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Topological quantum mechanics

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The quantum theory of a type of generally covariant field theory, that has no local degrees of freedom, is described. Physical observables that capture topological properties of the manifold are identified and a representation of their Poisson algebra is constructed to obtain the quantum theory. A non-Abelian generalization to SU(2) is also discussed in a similar way.

INTRODUCTION

There has been renewed interest in $2+1$ gravity recently following the work by Witten,[1] who showed that this theory can be described by a Chern-Simons action for the group ISO(2,1). Horowitz has presented a class of exactly solvable diffeomorphism-invariant theories[2] that are of a similar type. These theories are finite-dimensional dynamical systems in which, when viewed from a canonical point of view, the diffeomorphism constraints on the phase spaces of the theories do not appear explicitly. Rather, the diffeomorphisms are generated by linear combinations of other constraints (that arise naturally) on the phase spaces. It is this fact that makes this class of theories, which includes $2+1$ gravity,[2-5] exactly soluble classically and quantum mechanically. It also appears at present (from the known examples) that this type of simplification (diffeomorphisms appearing implicitly) occurs only if the theories are finite-dimensional. For example, this situation does not arise in $3+1$ gravity.

For four-dimensional diffeomorphism-invariant theories, there is another way of dealing with diffeomorphism constraints in the quantum theory. This is via the loop representation for the quantum theory,[6] which was developed for general relativity in the context of the new Hamiltonian variables discovered by Ashtekar.[7] The representation space is the space of complex-valued functions of loops on the spacelike surfaces. In this representation, it is possible to construct, via the Dirac procedure, a class of physical states for general relativity. These states are functions of knot classes of the loops. Specifically, the functions of knot classes of loops are diffeomorphism-invariant and it turns out that these are also annihilated by the Hamiltonian constraint.

In this paper I first discuss the quantization of an Abelian topological theory[2] via the loop representation. A non-Abelian generalization to SU(2) is then described. The motivations for this work are (i) to study the loop representation in a four-dimensional generally covariant theory that is much simpler than general relativity, (ii) to see explicitly what quantum-mechanical systems result by constructing a representation of a Poisson algebra of topological observables, and (iii) to see what differences (if any) occur in the characterization of the physical states if the diffeomorphism constraints are treated explicitly as opposed to implicitly as discussed above (where the diffeomorphisms appear as linear combinations of other constraints, and the Dirac conditions for these latter constraints are used for canonical quantization).

ABELIAN THEORY

Consider the (Abelian) theory described by the action

$$\int {}^4F\wedge B \ . \tag{1}$$

Here ${}^4F=d\,{}^4A$, the curvature of an Abelian connection ${}^4A$, and B is a two-form. The action is invariant under the usual gauge transformations $A\rightarrow A+d\Lambda$, and also under $B\rightarrow B+d\omega$ where Λ and ω are, respectively, arbitrary functions and one-forms on the manifold. The equations of motion are

$$ {}^4F=0, \quad dB=0 \ . \tag{2}$$

Assuming the spacetime topology to be $\Sigma\times R$, where Σ is a compact three-manifold, one finds, upon performing a $3+1$ decomposition, that the canonically conjugate pair is A_a and $\tilde{E}^{\,a}=\epsilon^{abc}B_{bc}$, where A_a and B_{ab} are the pullbacks to Σ of ${}^4A_\mu$ and $B_{\mu\nu}$, and ϵ^{abc} is the Levi-Civita tensor density on Σ. The fundamental Poisson brackets is

$$\{A_a(x),\tilde{E}^{\,b}(y)\}=\delta_a^b\delta^3(x,y)$$

and the constraints (obtained by pulling back the equations of motion to Σ) are

$$F_{ab}=0, \quad \partial_a\tilde{E}^{\,a}=0 \ . \tag{3}$$

In this form, these constraints are of the same form as those of $2+1$ gravity.[3] A pair of physical observables (phase-space functions that have weakly vanishing Poisson brackets with the constraints) can be defined for the theory. These are parametrized by loops and closed two-surfaces in Σ and are

$$T^0[\gamma](A)=U_\gamma(s) \quad \text{and} \quad T^1[S](\tilde{E})=i\int_S d^2\sigma\, n_a\tilde{E}^{\,a} \ , \tag{4}$$

where $U_\gamma(s)=P\exp(\int_\gamma A)$ is the holonomy of A around the loop γ with the base point $\gamma(s)$ and S is an embedded two-surface[8] in Σ specified by the normal n_a. The Poisson brackets of these observables is nonzero only when the loop γ intersects the surface S. It is

$$\{T^0[\gamma](A), T^1[S](\tilde{E})\} = i\Delta(\gamma,S)T^0[\gamma](A) , \quad (5)$$

where

$$\Delta(\gamma,S) = \int_\gamma ds \int_S d^2\sigma\, \dot{\gamma}^a(s) n_a(\sigma) \delta^3(\gamma(s), S(\sigma)) .$$

The structure function $\Delta(\gamma,S)$ is a (diffeomorphism-invariant) constant that measures the linking number of the loop γ with the surface S.

One can construct the linear combination of the constraints

$$C_a = \tilde{E}^b F_{ab} - A_a \partial_b \tilde{E}^b \quad (6)$$

and verify that

$$\{C(N), C(M)\} = C(\mathcal{L}_N M) \text{ and } \{C(N), A_a\} = \mathcal{L}_N A_a , \quad (7)$$

where $C(N) = \int_\Sigma N^a C_a$ and $\mathcal{L}$ denotes the Lie derivative. A similar equation holds for the Poisson brackets of $C(N)$ with $\tilde{E}^a$. This shows that the C_a generate diffeomorphisms. At this stage it is important to point out that the constraints (3) imply the diffeomorphisms (6), but the converse is not true. Thus the constraints $F_{ab}=0$, although the same in number, are in this way "stronger than diffeomorphisms." The observables (4) are of course diffeomorphism-invariant.

I now describe the canonical quantization of this theory using a loop representation in a way similar to that used for $2+1$ gravity.[3] There are two ways to do this. One way is to use the constraints (3) and physical observables (4) and find a realization of the observable algebra on functions of loops (loop space representation). The other way, which is discussed briefly later, involves replacing the constraint $F_{ab}=0$ by the diffeomorphism constraint C_a. It may be argued that doing the latter is not relevant to the original theory since the diffeomorphism constraint does not arise directly in the $3+1$ decomposition of the action (1). While this is true, it is also true that the theory is diffeomorphism-invariant and investigating the differences in the resulting quantum theories may help in understanding a similar issue in $3+1$ gravity. [This issue is the following. In the Lagrangian basis for Ashtekar's variables due to Samuel,[9] the action is the integral of a four-form. In its $3+1$ decomposition the diffeomorphism (and Hamiltonian constraints) do not arise directly but are derived as the C_a was above. There is, however, another Lagrangian formulation for which this is not the case.[10]]

Considering first the situation with the constraint $F_{ab}=0$, one sees that on the constraint surface, the nontrivial observables $T^0[\gamma]$ will depend only on the set of *noncontractible loops* γ in Σ. Analogously, since the Gauss-law constraint is the same as $\partial_{[a}B_{bc]}=0$ (vanishing "curvature" of a two-form), the nontrivial observables $T^1[S]$ depend only on the set of closed, *noncontractible two-surfaces* S in Σ. The term *noncontractible* here, and in the rest of the paper, is used to signify nontrivial homology, since the constraints select the classes of loops and closed two-surfaces that are not boundaries.

There are thus as many T^0's as there are generators of the first homology group $H_1(\Sigma)$ and as many T^1's as there are generators of the second homology group $H_2(\Sigma)$. To see if these observables can form a basis for the phase space, one must check that the loop and surface observables are the same in number (since the phase space must be even-dimensional). That this is so follows from a theorem that $H_1(\Sigma)$ and $H_2(\Sigma)$ are isomorphic for a three-manifold (Poincaré duality). This will become clearer when some specific topologies are discussed below.

To construct a representation of the observable algebra, and hence obtain a quantum theory, consider complex-valued functions of the homotopy classes of (homologically nontrivial) loops, $\mathcal{A}[\gamma]$, that satisfy $\mathcal{A}[\gamma]=\mathcal{A}[\gamma^{-1}]$ and $\mathcal{A}[\alpha\cup\beta]=\mathcal{A}[\alpha\circ\beta]$. These relations reflect the parametrization invariance of the loops and the Abelian nature of the holonomies. The actions of the *operator* observables $\mathcal{T}$ are defined by

$$\mathcal{T}^0[\alpha]\mathcal{A}[\beta] = \mathcal{A}[\alpha\circ\beta] \quad (8)$$

and

$$\mathcal{T}^1[S]\mathcal{A}[\alpha] = \hbar\Delta(\alpha,S)\mathcal{A}[\alpha] , \quad (9)$$

where $\alpha\circ\beta$ is the homotopy class of the loop formed by composing the loops α and β (via their common base point). Note that in this realization of the algebra, the $\mathcal{T}^1$ is diagonal. From (8) and (9), the commutator is

$$[\mathcal{T}^0[\gamma], \mathcal{T}^1[S]] = -\hbar\Delta(\gamma,S)\mathcal{T}^0[\gamma] \quad (10)$$

using $\Delta(\alpha\circ\beta,S)=\Delta(\alpha,S)+\Delta(\beta,S)$. It follows from this that the Poisson algebra is recovered in the classical limit

$$\lim_{\hbar\to 0} \frac{1}{i\hbar}[\ ,\ ] = \{\ ,\ \} .$$

There is a natural inner product on this space, namely, that the functions of homotopy classes that are different eigenfunctions of $\mathcal{T}^1$ are orthogonal. The $\mathcal{T}^0$ and $\mathcal{T}^1$ are analogous to the raising and number operators for the simple harmonic oscillator as may be seen by comparing Eq. (10) with $[a^\dagger, N] = -a^\dagger$.

I now turn to the specific examples for spatial topology, $S^2\times S^1$ and T^3, and specialize the representation (8), (9) for these cases.

For the topology $S^2\times S^1$, there is only one (homologically) noncontractible loop (that which wraps around the S^1), and so there is one basic observable of the type T^0 that I will call $T^0[a]$. (There are more operators based on this loop labeled by integers $n, T^0[na]$, but it is sufficient to consider only the "generators".) Similarly, there is only one noncontractible closed two-surface in this space, namely, that which wraps around the S^2. The corresponding observable is $T^1[S^2]$. [Note that while any genus surface may be embedded into $S^1\times S^2$, any extra handles on the noncontractible S^2 may be shrunk away on the constraint surface without obstruction. Also, $H_1(S^1\times S^2)=H_2(S^1\times S^2)=Z$.]

The loop space states, $|n\rangle$ $(=|-n\rangle)$ are labeled by integers corresponding to the winding numbers n around the S^1. The definitions of the operators $\mathcal{T}$ (8), (9) become for this case

$$\mathcal{T}^0[a]|n\rangle=|n+1\rangle \quad \text{and} \quad \mathcal{T}^1[S^2]|n\rangle=\hbar n|n\rangle \ . \tag{11}$$

It is easily verified that the correct commutation relations are satisfied. The inner product is simply $\langle m|n\rangle=\delta_{n,m}$.

The T^3 case is a bit more complicated. There are three basic noncontractible loops (those that wrap around each S^1 in T^3). There are thus three T^0 observables, one for each loop: $T^0[a]$, $T^0[b]$, and $T^0[c]$. Similarly, there are three basic noncontractible two-surfaces and hence three T^1 observables. These correspond to the three two-tori in T^3 so that these observables may be specified by pairs of noncontractible loops: $T^1[a,b]$, $T^1[b,c]$, and $T^1[a,c]$. [Note that $H_1(T^3)=H_2(T^3)=Z^3$.]

The loop space is defined by the states $|n_1,n_2,n_3\rangle(=|-n_1,-n_2,-n_3\rangle)$, with the three integers n_i corresponding to winding numbers around each S^1. The six operators corresponding to the observables $\mathcal{T}^0$ and $\mathcal{T}^1$ act on these states as

$$\mathcal{T}^0[a]|n_1,n_2,n_3\rangle=|n_1+1,n_2,n_3\rangle$$

and

$$\mathcal{T}^1[a,b]|n_1,n_2,n_3\rangle=\hbar(n_1+n_2)|n_1,n_2,n_3\rangle \ , \tag{12}$$

with similar definitions for the others. It is again easy to verify that these satisfy the correct commutation rules (10). In particular, one finds that

$$[\mathcal{T}^0[a],\mathcal{T}^1[a,b]]=-\hbar\mathcal{T}^0[a]$$

and (13)

$$[\mathcal{T}^0[a],\mathcal{T}^1[b,c]]=0 \ .$$

The inner product is $\langle m_1,m_2,m_3|n_1,n_2,n_3\rangle=\delta_{m_1,n_1}\delta_{m_2,n_2}\delta_{m_3,n_3}$. The Hilbert space for T^3 is thus three copies of the one for $S^1\times S^2$.

NON-ABELIAN THEORY

I now describe, along similar lines, a non-Abelian generalization of this theory. The action (1) is now replaced by $\mathrm{Tr}\int^4 F\wedge B$ where F is the curvature of a connection A^i_μ where the indices i are internal Lie-group indices. The two-form $B^i_{\mu\nu}$ also acquires an internal index. The canonical breakdown shows that the phase-space variables are $(A^i_a,\tilde{E}^{ai})$ with $\tilde{E}^{ai}=\epsilon^{abc}B^i_{bc}$. The constraints are

$$F^i_{ab}=0, \quad D_a\tilde{E}^{ai}=0 \tag{14}$$

where D_a is the covariant derivative. There are again two sets of physical observables. One set is parametrized by loops, and the other by loops *and* closed two-surfaces in Σ. These are

$$T^0[\gamma](A)=\mathrm{Tr}[U_\gamma(s)]$$

and (15)

$$T^1[\{\beta\},S](A,\tilde{E})=i\int_S d^2\sigma\, n_a(\sigma)\mathrm{Tr}[\tilde{E}^a(\sigma)U_{\beta(\sigma)}(\sigma)] \ .$$

The second observable T^1 requires some explanation. Consider a loop $\beta(\sigma)$ (in Σ) attached to the surface S at the point σ, and at this point construct the trace $\mathrm{Tr}[\tilde{E}^a(\sigma)U_{\beta(\sigma)}(\sigma)]$. This quantity is clearly invariant under the transformations generated by the Gauss-law constraint. Consider now a neighboring point σ' on S and define a loop $\beta(\sigma')$ based at this point as follows: introduce a curve γ that connects σ and σ' and let $\beta(\sigma')=\gamma^{-1}\circ\beta\circ\gamma(\sigma')$ where $\circ$ denotes the usual composition of loops and γ^{-1} is the opposite traversal of the segment γ. (This, in fact, is the construction used to establish the isomorphism of the fundamental groups based at different points.) After constructing the holonomy associated with this new loop, the trace with $\tilde{E}$ is constructed as before. The base point of the original loop is smoothly extended in this way to all points of S, the holonomies at each point constructed and then traced in the above way. The integral is then a sum of all these contributions depending on the extended set of loops denoted $\{\beta\}$.

A few remarks are in order regarding this construction. Firstly, the $U_\beta(\sigma)$ are in this way extended to all points σ' in S. In particular, the construction does not change the homology class of the original loop. Secondly, this extension involves introducing arbitrary curves γ (for each point on S) which appear to imply an infinite set of such observables. This is true *off* the constraint surface only. As will be pointed out below, for a specific spatial topology, there is always a finite number of these observables *on* the constraint surface, where one is concerned only with loops that are *homologically* nontrivial. Lastly, there is an ambiguity inherent in the definition of T^1 when the two-surface S is other than S^2. This arises because there is more than one way of setting up a loop at a point σ' given one at σ, due to the arbitrary nature of the curve γ that connects these two points. In particular one may choose another curve γ' such that the closed curve $\gamma^{-1}\circ\gamma'$ is noncontractible. This situation is avoided for S^2 where all such closed curves are contractible. The remaining discussion for the non-Abelian theory is therefore restricted to the spatial topology $S^1\times S^2$.

To see that T^1 has vanishing Poisson brackets with the $F=0$ constraint, we first observe that

$$\{F(\Lambda),n_a(x)E^{ai}(x)\}=\varepsilon^{abc}n_a(x)D_b\Lambda^i_c(x)$$

[where $F(\Lambda)=\int_\Sigma\epsilon^{abc}\Lambda^i_aF^i_{bc}$]. Therefore,

$$\begin{aligned}\{F(\Lambda),T^1[\{\beta\},S]\}&=\int_S\mathrm{Tr}[U_\beta(d_b\Lambda_c+[A_b,\Lambda_c])]\epsilon^{abc}n_a d^2\sigma\\&=\int_S\mathrm{Tr}[d_a(U_\beta\Lambda_b)-(d_aU_\beta)\Lambda_b+U_\beta[A_a,\Lambda_b]]dS^{ab}\\&=\int_S\mathrm{Tr}[\Lambda_aD_bU_\beta]dS^{ab}=0 \ ,\end{aligned} \tag{16}$$

where $dS^{ab}=\epsilon^{abc}n_c d^2\sigma$. The third equality follows because $U_\beta \Lambda_b$ is a one-form and hence $\int_S d_a(U_\beta \Lambda_b)dS^{ab}=0$, and the last follows from the equation satisfied by the holonomy, namely $D_a U_\beta=0$.

Specializing now to the SU(2) case and using the fundamental representation for the traces, one finds that the observables form the following closed Poisson algebra

$$\{T^0[\alpha],T^1[\{\beta\},S]\}=i\Delta(\alpha,S)(T^0[\alpha\circ\{\beta\}]-T^0[\alpha\circ\{\beta\}^{-1}]) , \tag{17}$$

$$\{T^1[\{\alpha\},S],T^1[\{\beta\},S']\}=i\Delta(\alpha,S')(T^1[\{\alpha\circ\beta\},S]-T^1[\{\alpha\circ\beta^{-1}\},S])-i\Delta(\beta,S)(T^1[\{\beta\circ\alpha\},S']-T^1[\{\beta\circ\alpha^{-1}\},S']) , \tag{18}$$

where

$$T^1[\{\alpha\circ\beta\},S]=\int_S d^2\sigma\, n_a \mathrm{Tr}[U_{\alpha(\sigma)}(\sigma',\sigma)\tilde{E}^a(\sigma)U_{\alpha(\sigma)}(\sigma,\sigma')U_{\beta(\sigma')}(\sigma',\sigma')] \tag{19}$$

and $\Delta(\alpha,S)$ is as defined in Eq. (5). The precise path along the combination $\alpha\circ\beta$ between the surfaces S and S' that occurs in the Poisson brackets is indicated in Eq. (19). The relation $\mathrm{Tr}(A\tau^i)\mathrm{Tr}(B\tau^i)=\mathrm{Tr}(AB)-\mathrm{Tr}(AB^{-1})$, where A,B are SU(2) matrices and τ^i are the Pauli matrices, has been used in calculating the Poisson brackets.

As for the Abelian case, on the constraint surface these observables depend only upon the noncontractible loops and closed two-surfaces in Σ. A representation of their algebra may again be realized on complex valued functions of homotopy classes of loops, $\mathcal{A}[\gamma]$. However, there are now relations between the states induced by the equation $\mathrm{Tr}(A)\mathrm{Tr}(B)=\mathrm{Tr}(AB)+\mathrm{Tr}(AB^{-1})$ for SU(2) matrices, namely, $\mathcal{A}[\alpha\cup\beta]=\mathcal{A}[\alpha\circ\beta]+\mathcal{A}[\alpha\circ\beta^{-1}]$. The representation is defined by

$$\mathcal{T}^0[\alpha]\mathcal{A}[\beta]=\mathcal{A}[\alpha\circ\beta]+\mathcal{A}[\alpha\circ\beta^{-1}] , \tag{20}$$

$$\mathcal{T}^1[S,\alpha]\mathcal{A}[\beta]=\hbar\Delta(\beta,S)(\mathcal{A}[\alpha\circ\beta]-\mathcal{A}[\alpha\circ\beta^{-1}]) . \tag{21}$$

The $\mathcal{T}^1$ observable is now not diagonal (as was true for the Abelian case). Using these definitions it is straightforward to verify that the commutator algebra reduces to the Poisson algebra (17), (18) in the classical limit.

For the topology $S^1\times S^2$, on the constraint surface, there are again only two observables: $T^0[a]$ and $T^1[a,S]$. The representation (18), (19) specializes for this case with the definitions

$$\begin{aligned}\mathcal{T}^0[a]|n\rangle&=|n+1\rangle+|n-1\rangle ,\\ \mathcal{T}^1[a,S]|n\rangle&=\hbar n(|n+1\rangle-|n-1\rangle) .\end{aligned} \tag{22}$$

Using these one can verify the commutator $[\mathcal{T}^0[a],\mathcal{T}^1[a,S]]=-\hbar(\mathcal{T}^0[a^2]-\mathcal{T}^0[\circ])$, where $\mathcal{T}^0[\circ]=2$ and $\circ$ denotes the zero loop. The other commutators are zero. This is consistent with (17) and (18) for which the right-hand side (RHS) vanishes if the loops and surfaces are the same (as is the case here). The inner product may again be taken to be $\langle m|n\rangle=\delta_{m,n}$. However, neither operator is diagonal in this basis.

As pointed out above, the observable T^1 ceases to be well defined for any spatial topology for which the embedded noncontractible two-surfaces have genus greater than zero (S^2). Since the phase space for all cases is even-dimensional, there should exist observables linear in the momenta that are partners of the T^0 for any topology. It is not clear what these are.

The physical observables defined in Eq. (15) may be applied for any non-Abelian group. However, in order to calculate the Poisson brackets and see if the algebra is closed, one would need trace identities for the relevant group, analogous to those used for SU(2). It is these identities that determine the action of the observables in the loop representation. Furthermore, the number of these observables on the reduced phase space appears to be independent of the group since the reduction via the constraints depends only on the spatial topology. However if the trace identities for the relevant group are such that the Poisson algebra no longer closes, then there will be other observables generated via the Poisson brackets. The question of how many observables there are will then depend on the group and a hint toward the answer provided by the details of the Poisson brackets.

CONCLUSIONS

To conclude, I discuss briefly an alternative way of viewing the quantum theory for the Abelian case. This comes about by using the diffeomorphism constraints C_a (6) explicitly, rather than $F_{ab}=0$, to carry out the quantization. Consider now complex-valued functionals of loops (*not* homotopy classes of loops) $\mathcal{A}[\gamma]$, the physical states are determined by imposing the diffeomorphism constraints on this space via the Dirac procedure. The result, as for $3+1$ gravity,[6] is that the physical states are functions of the *knot classes* of loops on Σ. Further, on the constraint surface, the observable T^0 would depend on the knot classes as well, and T^1 on the diffeomorphism equivalence classes of closed two-surfaces in Σ. From this point of view, though, it is not clear at present how to define a representation of the observable algebra analogous to (8) and (9). The physical states that appear in this way seem, at first sight, to be the same as (at least a subset of) the ones for quantum gravity.[6] However, the loop space that one starts from in $3+1$ gravity is different in that the internal group there is SU(2), which results in certain linear relations among the unconstrained states. In the case of U(1) however, there are no relations. But it is not clear how the *physical* states, the functions of knot classes, retain information about the original internal group.

To summarize, the quantum theory of a particular generally covariant field theory has been described by constructing a representation of the Poisson algebra of its physical observables. Non-Abelian generalizations can be similarly discussed with some restrictions. It seems possible to quantize other topological field theories along these lines.

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