Exact solutions of a dynamical theory of gravity in 1+1 dimensions

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A Lagrangian-based, completely geometric theory of gravitation with nontrivial dynamics in two dimensions is *possible* with a non-Riemannian geometry for spacetime. A particular model with these features is discussed. General solutions of the vacuum field equations are presented in a variety of contexts (with and without singularities).

I. INTRODUCTION

Lower-dimensional theories of gravitation have elicited considerable attention from theorists over the past decade.[1] The rationale for those who wish to better understand gravity in four spacetime dimensions is that an analysis in fewer dimensions often enables one to see more clearly general features which may have four-dimensional (4D) analogs. Examples of the successful application of this approach abound in nongravitational field theory. One obstruction to its use for gravity stems from the observation that for other field theories it is often possible to model highly symmetric structures in the laboratory and thus directly search for lower-dimensional phenomena realized in four dimensions. This is somewhat more difficult for gravitation, although some cosmological applications (cosmic strings, domain walls) may realize some of these effects.

The explosion of interest in two-dimensional (2D) σ models (string theory) provided much of the impetus for the study of generally covariant structures of 2D base spaces (string world sheets) and the later generalizations to membranes, etc. Once the string action was identified as the area of the world sheet, it was natural to add higher-order geometrical invariants (such as curvature) to form more general actions.[2]

Whatever one's reasons for looking at gravity in 2D, one immediately encounters a rather significant obstacle: namley, the well-known triviality of general relativity (GR) in two dimensions. The precise statement of triviality is usually glossed over in the literature (for a more careful study, see Refs. 3 and 4). Writing the Riemann curvature in terms of the Riemannian metric and its derivatives, the so-called "second order" or "Hilbert" formalism,[5] necessitates imposing a metricity condition upon the connection in the theory, forcing it to be the Christoffel connection. An alternative to this approach is the "first-order" or "Palatini" formalism,[6] in which the metric and connection are both freely varied and are not *a priori* taken to be metrically compatible. Metricity follows at the level of the field equations. In all numbers of dimensions but two, the Hilbert and Palatini formalisms yield precisely the same (unique) dynamical theory: GR. The 2D triviality of GR is, in its second-order form,

$$R_{\alpha\beta}(\{\ \}(g_S)) \equiv \tfrac{1}{2} g_{S_{\alpha\beta}} R_S(\{\ \}) , \tag{1.1}$$

and so the Einstein (vacuum) field equations vanish identically. The Christoffel connection in (1.1) is completely determined by inverting the metricity relation ("compatibility condition"):

$$0 = g_{S_{\alpha\beta,\gamma}} - g_{S_{\sigma\beta}} \begin{Bmatrix} \sigma \\ \gamma\alpha \end{Bmatrix} - g_{S_{\alpha\sigma}} \begin{Bmatrix} \sigma \\ \beta\gamma \end{Bmatrix} . \tag{1.2}$$

In the above equations the subscript S is appended to the metric g as a reminder that in Riemannian geometry (GR) the metric tensor is necessarily symmetric.

Two comments about triviality need to be mentioned. The absence of field equations for g_S turns out to be a great boon for the string theorist and/or σ-model builder, for it ensures that the string dynamics is (at least classically) completely independent of the geometry of the world sheet. Equation (1.1), and hence the (trivial) Einstein equation which follows, is invariant under coordinate and conformal transformations. The first invariance follows from relativistic covariance. The second is peculiar to 2D and has as its implication that all 2D Riemannian spacetimes are locally conformally flat. Together these symmetries account for the three degrees of freedom in a rank-two symmetric tensor in 2D. This provides another practical manifestation of 2D triviality. The compatibility condition (1.2), however, is *not* preserved under the action of a conformal transformation. Fortunately, the effects of the conformal transformation on the connection cancel in (1.1).

The aforementioned triviality of 2D GR requires that "gravity on a line must be invented anew" (see Jackiw in Ref. 7). Several authors have responded to this challenge by proposing the constant-curvature equation

$$0 = R_S(\{\ \}) - \Lambda , \tag{1.3}$$

as the vacuum-field equation appropriate to 2D gravity.[7,8] As in GR, this is a geometrical second-order field equation. Invoking the local conformal flatness property of 2D Riemannian spaces, the metric appearing in (1.3) may be written in conformal form

$$g_{S_{\alpha\beta}}(x)=e^{2\phi(x)}\eta_{\alpha\beta}\ , \qquad (1.4)$$

in which case the constant-curvature equation (1.3) reduces to the Liouville equation for the conformal field $\phi(x)$, viz.,

$$0=\eta^{\alpha\beta}\phi_{,\alpha\beta}+\tfrac{1}{2}\Lambda e^{2\phi}\ . \qquad (1.5)$$

This model is integrable (which serves to illustrate the utility of the lower-dimensional approach), and so the 2D Liouville theory of gravity is (in principle) completely solved.

In order to derive the proposed gravitational field equation (1.3) by means of a variational principle, an invariant action is constructed by introducing a Lagrange multiplier field $N(x)$:

$$I_{\text{Liouville}}=\int d^2x\sqrt{-g_S}(R-\Lambda)N\ . \qquad (1.6)$$

Obviously, the field equation (1.3) follows from the $\delta/\delta N$ variation of $I_{\text{Liouville}}$. Less obviously, but fortuitously, the variation with respect to $g_{S_{\mu\nu}}$ leads to an equation of motion for $N(x)$ which does not constrain the metric (Liouville) solutions.[7] The (rather steep) price that has been paid in order to apply the action principle is the introduction of a *nongeometrical field.* That $N(x)$ must be dynamical for a nontrivial theory is related to yet another manifestation of the 2D triviality of GR. In 2D the usual Hilbert action $I_H=\int d^2x\sqrt{-g_S}R_S(\{\ \})$ is a topological invariant (the Euler characteristic) and thus cannot possibly yield nontrivial local-field equations.

A natural extension of the Liouville model to the nonvacuum case is to replace Λ with $8\pi GT$ (where T is the trace of the energy-momentum tensor and G is Newton's constant) in the field equation (1.3), viz.,

$$R=8\pi GT\ . \qquad (1.7)$$

Models of this type exhibit many features analogous to four-dimensional general relativity, including gravitational radiation, Friedmann-Robertson-Walker (FRW) cosmological solutions, a post-Newtonian expansion, gravitational collapse, and black-hole thermodynamics. However, it is still generated by a nongeometrical Lagrangian.[9,10]

General relativity is the *unique* completely geometric theory of gravitation in Riemannian spacetime. In order for alternative theories to be distinct, they must either sacrifice the aesthetic appeal of a *fully* geometric formulation (as seen above in the case of the Liouville model) or must generalize the geometric structure of spacetime. We choose the latter approach and construct fully geometric, Lagrangian-based, non-Riemannian theories of gravitation (in any number of dimensions) by means of the method of algebraic extension[11] (AE). In $1+1$ dimensions this provides us with the *first ever* (to our knowledge) instance of a fully geometric *and* dynamical theory.[4] In the non-Riemannian geometries described by AE, the fundamental tensor (metric) is allowed to be nonsymmetric $g_{\alpha\beta}\neq g_{\beta\alpha}$ as is the connection $\Gamma^{\alpha}_{\beta\gamma}\neq\Gamma^{\alpha}_{\gamma\beta}$. Clearly, then we are dealing with non-Riemannian geometry.

In this paper we demonstrate how to construct a completely geometric Lagrangian-based dynamical theory of gravitation in two dimensions and exhibit some exact solutions of the theory. In Sec. II we review the method of algebraic extension, with some specialization to the 2D case. We derive the field equations using two independent methods and demonstrate their equivalence. We show that these equations have nontrivial dynamics even if the cosmological constant is zero. In Sec. III we obtain the most general possible exact solutions in conformal gauge using "null" coordinates. In Sec. IV we demonstrate that solutions with event horizons exist, and we construct some of these. One class of solutions is equivalent to a class of black-hole solutions found in the theory (1.7). Section V is concerned with the construction of more solutions in another coordinate "gauge." Finally, in Sec. VI we explicitly demonstrate that our new model does not contain the Liouville model. We summarize in a brief concluding section.

II. ALGEBRAICALLY EXTENDED HILBERT THEORY OF GRAVITATION IN TWO DIMENSIONS

The method of algebraic extension[11] (AE) was developed so as to provide a framework in which nontrivial geometric Lagrangian-based extensions of general relativity could be studied. The "algebraic" in AE derives from the fact that an algebraic structure is imprinted upon the tangent space to the underlying spacetime manifold. Tensors and geometric objects (especially the metric and connection) assume algebra values. The "extension" is in part a recognition of the success that GR has achieved in describing weak gravitational fields (for example, solar system tests). Built into the AE program is the requirement that GR is *the* AE theory in the case that the algebraic structure is just $\mathcal{R}$. On the other hand, extension is also a reference to the primary role played by geometry in the generalization from Riemannian spacetime (GR) to the fundamentally non-Riemannian AE spacetime.

AE may be applied in any number of dimensions, and since the algebraic structure is imposed upon the tangent space, the dimension of the underlying manifold (spacetime) is unchanged. Enforcing compatibility between the algebraic structure and the (extended) metric and connection restricts the number of possible choices of algebra to just five: $\mathcal{R}$ the real numbers, $\mathcal{C}$ the complex, $\mathcal{E}$ the hypercomplex, $\mathcal{Q}$ the quaternions, and $\mathcal{H}$ the hyperquaternions.

For the algebraically extended Hilbert theory of gravitation (AHG) the specific algebra that we choose is $\mathcal{E}$ (the hypercomplex). The canonical basis elements (generators) of $\mathcal{E}$ are $\{(1,e)|e^2=+1\}$. Note how this structure differs from the complex algebra. Algebraic "conjugation" is defined in the usual manner when acting on the generators $\{(1^*,e^*)=(1,-e)\}$. In order that the spacetime and algebraic structures be compatible, it is necessary that the "extended fundamental tensor" (hereafter called the "metric") be Hermitian under algebraic conjugation, viz.,

$$g^*_{\alpha\beta}=g_{\beta\alpha} . \tag{2.1}$$

Clearly, then, the symmetric part of $g_{\alpha\beta}$ is $\mathcal{R}$ valued, while the skew part is e valued. The Hermiticity of g is the AE generalization of the symmetry requirement (of g_S) in Riemannian gravitational theory (GR). A further requirement that we impose is that the metrically compatible connection also be Hermitian:

$$\Gamma^{\alpha *}_{\beta\gamma}=\Gamma^{\alpha}_{\gamma\beta} , \tag{2.2}$$

which implies that only theories based on the $\mathcal{E}$ algebra can satisfy positivity of energy requirements (in flat space).[12] It may be possible to discard the Hermiticity requirement (2.2) and still have positivity of energy, permitting use of the $\mathcal{C}$ and $\mathcal{Q}$ algebras.[13] The structure and consequences of these non-Hermitian theories remain largely unexplored.

The properties of the $\mathcal{E}$ algebra, coupled with the Hermiticity of g and Γ, make it possible in practice to "ignore" the algebraic derivation and treat the AHG as a completely real, nonsymmetric theory. This is tantamount to requiring all other fields in the theory to assume values in the real tangent space only. In the sequel we take this viewpoint, but it will often be clear from context how the algebra generators could be reinstated. Now that g and Γ are no longer symmetric (as they are in Riemannian geometry), the order of indices in expressions will assume a crucial importance. A fortunate aspect of the $\mathcal{E}$ algebra is that it is commutative. If this was not the case, then the factor ordering of terms would become significant also.

In keeping with the Hilbert principle, the (generic n-dimensional) AHG gravitational degrees of freedom are described by the metric g and its derivatives. The metrically compatible connection is *completely* defined through the AE analog of (1.2):

$$0=g_{\alpha\beta,\gamma}-g_{\sigma\beta}\Gamma^{\sigma}_{\gamma\alpha}-g_{\alpha\sigma}\Gamma^{\sigma}_{\beta\gamma} . \tag{2.3}$$

From the connection implicitly defined in (2.3), various curvature tensors may be constructed in manners analogous to the familiar Riemannian case.[11]

There are, however, a number of technical points which must be discussed. First is the definition of a contravariant g by identification with the "inverse" of the g covariant:

$$g^{\alpha\rho}g_{\beta\rho}=\delta^{\alpha}_{\beta}=g^{\rho\alpha}g_{\rho\beta} . \tag{2.4}$$

Armed with (2.4), $g^{\alpha\beta}$ and $g_{\alpha\beta}$ may be used to (carefully) raise, lower, and contract indices. Second, the curvature tensor is naturally defined by the following combination of connection terms:

$$R^{\alpha}{}_{\beta\gamma\delta}=\Gamma^{\alpha}_{\beta\delta,\gamma}-\Gamma^{\alpha}_{\gamma\delta,\beta}+\Gamma^{\sigma}_{\beta\delta}\Gamma^{\alpha}_{\gamma\sigma}-\Gamma^{\sigma}_{\gamma\delta}\Gamma^{\alpha}_{\beta\sigma} . \tag{2.5}$$

The curvature tensor is $[\beta\gamma]$ skew by definition, but since torsion is necessarily present in the connection [see (2.3)], not all of the symmetries familiar from the Riemannian case will now hold. Third, a specific instance of the absence of the familiar symmetries leading to richer possible structures occurs when one contracts the curvature tensor. There exist two distinct ways to do this, unlike the case in GR. One yields the AE generalized Ricci tensor, while the other results in the so-called "second curvature":

$$R_{\delta\beta}=R^{\lambda}{}_{\beta\lambda\delta}=\Gamma^{\lambda}_{\beta\delta,\lambda}-\Gamma^{\lambda}_{\lambda\delta,\beta}+\Gamma^{\sigma}_{\beta\delta}\Gamma^{\lambda}_{\lambda\sigma}-\Gamma^{\sigma}_{\lambda\delta}\Gamma^{\lambda}_{\beta\sigma} , \tag{2.6}$$

$$P_{\beta\gamma}=R^{\lambda}{}_{\beta\gamma\lambda}=\Gamma^{\lambda}_{\beta\lambda,\gamma}-\Gamma^{\lambda}_{\gamma\lambda,\beta} . \tag{2.7}$$

In GR the trace of the Christoffel connection $\{^{\lambda}_{\alpha\lambda}\}$ is a total derivative, and hence the "curl" in (2.7) vanishes identically, i.e., $P_{\beta\gamma}(\{\ \})\equiv 0$. In AHG we do not use P in the construction of the gravitational action for reasons having to do with the 4D model.[12] Finally, the curvature scalar is the trace of the Ricci tensor:

$$R(\Gamma(g))=g^{\alpha\beta}R_{\alpha\beta}(\Gamma(g)) . \tag{2.8}$$

When we form the AE Hilbert action using R, we arrive at an action which is purely real (or any e-valued parts must be surface terms only), so as not to face the problems associated with a complex Lagrangian.

The AHG gravitational action is (generically)

$$I_H=\int d^nx\sqrt{-g}R(\Gamma) . \tag{2.9}$$

We make provision for more general action contributions by allowing for a "cosmological constant" term and a matter contribution:

$$I_\Lambda=\int d^nx\sqrt{-g}(-\Lambda) , \tag{2.10}$$

$$I_M=\int d^nx\sqrt{-g}\mathcal{L}_M . \tag{2.11}$$

The sign of Λ is chosen so as to conform to the usual conventions. When we vary the total action $I_{H\Lambda M}=I_H+I_\Lambda+I_M$ with respect to $g^{\mu\nu}$, we get the AHG field equations

$$\frac{\delta I_{H\Lambda M}}{\delta g^{\mu\nu}}=\sqrt{-g}(R_{\mu\nu}+X_{\mu\nu}-\tfrac{1}{2}g_{\mu\nu}R+\tfrac{1}{2}g_{\mu\nu}\Lambda-T_{\mu\nu}) . \tag{2.12}$$

In this expression we have written $T_{\mu\nu}$ as shorthand for $-(1/\sqrt{-g})\delta I_M/\delta g^{\mu\nu}$, while

$$X_{\mu\nu}=\Gamma^{\lambda}_{\nu\mu}\Gamma_{\lambda}-\Gamma_{\nu,\mu}-\Gamma_{\nu}\Gamma_{\mu} . \tag{2.13}$$

The "torsion vector" $\Gamma_\alpha=\Gamma^{\lambda}_{[\alpha\lambda]}$ is the trace of the torsion. The torsion is tensorial since it corresponds to a difference between two connections.

The field equations (2.12) are, even in the vacuum case (i.e., $T_{\mu\nu}=0$) and for a vanishing cosmological constant, very complicated and unhappily not a completely independent set. Coordinate invariance, the backbone of a relativistic theory, implies that there exist Bianchi identities relating curvature components and hence the field equations (FE's). AHG has been shown to possess *consistent* Bianchi identities,[14] where the notion of consistency is between the set of identities derived by insisting that $I_{H\Lambda}$ be invariant under infinitesimal coordinate transformations (which vanish on the boundary of the spacetime, so as to make the contributions due to surface terms equal to zero) and the set of identities which result direction from the symmetries of the generalized curvature tensor. In 2D the situation is markedly different, because in 2D the Bianchi identities are identically satisfied, and thus the four (in 2D) gravitational FE's contained in (2.12) are independent. The easiest way to argue this is to note that the Bianchi identities may be expressed so as to depend on three index antisymmetry, which (in 2D) makes them devoid of content.

At this stage we begin to exploit the considerable simplifying power which arises as a consequence of working in 2D. In particular, we adopt a "GR-plus" (GR+) formalism to no small advantage in the sequel. GR+ is like a background-field approach, in that one writes the full non-Riemannian theory (AHG) expressed entirely in terms of Riemannian- (GR-) based theory and its covariant objects. For all other dimensions higher than two, writing the "plus" bits in terms of Riemannian covariant objects gets exceedingly complicated, and success is limited to the use of "perturbative methods", where the higher-order skew contributions are discarded. In 2D the GR+ treatment is exact, and no (seriously) restrictive assumptions need be made about the size of the "plus" bit.

In 2D there is but one independent antisymmetric component of the full covariant metric:

$$g_{\alpha\beta}=g_{S_{\alpha\beta}}+\omega\epsilon_{\alpha\beta}\ ,\quad \epsilon_{01}=+1\ ,$$
$$=g_{S_{\alpha\beta}}+\frac{\omega}{\sqrt{-g_S}}e_{S_{\alpha\beta}}\ ,\quad e_{S_{\alpha\beta}}=\sqrt{-g_S}\epsilon_{\alpha\beta}\ . \tag{2.14}$$

The Levi-Civita (permutation) symbol $\epsilon_{\alpha\beta}$ is a tensor density of weight -1, and so $e_{S_{\alpha\beta}}$ transforms as a tensor (weight zero). The determinant of $g_{\alpha\beta}$, so necessary for the volume element and the calculation of the inverse, is

$$-g=-g_S+\omega^2=-g_S\left[1-\frac{\omega^2}{-g_S}\right]\ . \tag{2.15}$$

Equation (2.15) will (with our Minkowskian conventions) be positive (zero, negative) for $\omega/\sqrt{-g_S}<1$ ($=1,>1$). Our fondness for Minkowski space leads us to confine ourselves to the $\omega/\sqrt{-g_S}<1$ regime. The contravariant g, formed using the definition (2.4), turns out to be

$$g^{\alpha\beta}=\frac{-g_S}{-g}\left[g_S{}^{\alpha\beta}+\frac{\omega}{\sqrt{-g_S}}e_S{}^{\alpha\beta}\right]\ , \tag{2.16}$$

where $g_S{}^{\alpha\beta}$ is the inverse of $g_{S_{\alpha\beta}}$, $e_S{}^{01}=\epsilon^{01}/\sqrt{-g_S}=-1/\sqrt{-g_S}$, and the sign of the skew term follows from the transpose implicit in (2.4).

The factors of $(-g_S/-g)=(1-\omega^2/-g_S)^{-1}$, which will occur often in GR+ expressions, coupled with the restriction $\omega/\sqrt{-g_S}<1$, suggest a natural representation in terms of hyberbolic functions, viz., $\omega/\sqrt{-g_S}=\tanh(\theta)$:

$$g_{\alpha\beta}=g_{S_{\alpha\beta}}+\tanh(\theta)e_{S_{\alpha\beta}}\ , \tag{2.17}$$

$$g^{\alpha\beta}=\cosh^2(\theta)[g_S{}^{\alpha\beta}+\tanh(\theta)e_S{}^{\alpha\beta}]\ . \tag{2.18}$$

Furthermore, a very convenient quantity for the exposition and solving of the FE's is

$$A=\ln[\cosh(\theta)]\ . \tag{2.19}$$

The compatibility condition (2.3) may be inverted and the full connection solved for.[4] In our GR+ notation, the result is

$$\Gamma^{\alpha}_{\beta\gamma}=\left\{\begin{matrix}\alpha\\ \beta\gamma\end{matrix}\right\}+\kappa^{\alpha\rho}_{\beta\gamma}A_{|\rho}-\delta^{\alpha}_{\beta}\Gamma_{\gamma}+\delta^{\alpha}_{\gamma}\Gamma_{\beta}\ . \tag{2.20}$$

The Christoffel connection found in (2.20) is formed in the usual manner, from the Riemannian metric g_S. The term $\kappa^{\alpha\rho}_{\beta\gamma}=[2g_{S_{\beta\gamma}}g_S{}^{\alpha\rho}-\delta^{\alpha}_{\beta}\delta^{\rho}_{\gamma}-\delta^{\alpha}_{\gamma}\delta^{\rho}_{\beta}]$ is a $(\beta\gamma)$ symmetric combinatorial factor familiar to those who have studied the perturbative expansion of a gravitational field on a curved background. The bar (|) denotes covariant differentiation with respect to the Christoffel connection. Of course, since A is a scalar, the covariant derivative with respect to any connection is simply the partial derivative, and so this distinction need not be made yet, but we do so to emphasize the GR+ approach. The full torsion is provided for by the final two ($[\beta\gamma]$ skew) terms, with the torsion vector given in terms of the hyberbolic parameter θ by

$$\Gamma_{\alpha}=-g_{S_{\alpha\lambda}}e_S{}^{\lambda\rho}\theta_{|\rho}\ . \tag{2.21}$$

At first glance this would appear to be a surprising result (that the entire torsion can be written in terms of its trace), until one realizes that in 2D they each contain the same number of independent components.

Now that the full connection has been determined, the sourceless AHG action may be constructed using the previously derived equations. It is

$$I_{H\Lambda}=\int d^2x\sqrt{-g}\,(R-\Lambda)$$
$$=\int d^2x\sqrt{-g_S}e^{A}[R_S(\{\ \})+2\Box_S A-e^{-2A}\Lambda]\ . \tag{2.22}$$

Note that (2.22) manifestly obeys the AE dictum that the action be real. The Riemannian "box operator" ($\Box_S$) acting on A is

$$\Box_S A=g_S{}^{\alpha\beta}A_{|\alpha\beta}=\frac{1}{\sqrt{-g_S}}\partial_\alpha(\sqrt{-g_S}g_S{}^{\alpha\beta}\partial_\beta A)\ , \tag{2.23}$$

coinciding with the usual definition of the (Riemannian) Laplacian acting on a scalar function. Implicit in the formula (2.22) is the relation

$$R(\Gamma)=e^{2A}[R_S(\{\ \})+2\Box_S A]\ . \tag{2.24}$$

The non-Riemannian cosmological constant has become, by virtue of the skew contribution to the volume element, a "*cosmological variable*" from the GR+ perspective. Even in the case of vanishing Λ, the "kinetic(like) term" for A in (2.22) will impart dynamics to the theory.

Remaining true to the GR+ perspective, we vary (2.22) with respect to the fields $g_S{}^{\mu\nu}$ and A, which account for the (four) degrees of freedom in the model,

$$\frac{\delta I_{H\Lambda}}{\delta A}=0=R_S(\{\ \})+4\Box_S A+2g_S{}^{\alpha\beta}A_\alpha A_\beta+e^{-2A}\Lambda\ , \tag{2.25}$$

once we have discarded extraneous solutions ($A\to-\infty$ and $\sqrt{-g_S}=0$):

$$\frac{\delta I_{H\Lambda}}{\delta g_S{}^{\mu\nu}}=0=\kappa^{\alpha\beta}_{\mu\nu}(e^{A})_{|\alpha\beta}-\begin{bmatrix}\alpha\beta\\ \mu\nu\end{bmatrix}e^{A}A_{|\alpha}A_{|\beta}+\tfrac{1}{2}g_{S_{\mu\nu}}e^{-A}\Lambda\ , \tag{2.26}$$

discarding the possibility of $\sqrt{-g_S}=0$. The expression for $\kappa^{\alpha\beta}_{\mu\nu}$ was given above; $[^{\alpha\beta}_{\mu\nu}]=\delta^\alpha_\mu\delta^\beta_\nu+\delta^\alpha_\nu\delta^\beta_\mu-g_{S_{\mu\nu}}g_S{}^{\alpha\beta}=g_{S_{\mu\nu}}g_S{}^{\alpha\beta}-\kappa^{\alpha\beta}_{\mu\nu}$ is a combinatorial factor whose form arises from the $g_{S_{\mu\nu}}$ variation of $\sqrt{-g_S}g_S{}^{\alpha\beta}$ [and which, if contributing to the connection (contracted with a vector field), is generically associated with the presence of conformal symmetry]. Some care must be taken in the derivation of (2.26); in particular, one must take the symmetrized derivative and account for the identity (1.1). The (three) distinct equations embodied by (2.26) are usefully split into trace and traceless parts, viz.,

$$g_S{}^{\alpha\beta}(A_{|\alpha\beta}+A_{|\alpha}A_{|\beta})+e^{-2A}\Lambda=0=\Box_S(e^{A})+e^{-A}\Lambda\ , \tag{2.27}$$

$$\begin{bmatrix}\alpha\beta\\ \mu\nu\end{bmatrix}(A_{|\alpha\beta}+3A_{|\alpha}A_{|\beta})=0=\begin{bmatrix}\alpha\beta\\ \mu\nu\end{bmatrix}(e^{3A})_{|\alpha\beta}\ . \tag{2.28}$$

Two (equivalent) means of representation have been provided for the g_S FE's for later convenience. Substituting the trace of the g_S FE into the A FE allows the elimination of the explicit Λ dependence:

$$0=R_S(\{\ \})+g_S{}^{\alpha\beta}(3A_{|\alpha\beta}+A_{|\alpha}A_{|\beta})\ . \tag{2.29}$$

Another readily determined consequence of these two equations is that

$$R-\Lambda=0\ , \tag{2.30}$$

the AHG analog of the constant curvature equation in the Liouville model.[7,8] Because of the non-Riemannian nature of R, the content of Eq. (2.30) differs dramatically from the Riemannian Liouville theory (1.3).

Obviously, in order for the GR+ approach to be successful, the full AHG FE's (2.12) must agree precisely with the above GR+ FE's [Eqs. (2.27)–(2.29)]. As a preliminary, we note that

$$R_{\mu\nu}(\Gamma)+X_{\mu\nu}(\Gamma)\equiv(R+X)_{\mu\nu}(\Gamma)=R_{\mu\nu}(\Gamma_S)\ , \tag{2.31}$$

where $\Gamma_S{}^{\alpha}_{\beta\gamma}=\Gamma^{\alpha}_{(\beta\gamma)}$ is the symmetric part of the full connection. We see from (2.31) that the presence of the X ("extra") term in the AHG gravitational field equation serves to precisely cancel the direct torsion contribution from $R_{\mu\nu}$. Indirect contributions from the skew sector enter via the non-Christoffel part of Γ_S. Furthermore,

$$(R+X)_{\mu\nu}=\tfrac{1}{2}g_{S_{\mu\nu}}R_s(\{\ \})-A_{|\mu\nu}-3A_{|\mu}A_{|\nu}+2g_{S_{\mu\nu}}g_S{}^{\alpha\beta}(A_{|\alpha\beta}+A_{|\alpha}A_{|\beta})\ , \tag{2.32}$$

again invoking (1.1). The full (AHG) trace of $X_{\alpha\beta}$ is

$$X=g^{\alpha\beta}X_{\alpha\beta}=e^{2A}g_S{}^{\alpha\beta}(A_{|\alpha\beta}+A_{|\alpha}A_{|\beta})\ . \tag{2.33}$$

The full AHG vacuum (so $T_{\mu\nu}=0$) FE is (2.12):

$$0=(R+X)_{\mu\nu}-\tfrac{1}{2}g_{\mu\nu}(R-\Lambda)\ . \tag{2.12$'$}$$

The full (AHG) trace of (2.12′) yields at $X+\Lambda=0$, which is precisely the content of (2.27). Now, the traceless part of (2.12′) reads

$$0=(R+X)_{\mu\nu}-\tfrac{1}{2}g_{\mu\nu}(R+X)\ . \tag{2.12$''$}$$

The skew part of (2.12″) provides $0=R+X=R-\Lambda$, which exactly reproduces Eqs. (2.29) and (2.30). Finally, the symmetric part of (2.12″) is the Einstein-like equation

$$0=R_{\mu\nu}(\Gamma_S)-\tfrac{1}{2}g_{S_{\mu\nu}}g_S{}^{\alpha\beta}R_{\alpha\beta}(\Gamma_S)\ , \tag{2.34}$$

which is not trivial because the symmetric connection has a non-Christoffel part. Analysis of (2.34) shows that it reduces to Eqs. (2.28) when written in GR+ notation.

Therefore, we have explicitly demonstrated that the GR+ system of FE's is identical to the set obtained directly from the AHG variation specialized to 2D. The set of equations that we choose to consider for the purposes of solving them are comprised of (2.27)–(2.29). These (four independent) equations are not related by Bianchi identities as discussed earlier. In the remainder of this paper, we construct sets of solutions to these field equations.

III. CONFORMAL SOLUTIONS

The freedom to choose coordinates allows us to everywhere (locally) write $g_{S_{\alpha\beta}}$ in conformally flat form [cf., Eq. (1.4)],

$$\bar{g}_{S_{\alpha\beta}}=e^{2\phi(\bar{x})}\eta_{\alpha\beta}\ ,\quad \eta_{\alpha\beta}=\mathrm{diag}(-1,+1)\ , \tag{3.1}$$

with respect to "natural conformal (*C*) coordinates" $(\bar{x}^0,\bar{x}^1)$. In fact, the "everywhere" in the preceding statement refers to regions free of singularities and/or event horizons. Such singular points are mapped to ∞ in the conformal coordinate system. The $\bar{x}^\alpha$ coordinates are not optimally suited for the analysis of our problem, and so we perform a linear transformation to "null conformal (NC) coordinates":

$$\tilde{x}^{\mp}=\frac{1}{\sqrt{2}}(\mp\bar{x}^0+\bar{x}^1)\ . \tag{3.2}$$

Under the transformation to NC the metric (3.1) reads

$$\tilde{g}_{S_{\alpha\beta}}=e^{2\phi(\bar{x})}\begin{bmatrix}0&1\\1&0\end{bmatrix}\ . \tag{3.3}$$

The inverse (3.3) is just

$$\tilde{g}_S{}^{\alpha\beta}=e^{-2\phi(\tilde{x})}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix} . \tag{3.4}$$

The Christoffel connection in NC coordinates is

$$\begin{Bmatrix}\tilde{\alpha}\\ \beta\gamma\end{Bmatrix}=\begin{bmatrix}\tilde{\alpha}\tilde{\sigma}\\ \beta\gamma\end{bmatrix}\phi_{,\sigma}=(\delta^{\alpha}_{\beta}\delta^{\sigma}_{\gamma}+\delta^{\alpha}_{\gamma}\delta^{\sigma}_{\beta}-\tilde{g}_{S\beta\gamma}\tilde{g}_S{}^{\alpha\sigma})\phi_{,\sigma} . \tag{3.5}$$

The form of this expression is canonical for conformally flat representations. We note that our NC system of coordinates is harmonic, since $\tilde{g}_S{}^{\beta\gamma}[^{\tilde{\alpha}\tilde{\sigma}}_{\beta\gamma}]\equiv 0$. The Ricci tensor is

$$R_{\mu\nu}(\{\tilde{\ }\})=-\tilde{g}_{S_{\mu\nu}}\tilde{g}_S{}^{\alpha\beta}\phi_{,\alpha\beta}=-\tilde{g}_{S_{\mu\nu}}\tilde{\Box}_S\phi , \tag{3.6}$$

where the second equality follows as a consequence of the harmonic nature of the coordinate system. Of course, Eq. (1.1) holds and simplifies the left-hand side of (3.6).

The feature which makes the NC system most advantageous is the simplification which ensues from

$$\tilde{\Box}_S Q=e^{-2\phi}(2Q_{,-+}) \tag{3.7}$$

for any scalar function Q. In (3.7) the comma signifies partial differentiation with respect to the following (index) argument(s). It is cumbersome to write the commas explicitly, and we will be most often dealing with scalar quantities, so often that we will drop the commas, in which case the index subscripted onto the scalar denotes the derivative. It will be clear from context, and in the event of ambiguity, we will restore the commas.

When the field equations from Sec. II are expressed in their NC form, ϕ occurs only in the combination $A-\phi$, which we denote by B. In addition, we will write $A=\ln a$ and $B=-\frac{1}{2}\ln b$, where it serves to make the expressions simpler. The field equations (2.27)–(2.29), when specialized to the present case, read

$$0=2a_{-+}+\Lambda ab , \tag{3.8}$$

$$0=2B_{-+}+\tfrac{1}{2}\Lambda e^{-2B} , \tag{3.9}$$

$$0=A_{--}+2B_{-}A_{-}+A_{-}A_{-} ,\quad 0=b^2\left[\frac{a_-}{b}\right]_- , \tag{3.10}$$

$$0=A_{++}+2B_{+}A_{+}+A_{+}A_{+} ,\quad 0=b^2\left[\frac{a_+}{b}\right]_+ . \tag{3.11}$$

As was stated above, the subscripts in (3.8)–(3.11) signify partial differentiation with respect to the appropriate coordinate variable.

The first solutions that we will construct are ones which have $\Lambda=0$. The most important aspect of this analysis is the demonstration of nontrivial (vacuum) gravitational dynamics even in the case of vanishing cosmological constant. Next, we will tackle the $\Lambda\neq 0$ case. Naturally, these solutions will possess a greater richness of structure than the $\Lambda=0$ solutions.

In the case of vanishing Λ, Eq. (3.8) and (3.9) are greatly simplified, while (3.10) and (3.11) remain unchanged. The solutions to (3.8) and (3.9) are easily obtained:

$$a=m(\tilde{x}^-)+p(\tilde{x}^+) , \tag{3.12}$$

$$B=M(\tilde{x}^-)+P(\tilde{x}^+) , \tag{3.13}$$

for (four) unknown functions $\{m,p,M,P\}$, each of only one coordinate. Substitution of (3.12) and (3.13) back into (3.10) and (3.11) results in

$$0=m_{--}+2m_{-}M_{-} , \tag{3.14}$$

$$0=p_{++}+2p_{+}P_{+} . \tag{3.15}$$

The partial derivatives in (3.14) and (3.15) are now (in fact) total derivatives. Both of the above equations integrate immediately to

$$k_-=e^{2M}m_- , \tag{3.16}$$

$$k_+=e^{2P}m_+ , \tag{3.17}$$

for nonzero constants k_-,k_+. (Here the subscripts denote the respective constants.) In the event that one (or both) of these constants actually vanishes, the implication is that the solution being constructed is independent of that coordinate.

Having proceeded as far as is possible by integration, the remaining task is to ensure the consistency of the solutions. Let

$$k=k_-k_+=e^{2(M+P)}m_-p_+=e^{2B}m_-p_+ , \tag{3.18}$$

and hence

$$B=-\tfrac{1}{2}\ln(k^{-1}m_-p_+) . \tag{3.19}$$

Recalling that $A=\ln a$ and $\phi=A-B$, the general solution for 2D (vacuum) AHG in NC "gauge," and with $\Lambda=0$, is

$$A=\ln(m+p) ,\qquad 2\phi=\ln[k^{-1}(m+p)^2m_-p_+] , \tag{3.20}$$

for two arbitrary functions $m(\tilde{x}^-),p(\tilde{x}^+)$. It is clear that while $R(\Gamma)$ must equal zero [recall Eq. (2.30)], the solutions (3.20) do not reduce to triviality. In fact,

$$R_S(\{\tilde{\ }\})=+\frac{4k}{(m+p)^4}=-2\tilde{\Box}_S A \tag{3.21}$$

is nowhere zero, without violating the conditions which ensure freedom from singularities.

While the introduction of NC coordinates has allowed the problem to be solved, the physical nature of the solutions is (to say the least) somewhat obscure. One conclusion that we are able to draw is that a restriction of the solutions (3.20) to a GR plus infinitesimal case over an extended region (such as a small non-Riemannian perturbation about a GR background) requires an unnatural accommodation. To wit, $\cosh\theta\to 1$ requires $(m+p)\to 1$, but m and p are each functions of a single (different) coordinate variable, and so they must each tend toward constant values. In this case their derivatives approach zero, and hence $2\phi\to-\infty$.

Now we examine the situation in which the theory is independent of one of the coordinates ($\tilde{x}^-$, say). Equations (3.8)–(3.10) are consistent only if $\Lambda=0$ and have no content. The remaining Eq. (3.11) is integrable (provided $b^2\neq 0$), with the result that $a_+=kb$. Hence the general solution is

$$\begin{aligned} A&=\ln a\ , \\ 2\phi&=\ln(k^{-1}a^2a_+)\ , \end{aligned} \tag{3.22}$$

for $a=a(\tilde{x}^+)$ arbitrary. The above is also true if one tries to take $A=A(\tilde{x}^+)$, while leaving $B=B(\tilde{x}^-,\tilde{x}^+)$ [i.e., $\phi=\phi(\tilde{x}^-,\tilde{x}^+)$] since one soon discovers that B (and hence ϕ) are forced to be $\tilde{x}^-$ independent.

Motivated by an analysis of 2D AE Palatini gravity, which has a trivial g_S sector and $\omega/\sqrt{-g_S}=\text{const}$,[4] one may wish to attempt to set $A=\text{const}$ here. In this case the field equations require $\Lambda=0$, just as in the above case of one coordinate independence, and only (3.9) has physical content: $0=B_{-+}$, and so

$$\phi=M(\tilde{x}^-)+P(\tilde{x}^+)\ . \tag{3.23}$$

(We have absorbed some constant factors into the arbitrary functions M and P.) The result (3.23) is consistent with 2D triviality since we have $R(\Gamma)=0$ from (2.30), $R_S(\{\tilde{\ }\})=0$ by combining (3.23) with (3.6) and (3.7), and $\tilde{\Box}_S A=0$ by assumption. A more complete analysis of the $\Lambda=0$, $A=\text{const}$ case will be undertaken in Sec. IV.

The task of solving (3.8)–(3.11) when $\Lambda\neq 0$ begins with the recognition of (3.9) as the Liouville equation for B. This equation integrates to (we write $B=-\frac{1}{2}\ln b$)

$$b=\frac{F_-(\tilde{x}^-)G_+(\tilde{x}^+)}{[1+(\Lambda/4)FG]^2}\ , \tag{3.24}$$

where $F(G)$ is an arbitrary function of $\tilde{x}^-(\tilde{x}^+)$ only, and the subscripts on the functions in the numerator denote partial differentiations. Dividing (3.10) and (3.11) by $b^2\neq 0$ allows for immediate (partial) integration:

$$\frac{a_-}{b}=g(\tilde{x}^+)\ ,\quad \frac{a_+}{b}=f(\tilde{x}^-)\ , \tag{3.25}$$

for functions of integration $f(\tilde{x}^-),g(\tilde{x}^+)$. The set of functions $\{F,G,f,g\}$ cannot all be arbitrary and independent. Relations between them follow from the remaining field equation (3.8). After a careful analysis, we get

$$a=\frac{KF+LG+M[1-(\Lambda/4)FG]}{1+(\Lambda/4)FG}\ , \tag{3.26}$$

where K,L,M are constants, and F,G appear in (3.24). The "physical quantities" A and ϕ are determined from (3.24) and (3.26) to be

$$\begin{aligned} A&=\ln\left[\frac{KF+LG+M[1-(\Lambda/4)FG]}{1+(\Lambda/4)FG}\right]\ , \\ 2\phi&=\ln\left[\frac{\{KF+LG+M[1-(\Lambda/4)FG]\}^2F_-G_+}{[1+(\Lambda/4)FG]^4}\right]\ . \end{aligned} \tag{3.27}$$

Equations (3.27) comprise the complete, most general set of (vacuum) solutions to 2D AHG expressed in NC coordinates (singularity-free). Note that the $\Lambda\to 0$ limit of the solutions (3.27) reproduces the $\Lambda=0$ solution set (3.20).

To illustrate the nontriviality of (3.27), we calculate

$$R_S(\{\tilde{\ }\})=e^{-4A}(4KL+\Lambda M^2)+2\Lambda e^{-2A}\ , \tag{3.28}$$

$$2\tilde{\Box}_S A=-e^{-4A}(4KL+\Lambda M^2)-\Lambda e^{-2A}\ . \tag{3.29}$$

Recalling (2.24), we see from (3.28) and (3.29) that $R=\Lambda$ as required by (2.30).

Again, the physical interpretation of the solutions is hampered by the "ungenerous nature" of the NC coordinate system. Any attempt to posit solutions which are independent of one of the coordinates cannot be accomplished here (while keeping $\Lambda\neq 0$). Trying to fix boundary conditions in order to restrict the arbitrary functions which appear in (3.27) quickly becomes a daunting task. The reason is that the physically comfortable ideas of large spatial distance "asymptotic" behavior is nearly impossible to express in NC coordinate terms.

IV. SCHWARZSCHILD-LIKE SOLUTIONS

In order to accommodate singularities (at coordinate values other than $\mp\infty$), we choose coordinates so that

$$g_{\text{Sch}_{\alpha\beta}}=\begin{bmatrix} -\dfrac{1}{U} & 0 \\ 0 & U \end{bmatrix}\ , \tag{4.1}$$

which is analogous to the standard coordinates used in exhibiting the Schwarzschild solutions in (3+1)-dimensional GR. The coordinates we use here are labeled (t,r) for distinction from the previous cases. The Schwarzschild metric is *static* (i.e., $g_{\text{Sch}_{tr}}\equiv 0$). One-dimensional "spherical symmetry" may be imposed by restricting r to strictly non-negative values or replacing r by $|r|$ in the solutions to be obtained. This is a nontrivial modification of the problem because it is predicated on the existence of the point in spacetime about which the symmetry holds. Such a point must be a δ-function "source" for the field equations; i.e., there is a "gravitational charge" located at the origin of the spherical symmetry, and this integrates to give the $|r|$ behavior of the symmetric solutions.

The determinant of the symmetric metric (4.1) is $-g_{\text{Sch}}=1$, while the inverse is simply

$$g_{\text{Sch}}{}^{\alpha\beta}=\begin{bmatrix} -U & 0 \\ 0 & \dfrac{1}{U} \end{bmatrix}\ . \tag{4.2}$$

The Christoffel connection is found to be

$$\begin{aligned} &\begin{Bmatrix} t \\ tt \end{Bmatrix}=-\frac{1}{2}\frac{U_t}{U}\ ,\quad \begin{Bmatrix} r \\ rr \end{Bmatrix}=\frac{1}{2}\frac{U_r}{U}\ , \\ &\begin{Bmatrix} t \\ tr \end{Bmatrix}=-\frac{1}{2}\frac{U_r}{U}\ ,\quad \begin{Bmatrix} r \\ tr \end{Bmatrix}=\frac{1}{2}\frac{U_t}{U}\ , \\ &\begin{Bmatrix} t \\ rr \end{Bmatrix}=\frac{1}{2}UU_t\ ,\quad \begin{Bmatrix} r \\ tt \end{Bmatrix}=\frac{1}{2}\left[\frac{1}{U}\right]\left[\frac{1}{U}\right]_r\ . \end{aligned} \tag{4.3}$$

Here, as usual, the subscripts denote partial derivatives.

Applying (4.2) and (4.3), we form the Riemannian curvature

$$R_S(\{\ \})=U_{tt}-\left[\frac{1}{U}\right]_{rr} . \tag{4.4}$$

The Schwarzschild coordinate "gauge" is not harmonic, and so the Riemannian box operator is not as simply expressed as it was in the conformal case (C or NC). In fact, the expression for the Schwarzschild Laplacian is (acting on some scalar function Q)

$$\Box_S Q(t,r)\equiv g_{\rm Sch}{}^{\alpha\beta}Q_{|\alpha\beta}=(-UQ_t)_t+\left[\frac{1}{U}Q_r\right]_r . \tag{4.5}$$

Now the necessary tools have been assembled, with which we proceed to write the generally (GR+)-covariant AHG field equations in their Schwarzschild form:

$$0=[-U(e^A)_t]_t+\left[\frac{1}{U}(e^A)_r\right]_r+e^{-A}\Lambda , \tag{4.6}$$

$$0=(e^{3A})_{tt}+\frac{1}{U^2}(e^{3A})_{rr} , \tag{4.7}$$

$$0=[U(e^{3A})_t]_r+U^2\left[\frac{1}{U}(e^{3A})_r\right]_t , \tag{4.8}$$

$$0=[U_{tt}-U_t 3A_t-U(3A_{tt}+A_tA_t)]-\left[\left[\frac{1}{U}\right]_{rr}-\left[\frac{1}{U}\right]_r 3A_r-\left[\frac{1}{U}\right](3A_{rr}+A_ra_r)\right] . \tag{4.9}$$

Even with the simplification which ensues from the use of a static metric, these equations are at best quite tedious to solve in general, and so we further restrict ourselves to an examination of the *fully stationary* case. Fully stationary means AHG stationary (viz., $g_{\alpha\beta,t}=0$). In the GR+ notation, this entails

$$U_t=0=A_t , \tag{4.10}$$

for the GR+ (Schwarzschild) fields U, A. Condition (4.10) is perhaps not as Draconian as it may at first appear, since the general 2D vacuum solution in regions free from singularities is known from Sec. III of this paper; the "Schwarschild gauge" solutions need only be applied to (small) regions enclosing (coordinate) singularities—with appropriate matching at the boundaries. It is always possible to define a stationary coordinate system in such a region.

Under condition (4.10) the field equation (4.8) loses all meaning, while (4.7) leads to

$$0=(e^{3A})_{rr} , \tag{4.11}$$

which can be immediately integrated to determine

$$A=\tfrac{1}{3}\ln(c+kr) , \tag{4.12}$$

for real constants c,k. Equation (4.12) appears ingenuous, but must be treated carefully. Whatever the signs of c and k, there will be some region (with boundary) for which $c+kr\le 0$ if we allow $r\in(-\infty,\infty)$. However, if we impose S^0 symmetry (i.e., replace r by $|r|$), then we can avoid imaginary A by choosing $k>0$, but at the cost of restricting r to the half-line. If $k=0$, $c>0$, then A is constant for all r. This particular case is examined in detail later in this section.

We seek to examine the behavior of (4.12) in various limits, without restricting r. For convenience we introduce $q=(c+kr)^{1/3}$ and study the limits $q\to+\infty,1,0$ and the case $q<0$. From the definition of A, we see that $\omega/\sqrt{-g_S}=(1-q^{-2})^{1/2}$. In the limit of $q\to\infty$, $\omega/\sqrt{-g_S}\to 1^-$. This limit is equivalent to a large $\pm r$ limit and demonstrates that the 2D spacetime is not asymptotically Riemannian at large distances from the origin. Fortunately, the GR+ treatment is exact in 2D; otherwise, this result might have case serious doubts upon the validity of a perturbative approach. The behavior of $\omega/\sqrt{-g_S}$ is at least softer than the linear (confining) potentials that often occur in 1D problems (since the field lines cannot spread out). In the limit of $q\to 1^+$, $\omega/\sqrt{-g_S}\to 0^+$, the Riemannian limit. Therefore, the 2D AHG Schwarzschild space is $\approx$ Riemannian in the vicinity of $c+kr\to 1^+$, which is in the small r region. This is completely unlike the situation in four dimensions. The final limit that we wish to consider is that of $q\to 0$; however, we see that once $q<1$, then $\omega^2/-g_S<0$ and the analysis (the theory) breaks down. Furthermore, as $q\to 0$, $\omega^2/-g_S\to-\infty$, which is very pathological. Although (4.12) is incapable of accommodating negative values of q, we see that the physical quantity $\omega/\sqrt{-g_S}$ is unaffected by $q\to -q$. [The same will be true for the symmetric part $1/U$; see (4.14).] Hence there are two physical regions of parameter space: $q\le -1$ and $q\ge +1$ separated by an excluded region $-1<q<1$. By shifting the coordinate variable r, we can collapse this singular region to the origin (in r space).

We shall assume that we are in one of the physical regions (say $k,r>0$) and proceed with the analysis of the remaining field equations. Together, (4.10) and (4.12) turn (4.6) into

$$0=\Lambda(c+kr)^{-1/3}+\left[\frac{1}{U}[(c+kr)^{1/3}]_r\right]_r , \tag{4.13}$$

which upon integration becomes

$$c_1=\frac{1}{2}\Lambda\frac{3}{k}(c+kr)^{2/3}+\frac{1}{U}\frac{k}{3}(c+kr)^{-2/3} . \tag{4.14a}$$

The constant of integration is labeled c_1. Rearranging (4.14a), we obtain the solution

$$\frac{1}{U}=-\frac{1}{2}\Lambda\left[\frac{3}{k}\right]^2(c+kr)^{4/3}+c_1\frac{3}{k}(c+kr)^{2/3} . \tag{4.14b}$$

The above solutions for A and $1/U$ together satisfy the final remaining field equation (4.9), subject to (4.10).

There are three particular cases of these general solutions (4.12) and (4.14b) that we will now examine in some detail. The first case occurs if the constant k appearing in (4.12) and subsequently is equal to zero [contrary to the assumption made in deriving (4.14b)]. The second special case arises when we set $\Lambda=0$, while keeping $c_1\neq 0$, $k\neq 0$, and the third is $c_1=0$, while $\Lambda,k\neq 0$.

When we attempt to set $k=0$ in the process of deriving (4.12), we arrive at the solution $A=\text{const}$. The field equation (4.6) then demands that $\Lambda=0$ in order to remain consistent. We recall that this same result occurred in the conformal case. Here we will undertake the more detailed analysis postponed earlier. In order to obtain an equation for $1/U$, we turn to (4.9), which now reads

$$0=\left[\frac{1}{U}\right]_{rr} . \tag{4.15}$$

Therefore, the Schwarzschild-like solution in this particular case is (recall $\Lambda=0$), for a,b real constants,

$$A=\text{const} ,\qquad \frac{1}{U}=a+br . \tag{4.16}$$

If $b=0$, $a\neq 0$, then $g_{\text{Sch}_{\alpha\beta}}$ is just constant and can be simply transformed into $\eta_{\alpha\beta}$. We choose $b>0$ without loss of generality.

The solutions described by Eqs. (4.16) are rather uninteresting, since they correspond to "flat space." By recalling (2.30) we can conclude that $R(\Gamma)=0$ here, but this is not a sufficient condition for flatness. By comparing (4.4) with (4.15), we discover that $R_S(\{\ \})=0$. This is 2D, and so the *entire* Riemannian curvature tensor has but one independent component $R^{\alpha}{}_{\beta\gamma\delta}(\{\ \})=\frac{1}{2}[\delta^{\alpha}_{\gamma}g_{S_{\beta\delta}}-\delta^{\alpha}_{\beta}g_{S_{\gamma\delta}}]R_S(\{\ \})$, and we conclude that the spaces described by (4.16) are Riemannian flat. The full AHG curvature $R^{\alpha}{}_{\beta\gamma\delta}(\Gamma)$ is a GR+ combination of $R^{\alpha}{}_{\beta\gamma\delta}(\{\ \})$ plus terms which depend on derivatives of A. Clearly, then, the solutions (4.16) are flat.

An alternative means by which the Riemann flatness of $1/U=a+br$ may be demonstrated is to consider the following transformation of the canonical flat-space line element $ds^2=-dT^2+dR^2$:

$$T=\frac{2}{b}(a+br)^{1/2}\sinh\left[\frac{b}{2}t\right] ,\qquad R=\frac{2}{b}(a+br)^{1/2}\cosh\left[\frac{b}{2}t\right] . \tag{4.17a}$$

The result is precisely the Schwarzschild form with $1/U=a+br$. It is easy to argue that if $1/U=(a+br)^f$, only the case where $f\equiv 1$ is equivalent to (Riemannian) flat space.

Recall the discussion of spherical symmetry and gravitational charge at the beginning of this section. If r is replaced by $|r|$ in (4.16), then the solution corresponds to the solution of the (Riemannian) field equation (1.7) in the case of a point mass at the origin[9,10] and, in fact, may be understood as the "AHG extension" of that type of Riemannian solution. The parameter b is the gravitational charge (the mass). A complete discussion of this sort of AHG solution would require the introduction of an AHG (nonsymmetric) energy-momentum tensor. We shall not pursue this point any further.

The preceding dismissal of solutions (4.15) as flat, and hence uninteresting, was predicated on the assumption that there were no singularities. Whether the $k=0$ Schwarzschild metric will have singular behavior depends crucially on the sign of the constant a. (Actually, it depends on the relative signs of a,b, but we have chosen $b>0$.) If $a>0$, then there are no "event horizons"—points at which the diagonal metric coefficients change sign, and hence the roles of timelike and spacelike coordinates reverse. There are no singular points, because as was demonstrated above, the Riemannian and AHG curvature tensors vanish everywhere. If $a=0$, then the point $r=0$ has a coordinate singularity, but given our hypothesis of "spherical symmetry," the metric is not extendable through the origin, and so the "event horizon" at $r=0$ is physically irrelevant. If $a<0$, then there is an event horizon at the coordinate distance $r=|a/b|$. We note that the transformation to flat space (4.17a) breaks down at this point. If we restrict (4.17a) to the coordinate region $r>|a/b|$ and supplement it with

$$\hat{T}=\frac{2}{b}(-a-br)^{1/2}\sinh\left[\frac{b}{2}t\right] ,\qquad \hat{R}=\frac{2}{b}(-a-br)^{1/2}\cosh\left[\frac{b}{2}t\right] , \tag{4.17b}$$

for the region $r<|a/b|$, we see that we have two flat regions: $d\hat{s}^2=+d\hat{T}^2-d\hat{R}^2$ and $ds^2=-dT^2+dR^2$, joined at $r=|a/b|$. Note that the sense of timelike (TL) and spacelike (SL) coordinates has been interchanged.

The second special case is that of $\Lambda=0$, but $k,c_1\neq 0$. The steps leading to (4.12) are unaffected, while (4.13) and (4.14) are simplified by the elimination of their Λ dependence. Thus

$$A=\tfrac{1}{3}\ln(c+kr) ,\qquad \frac{1}{U}=c_1\frac{3}{k}(c+kr)^{2/3} . \tag{4.18}$$

Since $\Lambda=0$, these solutions must have $R(\Gamma)=0$ according to (2.30), but neither $R_S(\{\ \})$ nor $R^{\alpha}{}_{\beta\gamma\delta}(\Gamma)$ are zero (except in the limit as $r\to\infty$). In fact,

$$R_S(\{\ \})=2c_1\frac{k}{3}(c+kr)^{-4/3} . \tag{4.19}$$

A naive analysis of (4.18) would suggest that in analogy with the case first considered there will be no singularities in the model for $c>0$, a singular point at $r=0$ if $c=0$, and at $r=|c/k|$ for $c<0$. (Recall that we choose $k>0$.) However, we see from (4.19) that the singularity at $r=|c/k|$ for $c\leq 0$ is a point of infinite curvature. This singularity does not have an event horizon associated with it because $(\cdots)^{2/3}$ is an everywhere positive-(in)definite quantity. Recalling the discussion following (4.12), $|(c+kr)|\geq 1$ for physical solutions, and we avoid the pathological singularities.

The third special case is the complement to the second: $k>0$, $\Lambda\neq 0$, $c_1=0$, with the solution

$$A = \tfrac{1}{3}\ln(c+kr)\ ,$$
$$\frac{1}{U} = -\frac{1}{2}\Lambda\left[\frac{k}{3}\right]^2 (c+kr)^{4/3}\ . \tag{4.20}$$

The GR curvature is

$$R_S(\{\ \}) = 2\Lambda(c+kr)^{-2/3}\ , \tag{4.21}$$

and using this result, the full AHG curvature $R(\Gamma)=\Lambda$ is easily verified. The same comments about the singularities in the previous case [Eqs. (4.18) and (4.19)] hold also in the present case [Eqs. (4.20) and (4.21)]. The only point to be noted here is that the $g_{\rm Sch}$ parts of (4.18)–(4.21) rise and/or fall-off with different powers of r.

The different rates of change with r for the two special cases discussed above allows the formation of a bona fide event horizon in the general case [Eqs. (4.12) and (4.14)]. Specifically, $1/U$ in (4.14) passes through zero at the point r_0 determined by

$$\frac{3}{k}(c+kr_0)^{2/3} = \frac{2c_1}{\Lambda}\ . \tag{4.22}$$

This point r_0 will exist, provided that the signs of c_1 and Λ are the same, thereby making the right-hand side of (4.22) positive. By examining (4.14) the Λ term dominates at large values of r (and thus determines the asymptotic TL-SL sense of the coordinates), while the c_1 term dominates for $r \ll r_0$ (determining the TL-SL sense in the vicinity of the origin).

The AHG curvature is everywhere constant, while the GR curvature is given by the sum of (4.19) and (4.21), viz.,

$$R_S(\{\ \}) = 2c_1\frac{k}{3}(c+kr)^{-4/3} + 2\Lambda(c+kr)^{-2/3}\ . \tag{4.23}$$

At r_0,

$$R_S(\{\ \})|_{r_0} = \frac{3}{2}\,\frac{3}{k}\,\frac{\Lambda^2}{c_1}\ , \tag{4.24}$$

which is perfectly finite (and well behaved), leading to our identification of r_0 as an event horizon. Since Λ and c_1 are of the same sign, $R_S(\{\ \})$ will everywhere be positive or negative, while monotonically decreasing or increasing to zero as r approaches infinity.

V. COSMOLOGICAL SOLUTIONS

Another ansatz which bears examination is the "cosmological" (FRW) form for the GR metric:

$$g_{{\rm FRW}_{\alpha\beta}} = \begin{bmatrix} -1 & 0 \\ 0 & U^2 \end{bmatrix}\ . \tag{5.1}$$

Again, we label the coordinates (t,r). This choice for the form of the symmetric metric may also be described as "synchronous gauge." As in the $(3+1)$-dimensional FRW case, we demand that the spatial cross sections defined by constant t be maximally symmetric. This turns out to be the familiar assumption of global homogeneity. With the stronger assumption of global isotropy, the FRW metric (in N dimensions) is canonically decomposed into a temporal part, a radial part, and a $(N-1)$-dimensional spherically symmetric part, viz.,

$$ds^2 = -dt^2 + U^2(t)\left[\frac{1}{1-kr^2}dr^2 + r^2 d\Omega^2\right]\ .$$

The "cosmic scale factor" is labeled $U(t)$ in the following. The factor of $1/(1-kr^2)$ which appears in g_{rr} is necessary to account for the possible geometries (open, flat, closed) of the homogeneous N space. In $1+1$ dimensions, however, the maximally symmetric subspace is one dimensional (S^0 symmetric), and so there is no angular part to the line element. The r-dependent factor in g_{rr} may then be safely absorbed by means of a coordinate transformation (scaling) of r alone, and thus we may assume that $g_{{\rm FRW}_{\alpha\beta}}$ is completely independent of the spacial coordinate. Another way to see the validity of this assumption is to realize that since the spacelike subspace is only one dimensional, it does not have an intrinsic curvature, and so we can always choose a (locally) r-independent metric on it. However, there is a topological effect. For a closed universe the domain of r is compact (circular topology), while for the flat and open cases the domain is $-\infty < r < \infty$.

The determinant $\sqrt{-g_{\rm FRW}} = U(t)$, and the inverse of (5.1) is

$$g_{\rm FRW}{}^{\alpha\beta} = \begin{bmatrix} -1 & 0 \\ 0 & U^{-2} \end{bmatrix}\ . \tag{5.2}$$

The Christoffel connection for the FRW metric is easily deduced from its definition (1.2):

$$0 = \begin{Bmatrix} t \\ tt \end{Bmatrix} = \begin{Bmatrix} t \\ tr \end{Bmatrix} = \begin{Bmatrix} r \\ tt \end{Bmatrix} = \begin{Bmatrix} r \\ rr \end{Bmatrix}\ ,$$
$$\begin{Bmatrix} t \\ rr \end{Bmatrix} = UU_t\ , \quad \begin{Bmatrix} r \\ tr \end{Bmatrix} = \frac{U_t}{U}\ . \tag{5.3}$$

The Riemann curvature scalar is

$$R_S(\{\ \}) = 2\frac{U_{tt}}{U}\ . \tag{5.4}$$

The FRW coordinate system is *not* harmonic, leading to the following expression for the Laplacian:

$$\Box_S Q(t,r) = -\frac{1}{U}(UQ_t)_t + \frac{1}{U^2}Q_{rr}\ . \tag{5.5}$$

The vacuum field equations for the FRW-gauge GR+representation of 2D AHG are

$$0 = -\frac{1}{U}(Ua_t)_t + \frac{1}{U^2}a_{rr} + \frac{\Lambda}{a}\ , \tag{5.6}$$

$$0 = 2\frac{U_{tt}}{U} - \frac{1}{U}(U3A_t)_t - A_tA_t + \frac{1}{U^2}(3A_{rr} + A_rA_r)\ , \tag{5.7}$$

$$0 = U\left[\frac{1}{U}(a^3)\right]_{rt}\ , \tag{5.8}$$

$$0=U\left[\frac{1}{U}(a^3)_t\right]_t+\frac{1}{U^2}(a^3)_{rr} . \quad (5.9)$$

Despite the promising fact of (5.8), which determines the form of a to be

$$a^3=U(t)[f(t)+g(r)] , \quad (5.10)$$

the rest of the equations have proven impervious to solution. In order to simplify (5.6)–(5.9), we insist on AHG homogeneity and set

$$A_r=0=a_r . \quad (5.11)$$

With this restriction, (5.8) loses its status as an independent field equation, and (5.6), (5.7), and (5.9) become

$$0=a_{tt}+\frac{U_t}{U}a_t-\frac{\Lambda}{a} , \quad (5.12)$$

$$0=2\frac{U_{tt}}{U}-\frac{1}{U}(U3A_t)_t-A_tA_t , \quad (5.13)$$

$$0=\left[\frac{1}{U}(a^3)_t\right]_t . \quad (5.14)$$

The final equation (5.14) leads to

$$\kappa U=(a^3)_t , \quad (5.15)$$

for some constant of proportionality $\kappa\neq 0$. The precise value of the constant is irrelevant for the following. By differentiating (5.15) we can express U_t/U in terms of a, and thereby (5.12) becomes an equation for a alone:

$$0=(a^2)_{tt}-\Lambda . \quad (5.16)$$

The solution of (5.16) is

$$a^2=c_0+c_1t+\tfrac{1}{2}\Lambda t^2 , \quad (5.17)$$

where $c_{0,1}$ denote constants of integration.

In terms of t the cosmic scale factor is

$$\kappa U=3(c_0+c_1t+\tfrac{1}{2}\Lambda t^2)^{1/2}(c_1+\Lambda t) . \quad (5.18)$$

The field equation (5.13), which has hitherto been ignored, is satisfied for a and U given by (5.17) and (5.18). That this is so is also consistent with the constancy of the AHG curvature (2.30). The Riemannian curvature following from the FRW scale factor U is

$$R_S(\{\ \})=\frac{1}{a^2}\left[3\Lambda+\frac{(c_1+\Lambda t)^2}{2a^2}\right] . \quad (5.19)$$

In order that the solutions (5.17)–(5.19) be (physically) interpretable [e.g., $a^2\geq 0$ for mathematical consistency or the stronger requirement $a^2\geq 1$; cf. (2.19)], there will be relations among the parameters c_0, c_1, Λ, and/or restrictions on the allowed t domain. These features will be apparent as we analyze the three distinct cases of AHG vacuum cosmology determined by the sign of Λ.

The above analysis did not depend critically on the value of Λ, and so it suffices to take the $\Lambda\to 0$ limit of (5.17)–(5.19) for our first special case. When $\Lambda=0$,

$$U\sim(c_0+c_1t)^{1/2} ,$$
$$a\sim(c_0+c_1t)^{1/2} , \quad (5.20)$$
$$R_S(\{\ \})\sim a^{-4}\sim(c_0+c_1t)^{-2} .$$

Taking the positive roots and assuming that $c_1>0$, this describes a "softly" expanding universe. If the constant c_1 is negative, then this universe collapses. Either way, the domain of t values is restricted. For positive c_1, $t>-c_0/c_1$ is the weak limit (from U), while the stronger limit (from a) is $t>(-c_0+1)/c_1$. The results for $c_1<0$ follow by symmetry. Later, we will comment on the different limits on t arising from the symmetric/skew sectors. For the moment we simply note that the weak restriction allows $U\to 0$, which coincides with $R_S(\{\ \})\to\infty$ and pathological behavior for a, whereas the stronger restriction (on a) cuts out the geometrical singularity. We also note that a grows without bound as $t\to\infty$, and so the far future of this spacetime is essentially non-Riemannian.

If $\Lambda>0$, then

$$U\sim\left[c_0+c_1t+\frac{\Lambda}{2}t^2\right]^{1/2}(c_1+\Lambda t)\sim t^2 ,$$
$$a\sim\left[c_0+c_1t+\frac{\Lambda}{2}t^2\right]^{1/2}\sim t , \quad (5.21)$$
$$R_S(\{\ \})\sim +t^{-2} .$$

This case is like de Sitter space except that the "inflation" is of a power-law variety rather than the exponential behavior which arises in 3+1 dimensions. The particular features of this cosmology (big bang or bounce?) depend on relations among the parameters. An analysis of the U behavior of the solutions must also include consideration for the a behavior as well.

Zeros of U will occur at

$$t_U=-\frac{c_1}{\Lambda}\left[1\pm\left[1-\frac{2c_0\Lambda}{c_1^2}\right]^{1/2}\right] ,$$

and at $t_U=-c_1/\Lambda$. The first two t_U enumerated above are also zeros of a (pathological). Aside from the behavior of a, these roots change the sign of the argument in the square-root factor appearing in (5.21) (unless the root is a double root, in which case all three $t_U=-c_1/\Lambda$). An imaginary U changes the sign of U^2 (in the line element), which changes the signature of the spacetime metric. For positive Λ the asymptotic spacetime is non-Riemannian with Minkowskian signature. At the time of the first singularity, the spacetime suddenly becomes Euclidean. This "new" spacetime evolves and experiences a (Riemannian) metric singularity at its half-time. (An extreme form of "midlife crisis.") The curvature is well behaved at this point, and a is nonzero, and the TL-TL sense of the (Euclidean) spacetime is unaltered. Finally, at the third singularity the spacetime transforms back to a Minkowskian signature (Riemannian). Curvature singularities occur at both the birth and death of the Euclidean spacetime, along with very singular a behavior. In the $\Lambda=0$ case above, U^2 changes sign only once (with strongly singular behavior) and the interpretation is of a spacetime irrevocably changing its signature. It may be possible to (in some manner) interpret these solutions in terms of nucleating vacuum bubbles, but at present this has not been investigated.

For the evolutionary scenario just described to be realized, all three roots of U must be real. The sufficient condition for reality is

$$\frac{2c_0\Lambda}{c_1^2}\leq 1 \ . \tag{5.22}$$

Otherwise, if (5.22) does not hold, then a does not have zeros, and $U=0$ only at $t_U=-c_1/\Lambda$. Obviously, it is also necessary that $a^2>0$ (equivalently, that U is real). Fortunately, the failure of (5.22) will guarantee this. In the present case, $t_U=-c_1/\Lambda$ may be thought of as the "beginning of the universe." At this t_U the metric is singular, but a is at some nonzero minimum value, while the Riemannian curvature is finite and at an extremum. The Universe is then essentially non-Riemannian and forever expanding.

A loophole which will provide for a benign interpretation of the de Sitter case in general is to (by fiat) insist upon initial conditions $t_0, a(t_0), U(t_0)$ such that the singular behavior is avoided. This is unsatisfying, because it must necessarily make appeal beyond the model to account for these initial conditions.

If $\Lambda<0$, then much of the preceding analysis holds, but in the obverse. Equation (5.21) describes (with $\Lambda<0$ now) the spacetime. The Riemannian curvature will differ in sign, and its particular form will be slightly different [cf. (5.19)]. This case corresponds to "anti–de Sitter space" in that the effect of negative Λ makes a collapse inevitable (a Minkowskian big crunch). The formulas provided above the zeros of U and a are equally valid for negative Λ. In this case, however, the physical interpretation is of a Minkowskian universe, formed at the first singularity. This universe then evolves, also experiencing a "midlife crisis," which fortunately does not alter the TL-SL senses of the coordinates, and finally expires in the final singularity. It is not possible to alter this interpretation by avoiding the singularities and still describe a universe with a Minkowskian signature.

The usual cosmological assumption of homogeneity, extended to full AHG in (5.11), is not the sole approach that one may try to solve the field equations (5.6)–(5.9). A mathematical (rather than physically motivated) simplification is to substitute (5.10) into (5.9), multiply by U, and demand that the result be (t,r) separable. It turns out that

$$U^2\left[\frac{U_t}{U}\right]_t=\text{const} \tag{5.23}$$

is required. Provided that $U\neq$const, condition (5.23) is equivalent to setting the Riemann Christoffel curvature equal to a constant. Precisely this restriction and its consequences is the subject of the next section.

VI. LIOUVILLE-LIKE SOLUTIONS

A natural question which arises is the following: Do there exist circumstances in which the solutions of 2D AHG reduce to the Liouville model in the g_S sector? In other words, can 2D AHG accommodate the Liouville model? A quick perusal of the general solutions thus far described [Eqs. (3.20), (3.21); (3.27), (3.28); (4.18), (4.19); (4.20), (4.21); (4.14), (4.23); (5.17)–(5.19)] shows that in every case $R_S(\{\ \})$ is *not* a constant (with the exception of the flat solutions: $R_S(\{\ \})=0$ and $A=$const in (4.16)). Solutions (3.22) are restricted and Riemannian flat, and so it is impossible for them to accommodate (interesting) Liouville-like models.

Here we will force the issue and demand that the Christoffel curvature be constant, and investigate the consequences. The condition that we impose is (the Liouville equation)

$$0=R_S(\{\ \})-L \ . \tag{6.1}$$

Working in the NC coordinates described in the third section, this equation has the solution

$$e^{2\phi}=\frac{F_-G_+}{[1+(L/4)FG]^2} \ , \tag{6.2}$$

as determined in (3.24). Imposing (6.1) on the AHG field equations (2.27)–(2.29), we see that it is (2.29) which is most greatly affected. In fact, it becomes a Klein-Gordon(-like) equation for $q\equiv a^{1/3}=e^{A/3}$:

$$0=\tilde{\Box}_S q+\frac{L}{9}q \ . \tag{6.3}$$

The other field equations are less dramatically modified and read

$$0=q^3\tilde{\Box}_S(q^3)+\Lambda \ , \tag{6.4}$$

$$[e^{-2\phi}(q^9)_-]_-=0=[e^{-2\phi}(q^9)_+]_+ \ . \tag{6.5}$$

The following argument will describe explicitly how (6.3)–(6.5) *cannot* be solved in general, except for $q=$const, the flat case. The first step is to partially integrate (6.5), introducing functions of integration $f(\tilde{x}^-)$ and $g(\tilde{x}^+)$:

$$e^{-2\phi}(q^9)_-=g \ , \quad e^{-2\phi}(q^9)_+=f \ . \tag{6.6}$$

An important corollary of (6.6) is that

$$g(q^n)_+\equiv f(q^n)_- \ . \tag{6.7}$$

The $-,+$ derivatives of (6.7), when $n=1$, allows us to write the Laplacian of q in terms of f,g, viz.,

$$\tilde{\Box}_S q=f^{-1}2(gq_+)_+=g^{-1}2(fq_-)_- \ . \tag{6.8}$$

Our Klein-Gordon(-like) equation (6.3) allows us to invert (6.8), leading to

$$f=-\frac{9}{L}e^{-2\phi}\frac{2(gq_+)_+}{q} \ , \quad g=-\frac{9}{L}e^{-2\phi}\frac{2(fq_-)_-}{q} \ , \tag{6.9}$$

which can be substituted back into the defining relations for f and g. Already, this causes concern because [in order that (6.7) is maintained] we must have

$$2gq_+ - \frac{-L}{10} q^{10} + c = 2fq_- \ , \tag{6.10}$$

for some constant c. The functional form is quite constrained, and we have yet to exploit one of the field equations (6.4). When we apply the above results to the remaining equation, it boils down to

$$0 = \frac{7}{5} q^6 + \frac{4c}{-L} q^{-4} + \frac{3\Lambda}{L} \ . \tag{6.11}$$

Clearly, this equation has only q = const solutions, violating the assumption made at the outset of this analysis.

VII. CONCLUSIONS

Using the technique of algebraic extension, we have constructed and derived the field equations of a completely geometric Lagrangian-based dynamical theory of gravitation in two dimensions. The theory is dynamically nontrivial, even if the cosmological constant vanishes. In NC coordinates we found the most general exact solution of these field equations. Such solutions do not contain any singularities or event horizons. Solutions with these features are discerned among those constructed in the "Schwarzschild" and "cosmological" gauges of Secs. IV and V.

The results of our investigation leave us with mixed feelings. On the one hand, we have "invented a new theory of gravity on the line," yet we are disappointed by several aspects of this new theory. In retrospect our inability to reproduce the Liouville model except in its most trivial form is perhaps a consequence of demanding too much of 2D AHG. We want "to have our cake" (a new distinct model) and "to eat it too" (reproduce the Liouville results of Refs. 7–10). More seriously, though, is the manner in which the symmetric and skew degrees of freedom do not decouple (as we had hoped). A natural accommodation of models in which $A \to 0^+$ would have greatly pleased us, as well as guaranteed mathematical consistency [cf. (2.15)]. We may summarize by stating that the non-Riemannian geometry of 2D AHG does not lend itself to asymptotic "Riemanness."

It should be pointed out, however, that the same sort of behavior occurs in 4D AHG (which we proposed as an alternative theory of gravity). Briefly, the static spherically symmetric solutions of 4D AHG admit flat-space asymptotic behavior, but not the Schwarzschild behavior which is necessary in order to agree with solar system data.[12] By means of a perturbative expansion of the theory upon a curved Riemannian "background" spacetime,[15] this behavior has been understood to arise from a coupling between the skew field (a Kalb-Ramond scalar field[12]) and Weyl tensor component of the background Riemannian curvature. Hence 4D AHG is only able to comfortably match to asymptotic Riemannian geometries which are conformally flat. A parametrized post-Newtonian expansion of 4D AHG (Ref. 16) has corroborated this result. Even though the Weyl tensor does not exist in 2D, we have seen that 2D AHG does not admit uncoupled asymptotics.

It is precisely the existence of the skew field which is responsible for the nontrivial dynamics of the model. Other approaches (e.g., Liouville) must add the "extra" field(s) by hand, whereas in the case on non-Riemannian theories all of the fields are of geometric origin.

We close by pointing out that further generalizations of this theory are possible. We could have included matter effects as considered in Eq. (2.12) and mentioned in passing in Sec. IV, but have not done so, since it is unlikely that the qualitative behavior of the vacuum solutions will be altered. Another possibility is to include the trace of the 'second curvature' term (2.7) in the action (2.9). Such a term violates the positivity of energy requirements for (skew) wavelike excitations in the flat-space limit in four dimensions. However, in two dimensions such considerations are irrelevant since such excitations vanish. The inclusion of this term will have the effect of modifying the coupling of the A field to the symmetric metric.

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