z-\hat{\bar{P}}_\alpha(z)][y-\hat{\bar{P}}_\alpha(y)]} . \tag{3.15}$$

Here $\prod_\alpha''$ denotes the product over elements of $G^{(1+2)}$ except for the identity, where $\bar{P}_\alpha$ and $\bar{P}_\alpha^{-1}$ are counted only once.

As in the orbital sector, the bilinear coefficient in the exponent of the ghost sector can be written as

$$[\Gamma(J^{(1+2)}-1)]^{\mathrm{gh}}\equiv \sum_{\alpha\in G^{(1+2)}/\{1\}} D_{nm}^{(1)}(\Gamma\bar{P}_\alpha) =\frac{1}{(n-2)!(m+1)!}\left[\frac{\partial}{\partial x}\right]^{n-2}\left[\frac{\partial}{\partial y}\right]^{m+1} \times\left[\mathcal{G}(x,y)-\frac{1}{x-y}\right]_{x=y=0} , \tag{3.16a}$$

where

$$\mathcal{G}(x,y)=\sum_{\alpha\in G^{(1+2)}}\left[\frac{d\hat{\bar{P}}_\alpha(y)}{dy}\right]^{-1}\frac{1}{x-\hat{\bar{P}}_\alpha(y)}$$

or

$$=\sum_{\alpha\in G^{(1+2)}}\left[-\left[\frac{d\hat{\bar{P}}_\alpha(x)}{dx}\right]^2\frac{1}{y-\hat{\bar{P}}_\alpha(x)} +\sum_{m=0,\pm1}c_m^\alpha(x)y^{m+1}\right] . \tag{3.16b}$$

In the second line, $c_m^\alpha(x)$ is a certain function of x. Equation (3.16) indicates that the bilinear coefficients $n,m\geq 2$ are simultaneously the differential coefficients of the automorphic form with a weight of -1 in y and 2 in x.

With using these mathematical quantities, the nonplanar two-loop tadpole operator is rewritten as

$$\langle \mathcal{T}^{(2)}|=\int\frac{d^Dk_1}{(2\pi)^b}\int\frac{d^Dk_2}{(2\pi)^b}\int_0^1\frac{du}{u(1-u)}\frac{dv}{v(1-v)}\frac{dx}{x(1-x)}\frac{dy}{y(1-y)}\frac{1}{\Delta}\prod_{\alpha\in G^{(1+2)}}{}'\frac{1}{1-\omega_\alpha}\prod_{n=1}^\infty(1-\omega_\alpha^n)^{-(D-2)}$$
$$\times\langle 0a;q=3|\exp\left[-\tfrac{1}{2}\sum_{i,j=1,2}k_ik_j2\pi\,\mathrm{Im}\tau_{ij}-\sum_{j=1,2}\sum_{n=1}^\infty a_n\frac{\sqrt{n}}{n!}\left[\frac{\partial}{\partial z}\right]^n\phi_j(z)\Bigg|_{z=0}k_j\right.$$
$$-\tfrac{1}{2}\sum_{n,m=1}^\infty a_n\frac{\sqrt{nm}}{n!m!}\left[\frac{\partial}{\partial z}\right]^n\left[\frac{\partial}{\partial y}\right]^m\ln\frac{E(z,y)}{z-y}\Bigg|_{z=y=0}a_m$$
$$-\sum_{n,m=2}c_n\frac{1}{(n-2)!(m+1)!}\left[\frac{\partial}{\partial x}\right]^{n-2}\left[\frac{\partial}{\partial y}\right]^{m+1}$$
$$\left.\times\left[\mathcal{G}(x,y)-\frac{1}{x-y}\right]\Bigg|_{x=y=0}b_m\right] . \tag{3.17}$$

Before neglecting the cross terms in the ghost sector, $\langle \mathcal{T}^{(2)}|$ satisfies the BRST invariance in the form

$$\langle \mathcal{T}^{(2)}|TQ_B=0 \ .$$

This is also true after restricting the integration region, if one makes the divergences at the boundary regularize suitably as in the nonplanar self-energy case.

B. Integration region

In order to exclude a multiple counting of the same physical configurations, we next consider to restrict the integration region of Eq. (3.17). We will achieve this by taking account of the duality of the 5-Reggeon vertex and the periodicity associated with the loop part of the operator.

Let us first consider the duality property of the reduced 5-Reggeon vertex $\langle \tilde{V}_{12345}(x,y)|$ which is specified by the two Chan variables (x,y) with integration region $0\le x,y\le 1$ as shown in Fig. 5. The duality of the reduced 5-Reggeon vertex $\langle \tilde{V}_{12345}(x,y)|$ means that under the dual transformation

$$(x,y)\rightarrow \left[\frac{1-x}{1-xy},1-xy\right] , \tag{3.18}$$

the following relation is satisfied (Fig. 8):

$$\langle \tilde{V}_{12345}(x,y)|=\left\langle \tilde{V}_{23451}\left[\frac{1-x}{1-xy},1-xy\right]\right| . \tag{3.19}$$

By repeating this relation, it is easy to see the following chain of the duality relations:

$$\begin{aligned}\langle \tilde{V}_{12345}(x,y)| &=\left\langle \tilde{V}_{23451}\left[\frac{1-x}{1-xy},1-xy\right]\right| \\ &=\left\langle \tilde{V}_{34512}\left[\frac{1-y}{1-xy},x\right]\right| \\ &=\left\langle \tilde{V}_{45123}\left[y,\frac{1-x}{1-xy}\right]\right| \\ &=\left\langle \tilde{V}_{51234}\left[1-xy,\frac{1-y}{1-xy}\right]\right| \\ &=\langle \tilde{V}_{12345}(x,y)| \ .\end{aligned} \tag{3.20}$$

Note that each of the five configurations in the chain corresponds to one Feynman diagram. Furthermore one can show that the integration region for (x,y) can be decomposed into the five regions which corresponds to these

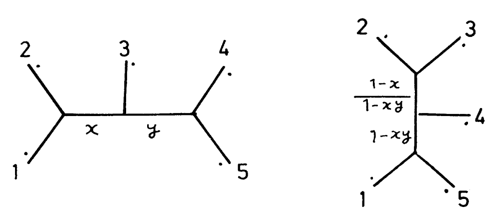

FIG. 8. A typical duality relation of the five-point vertex.

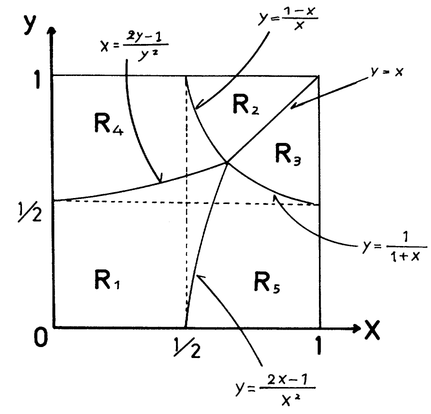

FIG. 9. The five regions corresponding to the five Feynman diagrams. Here the fixed point is $x=y=(-1+\sqrt{5})/2$.

five Feynman diagrams as depicted in Fig. 9:

$$\begin{aligned}\langle \tilde{V}_{12345}| &= \int_0^1 dx \int_0^1 dy \langle \tilde{V}_{12345}(x,y)| \\ &= \sum_{r=1}^{5} \int_{R_r} dx\, dy \langle V_{F_r}(x,y)| \ ,\end{aligned} \tag{3.21}$$

where $\langle V_{F_r}(x,y)|$ $(r=1-5)$ denotes the five vertices in Eq. (3.20). Note also that the five-point Koba-Nielsen amplitude has the poles corresponding to the five Feynman diagrams at the five points $(x,y)=(0,0)$, $(1,1),(1,0),(0,1),(1,1)$. Now it is clear that if one uses the left-hand side of Eq. (3.21) as the building block of the nonplanar two-loop tadpole, one has five times overcounting inevitably. We thus restrict the region $0\le x,y\le 1$ to one of the regions in Fig. 9. We chose, for example, R_1, which corresponds to the left configuration of Fig. 8.

We next consider the overcounting originated in the periodicity of the loop part. As can be easily found in Fig. 1(d), the nonplanar two-loop tadpole includes the nonplanar self-energy as a part of it. Let us consider to factorize this part. For doing this, one should find suitable variables in the nonplanar two-loop tadpole, which correspond to the two variables $\tilde{u}$ and $\tilde{v}$ used in the nonplanar self-energy. First note that due to the restriction to R_1, one can only consider the configuration illustrated in Fig. 5. As discussed in Sec. II the Schottky generator which corresponds to the nonplanar self-energy loop is given by $\hat{\bar{P}}(z)=\omega z+1$ [see Eq. (2.8)], where the multiplier ω is defined in terms of $\tilde{u}$ and $\tilde{v}$ by Eq. (2.9).

From Fig. 5, one can guess the generator of the nonplanar self-energy part is given by the combination

$$\bar{P}_3=\bar{P}_1^{-1}\bar{P}_2 \ . \tag{3.22}$$

This turns out to be true, if one changes the variables u and v in Eq. (3.1) to $\tilde{u}$ and $\tilde{v}$ in such a way that

$$\begin{aligned}\tilde{u} &=\frac{u(1-x)(1-y)}{1-ux-uy+uxy} , \\ \tilde{v} &=\frac{v}{1-\tilde{u}+\tilde{u}v} .\end{aligned} \tag{3.23}$$

These changes of variables are the results of the manipulations

$$\tilde{P}_1 \equiv V_1 \mathcal{P}(u) U_4 = (V_1 S^{-1}(x))\mathcal{P}(\tilde{u})\Gamma(V_4 S^{-1}(y))^{-1} , \tag{3.24a}$$

$$\tilde{P}_2 \equiv V_2 \mathcal{P}(v) U_5 = (V_2 S(\tilde{u}))\mathcal{P}(\tilde{v}) U_5 . \tag{3.24b}$$

The right-hand sides of Eq. (3.24) indicate that the use of the new variables u, v in the nonplanar two-loop tadpole corresponds to attaching the same gauge operators on the same position as the nonplanar self-energy (see Fig. 3). Then one can show that $\bar{P}_3$ becomes

$$\bar{P}_3 = \begin{bmatrix} \dfrac{\tilde{v}}{1-\tilde{v}} & 1 \\ \dfrac{\tilde{v}(1-\tilde{u})}{1-\tilde{v}} & 1 \end{bmatrix} \tag{3.25}$$

and that the multiplier ω_3 of P_3 is given by the same equation as of the nonplanar self-energy:

$$\frac{\omega_3}{(1+\omega_3)^2} = \tilde{u}\,\tilde{v}(1-\tilde{v}) . \tag{3.26}$$

This coincidence indicates the equivalence of $\bar{P}_3$ and $\tilde{P}$ up to a similarity transformation. In fact, one finds this transformation can be generated by the gauge operator $S(f)$ defined in Sec. II:

$$\tilde{P} = S(f)^{-1}\bar{P}_3 S(f) . \tag{3.27}$$

Now following the argument in Sec. II, one has an overcounting associated with this part, but this can be excluded in the same way by restricting the integration regions of $\tilde{u}$ and $\tilde{v}$ to $\frac{1}{2} \le \tilde{u} \le 1$ and $0 \le \tilde{v} \le \frac{1}{2}$ [see Eq. (2.11)].

As will be shown in the next paragraph, no more loops can be factorized from the nonplanar two-loop tadpole simultaneously with the above nonplanar self-energy. We are thus sure that there are no remaining symmetries which yield the overcounting.

We finally derive a nonplanar two-loop N-tachyon amplitude by saturating $\langle \mathcal{T}^{(2)}|$ with N-tachyon states $|p_1 p_2 \cdots p_N\rangle$ given by (2.18). In order to express the measure in terms of the one appearing in the nonplanar self-energy, we insert $1 = S(f) S(f)^{-1}$ between the external legs of the nonplanar self-energy part and the propagators $\mathcal{P}(x)$ and $\mathcal{P}(y)$ of the 5-Reggeon vertex. The gauge operator $S(f)$ changes the propagator variables x and y to

$$\tilde{x} = \widehat{S(f)^{-1}}(x), \quad \tilde{y} = \widehat{S(f)^{-1}}(y) ,$$

where

$$\tilde{x} = \frac{x\left[1 - \dfrac{f}{f-1}\right]}{1 - \dfrac{f}{f-1}x}, \quad \text{etc.} \tag{3.28}$$

Then the amplitude is given by

$$\begin{aligned} \mathcal{A}^{(2)\,\mathrm{NP}}_{N\ \mathrm{tach}}(p_1 p_2 \cdots p_N) &= \langle \mathcal{T}^{(2)} | p_1 p_2 \cdots p_N \rangle \\ &= \int_{\tilde{R}_1} \frac{d\tilde{x}\, d\tilde{y}(1-\tilde{x}\,\tilde{y})}{\tilde{x}^{\,2}(1-\tilde{x})\tilde{y}^{\,2}(1-\tilde{y})} \int_0^1 \frac{d\omega_3}{\omega_3^2} \int_{\omega_3}^1 \frac{d\xi}{\xi} \int \prod_{r=2}^{N} d\rho_r \theta(\rho_{r+1}-\rho_r) \\ &\quad \times \prod_\alpha{}' \prod_{n=1}^{\infty} (1-\omega_\alpha^n)^{-D} \prod_{n=2}^{\infty} (1-\omega_\alpha^n)^2 (\det \mathrm{Im}\tau)^{-D/2} \\ &\quad \times \prod_{1 \le r < s \le N} \left[E(\rho_s,\rho_r) \exp\left[-\tfrac{1}{2} \sum_{ij=1,2} \int_{\rho_r}^{\rho_s} d\phi_i (2\pi\, \mathrm{Im}\tau)^{-1}_{ij} \int_{\rho_r}^{\rho_s} d\phi_j \right] \right]^{p_r p_s} . \end{aligned} \tag{3.29}$$

C. Singularity

We here discuss the divergence originated in the partition function $\prod_\alpha \prod_{n=1}^{\infty} (1-\omega_\alpha^n)^{-D}$. As discussed in Ref. 21, the planar two-loop operator has only three sources of divergences associated with the multipliers ω_1, ω_2, and ω_{12}, which can independently reach 1, although there are an infinite number of primitive elements in the Schottky group $G^{(2\ \text{loop planar})}$. The reason why this happens is due to the existence of the linear dependence among the elements. In the nonplanar two-loop case, although the expression is superficially the same as the planar one except for the measure factor, the singularity structure is quite different; there may be only one independent source. That is to say, there is a stronger linear dependence in the nonplanar case. This may be seen first in the fact that the two loops in the planar case can be decomposed to the two one-loop tadpoles, while it is impossible in the nonplanar case. More minutely, this can be shown in the following way. In the nonplanar two-loop diagram, there are three possibilities in the way of pulling out a tube, corresponding to the three limits ω_1, ω_2, and ω_3 going to 1 as illustrated in Fig. 10. The tube may correspond to the propagation of closed string. Note that the diagram associated with the limit $\omega_3 \to 1$ just corresponds to the case which we have discussed as the factorization of the nonplanar self-energy operator. It is, however, not possible that these three occur simultaneously. This indicates the linear dependence among the three elements $\bar{P}_1$, $\bar{P}_2$, and $\bar{P}_3$. Indeed, by means of the isometric circles, it is also possible to show that in the case that ω_3 can go to one the other two multipliers ω_1, ω_2 cannot reach one, that is, if I_3^{-1} and I_3 are tangent with each other, the oth-

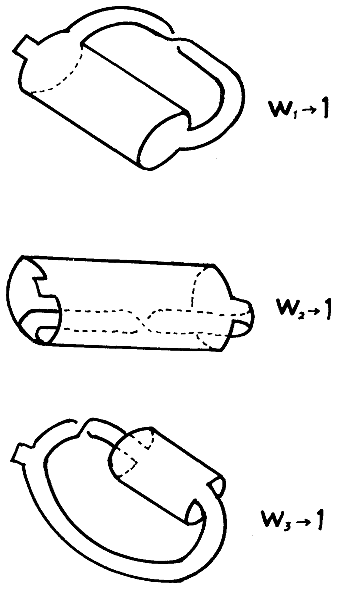

FIG. 10. The three possible ways of pulling a tube from the two-loop nonplanar tadpole diagram.

er pairs cannot become tangent.

The more complicated case, for example, the case corresponding to the multiplier $\omega_1^n \omega_3^m$, also cannot reach 1, because no tubes correspond to these limits are pulled out independently from the above discussed case as in the planar case.[21]

IV. CONCLUDING REMARKS

We have calculated the nonplanar self-energy and the nonplanar two-loop tadpole operators according to the BRST-invariant operator formalism of the open bosonic string. They satisfy the desired BRST relations which imply the decoupling of the unphysical modes from these operators. In particular, in the one-loop amplitude we have shown that the nontrivial ghost contribution makes the result be a correct one. In the nonplanar two-loop tadpole, we have determined the integration region by restricting the symmetries originated in the duality of the tree vertex and the periodicity of the loop part. It is, however, a future problem to prove the unitarity of the amplitude in such a determination of integration regions. It is also interesting to compare this issue with the argument of the modular invariance.

We now finished the evaluation of all primitive operators in the open-bosonic-string theory. We can compute all diagrams by using these operators and trees. In such evaluations our method of parametrizing the moduli space in terms of the Schottky variables is more powerful as compared with the Polyakov approach based on the Teichmuller parameters. Actually the evaluation of planar multiloop diagram has already been carried out by one of the authors[11] (H.K.) and Cristofano and coworkers,[13] and some rules for the computation of the general multiloop diagrams including nonplanar and nonorientable multiloops are obtained.[25]

However, our task cannot be finished. The remaining important problems appear in the unitarization program. We may summarize them in the following two problems.

The first problem we must study is the analysis of the singularity of the amplitude. In this paper, we have only developed the use of the primitive operators. The argument for the singularities arising from the primitive operators is unfortunately not enough to clarify the singularity structure of string amplitudes. Some new singularities can arise associated with more than two loops. The determination of the number of independent singularities has been confirmed only in the two-loop order. We discussed in Sec. III C that, in the two-loop order, three singularities in the planar tadpole and one singularity in the nonplanar tadpole have a correspondence to the independent ways of pulling out a tube. By extending this consideration to the general case, we can have a conjecture that in any type of amplitude-independent singularities can only arise associated with a set of loops from which tubes can be pulled out independently. Then a planar g-loop amplitude, for example, has $2g-1$ independent singularities. This conjecture is also natural, in the sense that these singularities are originated in the closed-string tachyon or dilaton tadpoles at zero momentum for the planar and nonorientable diagrams and closed-string intermediate states for the nonplanar diagrams.[26,27] As far as regularization of the amplitudes different schemes have been proposed in the one-loop level. It is shown that in order to cancel the divergences each scheme yields a different gauge group. The investigation of the regularization and the cancellation of the divergences in more than two loops may give us an idea to chose one of the schemes.

Even if we could overcome the above problem, we must further solve the problem of how to sum up the different diagrams in order to satisfy the perturbative unitarity. Because we do not yet have any reliable string field theory at present and there is no rule to determine the relative weight among different diagrams in our scheme, we must determine them such that the resulting sum is perturbatively unitary. Recently, this problem has been solved by introducing the completely symmetric expression of the N-point vertex in terms of the unified propagator.[28]

ACKNOWLEDGMENTS

One of the authors (H.K.) would like to thank Fuju-kai for financial support.

APPENDIX A: NOTATION

We summarize here our notation and properties of the $SL(2,R)$ operator and its infinite-dimensional representations. All matrices appearing in the operator formalism are the elements of $SL(2,R)$:

$$\Lambda=\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \det\Lambda=1 ,$$

with

$$\Lambda(z)=\frac{az+b}{cz+d}\ .$$

The infinite-dimensional representations of Λ are defined using the general representation

$$\mathcal{D}_{nm}^{(J,P-J)}(\Lambda)=\frac{1}{(m+P)!}\frac{d^{m+P}}{dz^{m+P}}\times\left[\left[\frac{d\hat{\Lambda}(z)}{dz}\right]^{-J}(\hat{\Lambda}(z))^{n+P}\right]\Bigg|_{z=0}\ . \tag{A1}$$

(1) *Orbital sector:*

$$D_{nm}^{(0)}(\Lambda)\equiv\sqrt{m/n}\,\mathcal{D}_{nm}^{(0,0)}(\Lambda),\quad n,m\geq 1\ , \tag{A2}$$

$$D_{n0}^{(0)}(\Lambda)=\frac{1}{\sqrt{n}}\left[\frac{b}{d}\right]^{n}\equiv(\Lambda)_0,\quad n\geq 1\ , \tag{A3}$$

$$D_{0n}^{(0)}(\Lambda)=\frac{1}{\sqrt{n}}\left[\frac{-c}{d}\right]^{n}\equiv{}_0(\Lambda),\quad n\geq 1\ , \tag{A4}$$

$$D_{00}^{(0)}(\Lambda)=-\ln|d|\equiv{}_0(\Lambda)_0\ . \tag{A5}$$

We often use the notation

$$(a|\Lambda|a)\equiv\sum_{n=1}a_nD_{nm}^{(0)}(\Lambda)a_m\ . \tag{A6}$$

The vector denoted $|\xi)$ whose nth component is $(1/\sqrt{n})\xi^n$ satisfies

$$(a|\xi)=\sum_{n=1}a_n\xi^n/\sqrt{n}\ , \tag{A7}$$

$$(\eta|\xi)=-\ln(1-\xi\eta)\ , \tag{A8}$$

and

$$\Lambda|\xi)=|\hat{\Lambda}(\xi))-|\hat{\Lambda}(0))\ . \tag{A9}$$

(2) *Ghost sector:*

$$D_{nm}^{(1)}(\Lambda)\equiv\mathcal{D}_{nm}^{(1,0)}(\Lambda)\ , \tag{A10}$$

and we also use

$$[c|\Lambda|b]\equiv\sum_{n,m=2}c_nD_{nm}^{(1)}(\Lambda)b_m\ . \tag{A11}$$

APPENDIX B: CALCULATION OF THE NONPLANAR TWO-LOOP TADPOLE OPERATOR

We give here the details of the calculation of the nonplanar two-loop tadpole operator given in Sec. III A. We begin by the calculation of the orbital sector, which is given explicitly as

$$\langle\mathcal{T}^{(2)x}|=\mathrm{Tr}^{14}\mathrm{Tr}^{25}\left[\langle 0a|\exp\left[-\sum_{r=1}^{5}(a^r|\mu_r)-\tfrac{1}{2}\sum_{\substack{r\neq s\\ r=1}}^{5}[a^r|\tilde{U}_r\tilde{V}_s|a^s)+p_rp_s\,{}_0(\tilde{U}_r\tilde{V}_s)_0]\right]|R_{11^+}\rangle\right]|R_{22^+}\rangle$$

$$=\langle 0a|\exp\left[-\tfrac{1}{2}\sum_{\substack{r\neq s\\ r=1}}^{5}p_rp_s\,{}_0(\tilde{U}_r\tilde{V}_s)_0-(a_3|\mu_3)\right]\mathcal{M} \tag{B1}$$

with $p_3=0$, $-p_1=p_4=k_1$, $-p_2=p_5=k_2$. We used the notations

$$|\mu_r)=\sum_{\substack{s=1\\ (s\neq r)}}^{5}(U_rV_s)_0p_s\ , \tag{B2}$$

where

$$\tilde{U}_1=\mathcal{P}(u)U_1,\quad \tilde{U}_2=\mathcal{P}(v)U_2,\quad \tilde{V}_1=V_1\mathcal{P}(u),\quad \tilde{V}_2=V_2\mathcal{P}(v)$$

and

$$\tilde{U}_r=U_r,\quad \tilde{V}_r=V_r,\quad \text{for } r=3,4,5\ .$$

In Eq. (B1), $\mathcal{M}$ is the trace part given by

$$\mathcal{M}=\mathrm{Tr}^{14}\mathrm{Tr}^{25}\left\{\langle 0a|\exp\left[-\tfrac{1}{2}(a^r|U_rV_s|a^5)-\sum_{\substack{r=1\\ (\neq 3)}}^{5}(a^r|\mu_r)\right]|R_{11^+}\rangle|R_{22^+}\rangle\right\}$$

$$=\int d^2\xi\,d^2\eta\,\exp[-\tfrac{1}{2}\mathbf{z}^TA\mathbf{z}+\mathbf{z}^T\mathbf{B}] \tag{B3}$$

with $\mathbf{z}^t=[\bar{\xi}\xi\bar{\eta}\eta]$. The matrix A and the vector $\mathbf{B}$ are given by

$$A=\begin{bmatrix} 0 & 1-\tilde{U}_1V_4 & \tilde{U}_1\tilde{V}_4 & -\tilde{U}_1V_5 \\ 1-U_4\tilde{V}_1 & 0 & -U_4\tilde{V}_2 & U_4V_5 \\ \tilde{U}_2\tilde{V}_1 & -\tilde{U}_2V_4 & 0 & 1-\tilde{U}_2V_5 \\ -U_5\tilde{V}_1 & U_5V_4 & 1-U_5\tilde{V}_2 & 0 \end{bmatrix} \equiv \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix} , \tag{B4}$$

$$\mathbf{B}=\begin{bmatrix} |\mu_1)+\tilde{U}_1V_3|a_3) \\ -|\mu_4)-U_4V_3|a_3) \\ |\mu_2)+\tilde{U}_2V_3|a_3) \\ -|\mu_5)-U_5V_3|a_3) \end{bmatrix} . \tag{B5}$$

After the integrations over $\xi,\bar{\xi}$ and $\eta,\bar{\eta}$ we get the trace part $\mathcal{M}$ as

$$\mathcal{M}=\det{}^{-D/2}A\,\exp(\tfrac{1}{2}\mathbf{B}^TA^{-1}\mathbf{B}) . \tag{B6}$$

Then we estimate $\det A$ and A^{-1} according to Appendix B of Ref. 16.

$$\det A=\det{}^2(1-\tilde{P}_1)\det{}^2(1-\tilde{P}_2)\det[1-(\tilde{J}_1-1)(\tilde{J}_2-1)] , \tag{B7}$$

where

$$\tilde{P}_1=V_1\mathcal{P}(u)U_4,\quad \tilde{J}_1=\sum_{l=-\infty}^{\infty}\tilde{P}_1^{\,l}\quad \tilde{P}_2=V_2\mathcal{P}(u)U_5,\quad \tilde{J}_2=\sum_{l=-\infty}^{\infty}\tilde{P}_2^{\,l} .$$

The inverse of A is given by

$$A^{-1}=\begin{bmatrix} \Sigma_{11}^{-1}\Sigma_{12}\Sigma_{22}^{-1} & -\Sigma_{11}^{-1} \\ -\Sigma_{22}^{-1} & \Sigma_{22}^{-1}\Sigma_{21}\Sigma_{11}^{-1} \end{bmatrix} \begin{bmatrix} 0 & \dfrac{1}{1-\Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12}\Sigma_{22}^{-1}} \\ \dfrac{1}{1-\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}\Sigma_{11}^{-1}} & 0 \end{bmatrix} . \tag{B8}$$

After the lengthy calculation with using the definition (B2), we get the exponent of (B1) as

$$\begin{aligned} -\tfrac{1}{2}\mathbf{B}^TA^{-1}\mathbf{B}= & -\tfrac{1}{2}(a^3|U_3\left[\tilde{J}_2\frac{1}{1-(\tilde{J}_1-1)(\tilde{J}_2-1)}\tilde{J}_1-1\right]V_3|a_3) \\ & -(a^3|U_3\tilde{J}_1\frac{1}{1-(\tilde{J}_2-1)(\tilde{J}_1-1)}|\mu_{14})-(a^3|U_3\tilde{J}_2\frac{1}{1-(\tilde{J}_1-1)(\tilde{J}_2-1)}|\mu_{25}) \\ & -\tfrac{1}{2}(\mu_{14}|(\tilde{J}_2-1)\frac{1}{1-(\tilde{J}_1-1)(\tilde{J}_2-1)}|\mu_{14})-(\mu_1|\sum_{l=0}(U_4\tilde{V}_1)^l|\mu_4) \\ & -\tfrac{1}{2}(\mu_{25}|(\tilde{J}_1-1)\frac{1}{1-(\tilde{J}_2-1)(\tilde{J}_1-1)}|\mu_{25})-(\mu_2|\sum_{l=0}(U_5\tilde{V}_2)^l|\mu_5)-(\mu_{14}|\frac{1}{1-(\tilde{J}_2-1)(\tilde{J}_1-1)}|\mu_{25}) , \end{aligned} \tag{B9}$$

where

$$|\mu_{14})=\sum_{l=0}\tilde{P}_1^{-l}U_1^{-1}|\mu_1)+\sum_{l=0}\tilde{P}_1^{\,l}U_4|\mu_4)\quad |\mu_{25})=\sum_{l=0}\tilde{P}_2^{-l}U_2^{-1}|\mu_2)+\sum_{l=0}\tilde{P}_2^{\,l}U_5|\mu_5) .$$

Inserting (B6) with (B7), (B10) into (B1), we get the orbital sector (3.2), where we use the relations

$$\bar{P}_1=V_3^{-1}\tilde{P}_1V_3,\quad \bar{P}_2=V_3^{-1}\tilde{P}_2^{-1}V_3 .$$

We turn to the calculation of the ghost zero-mode sector, which is given by

$$\begin{aligned} \prod_{r=1}^{5}\frac{Z_{r+2}-Z_r}{Z_{r+1}-Z_r}\mathrm{Tr}^{14}\mathrm{Tr}^{25}\Bigg[& \langle q=3|\prod_{r=1}^{5}[b_{-1}^r-b_0^r-D_{11}^{(1)}(U_rV_{r-1})b_0^{r-1}] \\ & \times\mathcal{P}_0^{(1)}(u)(b_0^1-b_{-1}^1)\mathcal{P}_0^{(2)}(v)(b_0^2-b_{-1}^2)|R_{11^+}\rangle|R_{22^+}\rangle\Bigg] , \end{aligned} \tag{B10}$$

where $\mathcal{P}_0(x)$ represents the zero-mode sector of $D_{nm}^{(1)}(\mathcal{P}(x))$, $n,m=0,\pm1$,

$$\mathcal{P}_0(x)=\begin{bmatrix} \frac{1}{x(x-1)} & \frac{-2}{x-1} & \frac{x}{x-1} \\ \frac{1}{x-1} & -\frac{x+1}{x-1} & \frac{x}{x-1} \\ \frac{x}{x-1} & -\frac{2x}{x-1} & \frac{x}{x-1} \end{bmatrix} . \tag{B11}$$

The action of $\mathcal{P}_0(x)$ under the constraint $b_0 - b_{-1}$ is

$$b_{-1}-b_0 \to \frac{x-1}{x} b_0, \quad b_0 \to \frac{x}{x-1}(b_1-b_0) . \tag{B12}$$

Apply to the trace formula given in Ref. 9, Eq. (B11) is reduced to

$$\prod_{r=1}^{5} \frac{Z_{r+2}-Z_r}{Z_{r+1}-Z_r} \frac{1-u}{u} \frac{1-v}{v} \langle q=3| \left[b^3_{-1} - b^3_0 + D^{(1)}_{11}(U_3V_2)\frac{v}{1-v} D^{(1)}_{11}(U_5V_4)\frac{u}{1-u} D^{(1)}_{11}(U_1V_5)\frac{v}{1-v} \right.$$
$$\left. \times D^{(1)}_{11}(U_2V_1)\frac{u}{1-u} D^{(1)}_{11}(U_4V_3) b^3_0 \right] . \tag{B13}$$

If we choose the set of Koba-Nielsen variables $(Z_1, Z_2, Z_3, Z_4, Z_5)$, as $(0, xy, y, 1, \infty)$ corresponding to the choice of 5-Reggeon vertex given in Fig. 5, Eq. (B14) can be written as

$$\frac{1}{\Delta}[b^3_{-1} - (1-\Delta^2) b^3_0]$$

with Δ being Eq. (3.8). The construction of $n \geq 2$ mode of the ghost sector is the same as that of the orbital sector.

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