Structure of singularities in the spherical gravitational collapse of a charged null fluid

Kayll Lake and T. Zannias
Department of Physics, Queen's University at Kingston, Kingston, Ontario, Canada K7L 3N6
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We examine the structure of the spacetime singularities formed during the radial infall of a coherent stream of charged "photons"—a piece of the charged Vaidya metric. Under the assumption of homotheticity, we are able to give an essentially complete account of the singularities which develop. The model derives its usefulness from the very rich structure it exhibits, including the generic appearance of naked singularities.

I. INTRODUCTION

The now classic theorems of Hawking and Penrose[1] have established the existence of singularities, in the sense of timelike and null geodesic incompleteness, under conditions appropriate to general-relativistic cosmology and gravitational collapse. The true nature of these singularities remains unclear. Progress, however, can be made from the study of known exact solutions of Einstein's equations modeling various collapse scenarios. Unfortunately, only a few such solutions are presently known, and these are characterized by a number of simplifying assumptions. Therefore, adopting them as candidates to describe realistic collapse is a dubious procedure. Nevertheless, they do offer the opportunity to explore the properties of singular spacetimes and, in the case of curvature singularities, to address issues such as global or local nakedness[2] and strength.[3] From this perspective it is clear that the more solutions we have available the better. It is in this spirit that we present here an essentially complete analysis of the singularity structure associated with the homothetic[4] spherical collapse of a charged null fluid.[5] We do not claim that the radial infall of a "self-similar" stream of charged "photons" represents a physically realistic collapse scenario (see also note added). Rather, the usefulness of the model we present derives from the rather remarkably rich structure it exhibits.

Our discussion is organized in the following way. After a brief review of the model in Sec. II, the restriction to homotheticity is imposed in Sec. III. We discuss boundary conditions in Sec. IV, and Sec. V presents the global features of the model which we discuss in Sec. VI.

II. SPHERICAL COLLAPSE OF A CHARGED NULL FLUID

We consider the radial infall of a coherent stream of charged "photons"—the charged Vaidya metric.[6] To review the dynamics we work in ingoing Bondi coordinates[7] (v,r,θ,ϕ), which give the line element

$$ds^2=2e^{\psi}dv\,dr-\left[1-\frac{2m}{r}\right]e^{2\psi}dv^2+r^2d\Omega^2\ , \tag{1}$$

where $m=m(v,r)$, $\psi=\psi(v,r)$, $d\Omega^2\equiv d\theta^2+\sin^2\theta\,d\phi^2$, and $r\geq 0$ is an affine parameter along the null generators of the v=const null hypersurfaces. The function m has an invariant meaning. In particular, it is defined by[8]

$$m=\frac{L}{2}(1-g^{\mu\nu}\nabla_\mu L\nabla_\nu L)\ , \tag{2}$$

where

$$L^2=\tfrac{1}{2}g_{\mu\nu}(\xi^\mu_{(1)}\xi^\nu_{(1)}+\xi^\mu_{(2)}\xi^\nu_{(2)}+\xi^\mu_{(3)}\xi^\nu_{(3)})\ ,$$

and $\xi_{(i)}$, $i=1,2,3$ are the spacelike Killing vector fields obeying the algebra of SO(3):

$$[\xi_{(i)},\xi_{(j)}]=\epsilon_{ijk}\xi_{(k)}\ .$$

Moreover, m is related to the gravitational energy within a given orbit of the SO(3) group.

The model considered in this paper is obtained from an energy-momentum tensor of the form

$$T^\mu_\nu=\rho l^\mu l_\nu+E^\mu_\nu\ , \tag{3}$$

where $l_\mu=-\delta^v_\mu$ (so that $l_\mu l^\mu=0$), E^μ_ν is related to the electromagnetic tensor $F_{\mu\nu}$ in the familiar way

$$E_{\mu\nu}=\frac{1}{4\pi}\left[F_{\mu\alpha}F_\nu{}^\alpha-\frac{1}{4}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}\right]\ , \tag{4}$$

and $F_{\mu\nu}$ obeys Maxwell's equations $\nabla_{[\alpha}F_{\mu\nu]}=0$, and $\nabla^\alpha F_{\alpha\mu}=-4\pi J_\mu$, where J_μ is the four-current vector.[9]

For the metric (1) Einstein's field equations reduce to

$$\frac{\partial m}{\partial v}=4\pi r^2T^r_v\ , \tag{5a}$$

$$\frac{\partial m}{\partial r}=-4\pi r^2T^v_v\ , \tag{5b}$$

and

$$\frac{\partial\psi}{\partial r}=4\pi rT_{rr}\ . \tag{5c}$$

Without loss of generality we take the vector potential

$$A_\alpha=\frac{e(v)}{r}\delta^v_\alpha\ , \tag{6}$$

where $e(v)$ is arbitrary. As a result, the nonvanishing components of $F_{\mu\nu}$ are $F_{rv}=-F_{vr}=e(v)/r^2$, and it follows from (4) that

$$E^{\mu}_{\nu}=\frac{e^2(v)}{8\pi r^4}\mathrm{diag}(-1,-1,1,1)\ . \tag{7}$$

From (3), (5b), and (7), then,

$$m=F(v)-\frac{e^2(v)}{2r}\ , \tag{8}$$

where F is arbitrary. In addition, with (5a) and (8),

$$4\pi r^2\rho=\dot{F}-\frac{e\dot{e}}{r}\ , \tag{9}$$

where an overdot denotes d/dv, and finally, with (5c), $\psi=\psi(v)$, which allows us to set $\psi=0$ in what follows. Note that

$$J_\alpha=\frac{1}{4\pi}\frac{\dot{e}}{r^2}l_\alpha\ , \tag{10}$$

while the remaining Maxwell equations and Bianchi identities are automatically satisfied. Thus there exists two degrees of freedom describing the injection process: $F(v)$ and $e(v)$ describe the injected mass and charge, respectively. However, we should point out that the resulting stress tensor does not in general obey the weak-energy condition. In fact, it is straightforward to show that for every vector field k^α, $k^\alpha k_\alpha=-a^2$, the following equality holds:

$$T_{\mu\nu}k^\mu k^\nu=\rho(l_\mu k^\mu)^2+\frac{1}{8\pi}\frac{e^2(v)}{r^4}\{a^2+2r^2[(k^\theta)^2+(k^\varphi)^2\sin^2\theta]\}\ . \tag{11}$$

Thus local violations of the weak-energy condition take place in the regions of spacetime characterized by $\rho<0$.[10]

With (8), the Kretschmann scalar ($K\equiv R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$, $R_{\alpha\beta\gamma\delta}$ the Riemann tensor) for the metric (1) reduces to

$$K=\frac{48}{r^6}\left[F^2-2\frac{e^2F}{r}+\frac{7}{6}\frac{e^4}{r^2}\right]\ , \tag{12}$$

and so for F and $e\neq0$ the metric is scalar-polynomial (SP) singular along $r=0$. Further analysis of the structure of this singularity is initiated by a study of the transverse radial null geodesic equation

$$\frac{dr}{dv}=\frac{1}{2}\left[1-\frac{2F(v)}{r}+\frac{e^2(v)}{r^2}\right]\ . \tag{13}$$

In general, Eq. (13) does not yield to analytic solution. If, however, $F\propto v$ and $e^2\propto v^2$, Eq. (13) becomes homogeneous and can be solved in terms of elementary functions. This simplification in fact has an invariant geometrical significance to which we now turn.

III. HOMOTHETICITY

We consider the metric (1) in the range $0\le v\le v_1$ and m given by Eq. (8) with $F=\lambda v$ ($\lambda=\mathrm{const}>0$) and $e^2(v)=\mu^2v^2$ ($\mu^2=\mathrm{const}$). It follows that the vector field

$$\xi=v\frac{\partial}{\partial v}+r\frac{\partial}{\partial r} \tag{14}$$

generates infinitesimal homothetic motions. That is,

$$\nabla_\mu\xi_\nu+\nabla_\nu\xi_\mu=2g_{\mu\nu}\ . \tag{15}$$

However, as was shown elsewhere,[11] if a homothetic geometry admits null geodesics which are simultaneously orbits of the infinitesimal generator, then as long as at one point along such an orbit $K\neq0$ and $R_{\alpha\beta}\xi^\alpha\xi^\beta\neq0$, the orbits necessary (if incomplete) terminate in a strong curvature singularity. (For the model considered in this paper, $K\neq0$, but $R_{\alpha\beta}\xi^\alpha\xi^\beta$ can be 0.) To apply these results we note that the null orbits obey

$$\xi^\alpha\xi_\alpha=J[2-f(J)J]=0\ , \tag{16}$$

where $J=v/r$ and

$$f(J)=\mu^2J^2-2\lambda J+1\ . \tag{17}$$

The orbit $J\equiv0$ is an incoming one and essentially corresponds to the null (geodesic) generators of the $v\equiv0$ initial null hypersurface.[12] On the other hand, it is easy to verify that any solution of the algebraic equation

$$fJ-2=0 \tag{18}$$

corresponds to an outgoing null orbit. Furthermore, along any solution of (18) we have

$$\nabla_\mu(\xi^\alpha\xi_\alpha)=-2(-\lambda J^2+J-3)\xi_\mu\ , \tag{19}$$

and so every null orbit is a null geodesic orbit. Furthermore, they are complete[12,13] provided

$$\lambda J^2-J+1=0\ . \tag{20}$$

Thus, by studying the spectrum of solutions of the algebraic equation (18) in combination with (19) and (20), the structure of the central singularity is exposed. However, before doing so, let us describe the model in more detail.

IV. BOUNDARIES TO THE NULL FLUID

For $v<0$ we take $F(v)=e(v)=0$, and for $v>v_1$ we take $\dot{F}=\dot{e}=0$, but F and e^2 positive definite. Thus, before the influx of charged null fluid, we have Minkowski space, whereas after we have a piece of the Reissner-Nordström solution. Whereas much progress has been made recently in the study of singular null surfaces,[14] the surfaces $v=0$ and v_1 considered here are simply boundary surfaces. In particular, for metrics of the form (1) with m given by (8) and $\psi=0$, it follows that the coordinates (v,u,θ,ϕ) are admissible ($g_{\alpha\beta}\in C^1$) across Σ defined by $\tilde{v}=v-\alpha=0$, where u is defined by

$$r=\frac{u-\tilde{v}}{2}-\frac{F(v)\tilde{v}}{u}+\frac{e^2(v)\tilde{v}}{u^2}>0\ , \tag{21}$$

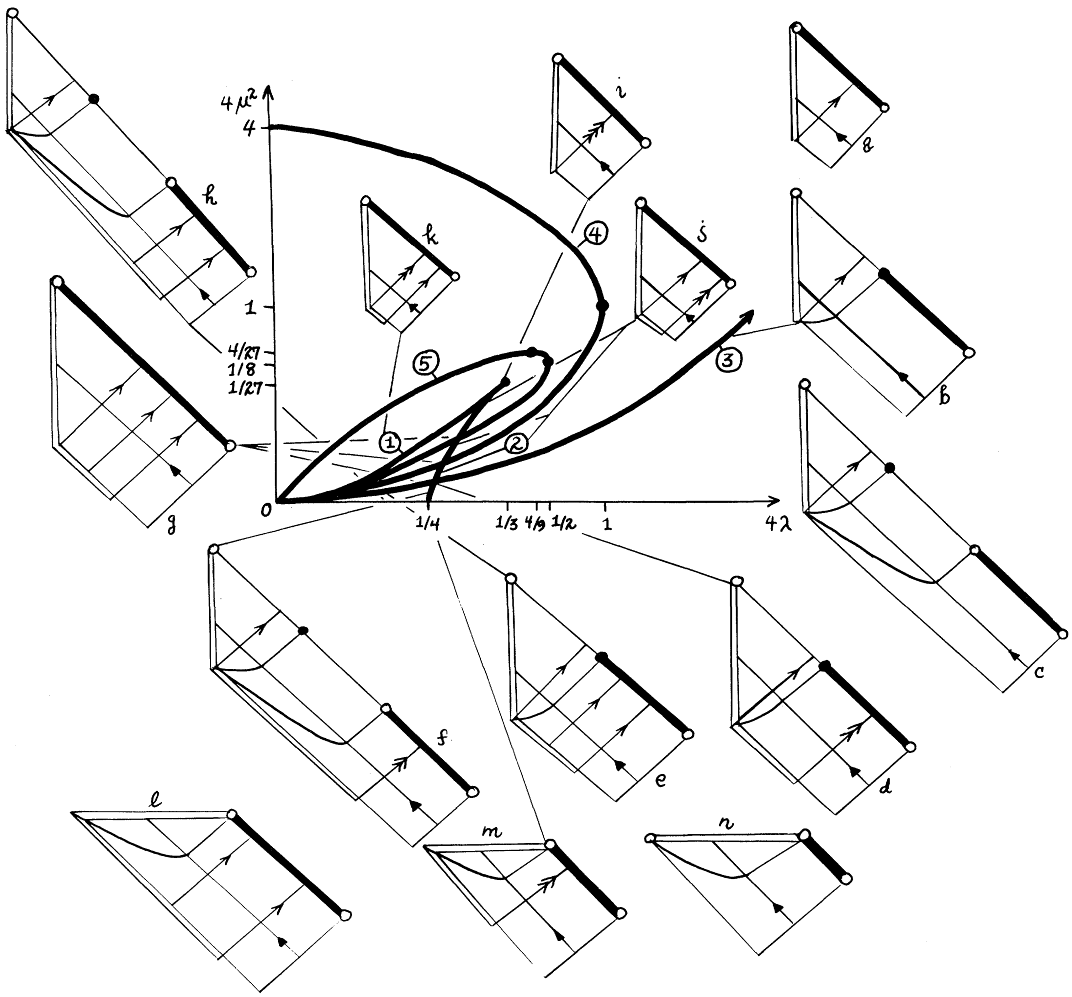

FIG. 1. Essential properties of the spacetimes. Some characteristic loci are shown in the $4\mu^2$-4λ plane. The loci, which are not drawn to scale, are defined as follows: Along 1 and 2 $\mu^2=(\lambda-\frac{1}{18})\{1\pm[1-16\lambda^2(\lambda-16)/3(\lambda-\frac{1}{18})^2]^{1/2}/3\}$ with + along 1 and − along 2, $\mu^2=\lambda^2$ along 3, $\lambda=|\mu|(1-|\mu|)$ along 4, and $\mu^2=\lambda[1\pm(1-8\lambda)^{1/2}]/4$ along 5. Some representative points are indicated. Note that 2 intersects 3, 4, and 5 at $(4\mu^2,4\lambda)=[(\frac{4}{27})^2,\frac{8}{27}]$, $(17-12\sqrt{2},\,8\sqrt{2}-11)$, and $(4/5^3,\frac{8}{25})$, respectively. Null orbits of the group are given by roots to $g(J)=Jf-2=0$. Along 1 and 2 there are two distinct roots, the coincident root being the largest along 1 and the smallest along 2. The loci intersect at $(\frac{1}{27},\frac{1}{3})$, where $g=[(J-6)/3]^3/4$. Outside the loci there is one root, whereas inside there are three distinct roots. The locus 4 gives complete null orbits and these have $J=1/|\mu|$. When there is more than one null orbit along 4, the largest root gives the complete orbit. The weak-energy condition is violated for $J>\lambda/\mu^2\equiv J_0$. Along the locus 5, J_0 is a null orbit, and when there is more than one null orbit along 5, J_0 is the largest root. For the orbits J_0 alone, $R_{\alpha\beta}\xi^\alpha\xi^\beta=0$. Within the locus 5 J_0 is timelike, whereas outside the locus it is spacelike. The local violation of the weak-energy condition is summarized in Fig. 2. Penrose diagrams for various regions of the $4\mu^2$-4λ plane are shown. Our conventions are the following: A double line indicates a singular boundary which, if it is null, is referred to as "shell focusing." The heavy boundary represents $\mathscr{I}^+$ (only the region within the Cauchy development is shown). The null boundary surface is indicated by a solid arrow. Null orbits are indicated by open arrows, the number of which gives the degree of degeneracy of the orbit. (That is, one for a root, two for a double root, and three at the intersection of locus 1 and 2.) On and below the locus 3, the roots to $f(J)=0$ are indicated. The corresponding trajectories J are spacelike within the null boundary and null exterior to it. Case *a* covers the region above 3 outside 1 and 2, *b* the locus 3 to the right of 2, *c* below 3 to the right of 2, *d* the intersection of 3 with 2, *e* along 3 to the left of 2, *f* along 2 below 3, *g* below 1 to the left of 2 above 3, *h* below 3 to the left of 2, *i* the intersection of 1 and 2; *j* along 2 above 3, and *k* along 1 to the left of 2. Also shown are the Vaidya cases $\mu=0$; *l* for $\lambda<\frac{1}{16}$, *m* for $\lambda=\frac{1}{16}$, and *n* for $\lambda>\frac{1}{16}$.

as long as F and e^2 are continuous.[15] For $v > v_1$ then, with the homothetic influx, we take $F = \lambda v_1$ and $e^2 = \mu^2 v_1^2$. From the properties of the Reissner-Nordström solution, it follows that for $\mu^2 > \lambda^2$ all outgoing radial null geodesics which cross v_1 reach $\mathcal{I}^+$. For $\mu^2 \le \lambda^2$ outgoing radial null geodesics which cross v_1 with $r > v_1/\lambda[1+(1-\mu^2/\lambda^2)^{1/2}]$ reach $\mathcal{I}^+$.

V. PROPERTIES OF THE SPACETIMES

The previous sections make clear the meaning of the spacetimes for which the metric is given by (1) with $\psi = 0$ and m given by (8), with

$$F(v) = \begin{cases} 0, & v \le 0 , \\ \lambda v, & 0 \le v \le v_1 , \\ \lambda v_1, & v_1 \le v , \end{cases}$$

and

$$e^2(v) = \begin{cases} 0, & v \le 0 , \\ \mu^2 v^2, & 0 \le v \le v_1 , \\ \mu^2 v_1^2, & v_1 \le v . \end{cases}$$

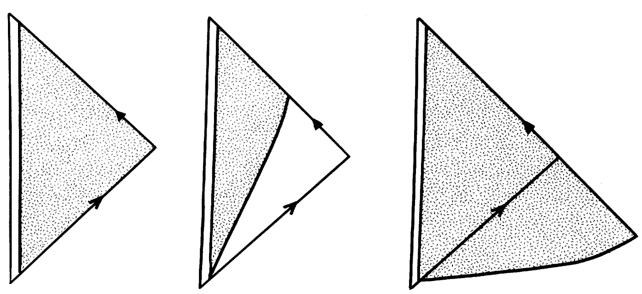

FIG. 2. Local violation of the weak-energy condition. Along with the conventions of Fig. 1, the shaded region indicates a local violation of the weak-energy condition. The region is bounded by $r=0$, $v=\alpha>0$ and $\mu^2 v = \lambda r$. (a) This shows the region associated with the locus 5 in Fig. 1. The region is visible from $\mathcal{I}^+$. (b) The region associated with the interior of the locus 5. [Along locus 1 the orbit shown is a double root, and at the intersection of 1 and 2 it is a triple root (see Fig. 1).] In all cases the region is visible from $\mathcal{I}^+$. (c) The region associated with the exterior of the locus 5. Along the locus 3 the lower boundary corresponds to $f=0$, whereas below 3 it is given by $f = 1-\lambda^2/\mu^2$. In these two cases alone the region is not visible from $\mathcal{I}^+$. Note that at no time does the region under consideration encompass the "shell focusing."

Figure 1 summarizes the essential global properties, and the caption gives an essentially complete discussion of these. Figure 2 demonstrates the extent to which these solutions violate the weak-energy condition.

VI. SUMMARY

We have examined the central shell-focusing singularity accompanying the collapse of a charged null fluid. Our analysis is based on the assumption that charged "photons" do not feel Coulomb repulsion, and consequently the mechanism responsible for the formation of the central null singularity is the same as the one characterizing the uncharged Vaidya model. (However, see also note added.) In the limit $\mu \to 0$ our results reduce to those obtained previously[16] (all results concerning the nakedness of the shell-focusing singularity characterizing the uncharged Vaidya model can be read from the $\mu \equiv 0$ axis of Fig. 1). However, as Fig. 1 shows, the presence of charge alters the global structure of the spacetime.

The model offers a very rich structure, as can be seen from Fig. 1. There are both incomplete and complete null orbits, the former being generic. Further, whereas $R_{\alpha\beta}\xi^\alpha\xi^\beta \neq 0$ is the generic condition of the model, $R_{\alpha\beta}\xi^\alpha\xi^\beta$ can be zero (see locus 5 of Fig. 1). This has the consequence that the shell-focusing singularity is weak[3] (as measured with respect to radial, but nonhomothetic, null geodesics) for $\mu^2 = 1/5^3$ and $\lambda = 2/5^2$. Otherwise, it is strong[3] (except along the single null orbit $J = \lambda/\mu^2$ along the locus 5). This behavior warrants further study in a more general context. This is under investigation.[17] It is also worth pointing out that our model shows that violation of the weak-energy condition does not necessarily prevent the formation of spacetime singularities. This further indicates that semiclassical gravity (where one expects violation of the weak-energy condition by the renormalized stress tensor) may not remove spacetime singularities. A full quantum theory of gravity may be needed.

Note added. After this paper was submitted for publication, we received a paper by Ori.[18] In this paper Ori carefully analyzes the equation of motion of null "charged photons." He shows that in general there exists a Lorentz force acting among the charged photons. When this force is taken into account, he concludes that the region where violation of the weak energy condition occurs [i.e., for $r < r_c$, $\rho(r_c)=0$; see Eqs. (9) and (11)] is avoided by the null shells. The spacetime is completed by joining along $r = r_c$ to another piece of an outgoing charged Vaidya solution.

In this regard we may note that neglecting the Lorentz force makes our model unrealistic. (However, we should keep in mind that it is a self-consistent solution of the Einstein equations which exhibits a remarkably rich singularity structure. As such, it may be valuable when one attempts to formulate the notion of cosmic censorship in concrete mathematical terms.) In addition, we may note that the global structure of the spacetime obtained by Ori is different than the one we have exhibited. However, the homothetic collapse still exhibits naked curvature singularities. In fact, by inspection of the conformal diagrams (see Figs. 1 and 2), it is easy to see that for any values of (λ,μ), for which the familiar interpretation exhibits naked singularities, the new interpretation will exhibit one too. (Of course, globally, the two solutions are characterized by a different structure.)

ACKNOWLEDGMENTS

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[1]See S. W. Hawking and G. F. R. Ellis, *The Large Scale Structure of Spacetime* (Cambridge University Press, Cambridge, England, 1973); F. J. Tipler, C. J. S. Clarke, and G. F. R. Ellis, in *General Relativity and Gravitation,* edited by A. Held (Plenum, New York, 1980); R. M. Wald, *General Relativity* (University of Chicago Press, Chicago, 1984).

[2]Globally or locally naked singularities here means singularities violating the weak or strong form of the cosmic-censorship hypothesis. Discussion of the two forms of this hypothesis can be found in R. Penrose, Riv. Nuovo Cimento **1**, 252 (1969); in *General Relativity, an Einstein Centenary Volume,* edited by S. W. Hawking and W. Israel (Cambridge University Press, Cambridge, England, 1979).

[3]Regarding the strength of singularities, see Tipler, Clarke, and Ellis (Ref. 1); C. J. S. Clarke and A. Królak, J. Geom. Phys. **2**, 127 (1986); R. P. A. C. Newman, Class. Quantum Grav. **3**, 527 (1986). The strength of singularities in spherically symmetric homothetic spacetimes has been examined by K. Lake, Phys. Rev. Lett. **60**, 241 (1988); B. Waugh and K. Lake, Phys. Rev. D **38**, 1315 (1988); **40**, 2137 (1989); V. Gorini, G. Grillo, and M. Pelizza, Phys. Rev. A **135**, 154 (1989); A. Ori and T. Piran, Phys. Rev. D **42**, 1068 (1990). See also R. P. A. C. Newman (above), and Ref. 16 (below).

[4]We refer to geometries as homothetic if they admit a homothetic Killing vector field. Often in the literature these are also referred to as "self-similar."

[5]The metric appropriate to the situation has been given by J. Plebanski and J. Stachel, J. Math. Phys. **9**, 269 (1968). A physical interpretation is given by W. B. Bonnor and P. C. Vaidya, Gen. Relativ. Gravit. **1**, 127 (1970).

[6]The charged Vaidya metric has been used by Sullivan and Israel to address issues such as the third law of black-hole mechanics. See B. T. Sullivan and W. Israel, Phys. Lett. **79A**, 371 (1980). Also see E. Poisson and W. Israel, Phys. Rev. D **41**, 1976 (1990).

[7]H. Bondi, M. G. J. van der Burg, and A. W. K. Metzner, Proc. R. Soc. London **A269**, 21 (1962).

[8]T. Zannias, Phys. Rev. D **41**, 3252 (1990). Independently, a covariant characterization of this mass has been advanced by Poisson and Israel (Ref. 6).

[9]We follow the sign conventions (for signature, Riemann, Maxwell's equations, etc.) of Wald (Ref. 1).

[10]One way to avoid violation of the weak-energy condition is by appropriate choice of the characteristic initial data. However, for homothetic geometries, this is not possible.

[11]K. Lake and T. Zannias, Phys. Rev. D **41**, 3866 (1990).

[12]Along the $J=0$ incoming orbit, we have $\nabla_\mu(\xi^\alpha\xi_\alpha)=2\xi_\mu$, $R_{\alpha\beta}\xi^\alpha\xi^\beta|_{J=0}=0$, and $K=0$, implying that it is an incomplete null orbit.

[13]For further discussion, see T. Zannias, Queen's University report, 1990 (unpublished).

[14]For recent work on singular null hypersurfaces, see C. J. S. Clarke and T. Dray, Class. Quantum Grav. **4**, 265 (1987); C. Barrabès, *ibid.* **6**, 581 (1989); C. Barrabès and W. Israel, University of Alberta report, 1990 (unpublished).

[15]This procedure is a generalization of the transformation introduced by W. G. Unruh, Phys. Rev. D **31**, 2693 (1985).

[16]See, for example, W. A. Hiscock, L. C. Williams, and D. M. Eardley, Phys. Rev. D **26**, 751 (1982); A. Papapetrou, in *A Random Walk in Relativity and Cosmology,* edited by N. Dadhich, J. K. Rao, J. V. Narlikar, and C. V. Vishveshwara (Wiley, New York, 1985); Y. Kuroda, Prog. Theor. Phys. **72**, 63 (1984). The strength (with respect to radial null geodesics) of the shell-focusing singularities for a general Vaidya spacetime has been examined by K. Rajagopal and K. Lake, Phys. Rev. D **35**, 1531 (1987). A more general analysis for outgoing causal geodesics, but restricted to the homothetic case, has been given by I. H. Dwivedi and P. S. Joshi, Class. Quantum Grav. **6**, 1599 (1989).

[17]K. Lake and T. Zannias (unpublished). Note that, unlike the Vaidya case, the shell-focusing singularity cannot be classified simply as weak or strong even with respect to radial null geodesics.

[18]A. Ori, Caltech report, 1990 (unpublished).