Heavy-quark mesons in a relativistic quark model

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The Coulomb-gauge QCD Hamiltonian, augmented by a term which produces linear, scalar confinement is used to generate kernels for the bound-state amplitudes and eigenvalues of heavy quarkonium. The model generates all the correct perturbative physics up to order α_s^4 and is applied to the $c\bar{c}$ and $b\bar{b}$ systems where good agreement with experiment is obtained.

I. INTRODUCTION

Generating the hadron spectrum from quantum chromodynamics (QCD) has proven to be a formidable task indeed. Unlike the situation in quantum electrodynamics (QED), the scale Λ at which the coupling becomes strong is precisely the region of interest, which in turn is set by the quark masses. Thus, since perturbation theory is not adequate, alternative nonperturbative methods must be used. To be sure, there are very encouraging results coming from numerical simulations of QCD on a space-time lattice. However even on rather modest lattices this approach reaches the envelope of present-day supercomputer technology.

On the other hand, potential models, although lacking the *ab initio* quality of the lattice approach, are easily interpreted and tractable.

Relativistic potential models (and strong coupling does imply relativistic as $v/c \sim \alpha$) fall into one of two classes. Relativistic corrections are either generated from a reduction of the explicitly covariant Bethe-Salpeter equation[1,2] or from a reduction of the scattering amplitude. In the latter approach relativistic kinematics are added to generate a Schrödinger-like equation.[3]

In this paper we are proposing a third approach based on the variational method in quantum field theory.[4] The method's attraction is twofold. Since, within the variational method, it is natural to work in the Hamiltonian formulation, the connection to ordinary nonrelativistic quantum mechanics is more immediate and the physical interpretation clearer than in the Bethe-Salpeter formalism. The method's second attraction is that it is inherently nonperturbative and one can obtain nonperturbative information[5] in field theories that are strongly coupled.

Although in principle we could apply the variational method to a direct solution of QCD,[6] we present a much more modest attempt here. Our variational *Ansatz* will not be sensitive to the non-Abelian structure of QCD. We will assume we have integrated out all the low-energy gluon degrees of freedom, which results in a linearly confining term of an effective field theory. We are guided by the Monte Carlo simulations which precisely do just this and dictate the strength, form and Lorentz structure of the long-range potential.[7]

Our *Ansatz* will however contain transverse gluons and as a result we will pick up all the physics of one-gluon exchange and thus generate results accurate to order α_s^4. We would like to stress that a model such as this is arbitrarily accurate for arbitrarily heavy-quark masses, as the system becomes insensitive to the confining potential and sits deep in the Coulomb well, modified by small relativistic corrections. The former point although obvious does not appear to be widely appreciated. The model in effect ceases to be a model of QCD but *is* QCD.

In Sec. II we present our model and *Ansatz* for arbitrary quantum numbers. The results and conclusions are presented in Sec. III where we compare with the experimental $c\bar{c}$ and $b\bar{b}$ mass values. We note results have been previously communicated.[8]

II. THE MODEL HAMILTONIAN AND ANSATZ

Our model Hamiltonian is the Hamiltonian of QCD in the Coulomb gauge augmented by a term which produces linear scalar confinement. Of course, in a true solution of the theory one would not need this additional term as all the nonperturbative confining physics is generated by the QCD Hamiltonian alone.

As our *Ansatz* is not explicitly sensitive to the non-Abelian terms of the Hamiltonian our effective Hamiltonian is given by

$$H = H_q + H_g + H_c + H_{qg} + H_s \ , \tag{1}$$

where

$$H_q = \int d^3x\, \bar{q}(\mathbf{x},0)(-i\nabla\cdot\boldsymbol{\gamma} + m_0)q(\mathbf{x},0) \ ,$$

$$H_g = \tfrac{1}{2}\int d^3x \{ \dot{\mathbf{A}}_a^2(\mathbf{x},0) + [\nabla\times\mathbf{A}_a(\mathbf{x},0)]^2 \} \ ,$$

$$H_c = \frac{\alpha_s}{2}\int d^3x\, d^3y\, q^\dagger(\mathbf{x},0)\frac{\lambda^a}{2}q(\mathbf{x},0) \times \frac{1}{|\mathbf{x}-\mathbf{y}|} q^\dagger(\mathbf{y},0)\frac{\lambda^a}{2}q(\mathbf{y},0) \ ,$$

$$H_{gq} = g_s \int d^3x\, \bar{q}(\mathbf{x},0)\boldsymbol{\gamma}\cdot\frac{\lambda_a}{2}\mathbf{A}_a(\mathbf{x},0)q(\mathbf{x},0) \ ,$$

$$H_s = \frac{3b}{8}\int d^3x\, d^3y\, \bar{q}(\mathbf{x},0)\frac{\lambda^a}{2} \times q(\mathbf{x},0)|\mathbf{x}-\mathbf{y}|\bar{q}(\mathbf{y},0)\frac{\lambda^a}{2}q(\mathbf{y},0) \ .$$

Note we have suppressed Dirac, color and flavor indices on the quark field operators. It is important to realize that, apart from H_s (the confining term), the above Hamiltonian (although not manifestly covariant) would generate, in old-fashioned time-ordered perturbation theory, results identical to covariant perturbation theory.

Our variational *Ansatz* for the color-singlet, bound-state, quark-antiquark system in the center-of-mass system is

$$|\text{meson}\rangle = |q\bar{q}\rangle + |q\bar{q}g\rangle \ , \tag{2}$$

where

$$|q\bar{q}\rangle = \sum_{\substack{\sigma\delta \\ i}} \int d^3p\, F(\mathbf{p},\sigma,\delta) b_i^\dagger(\mathbf{p},\sigma) d_i^\dagger(-\mathbf{p},\delta)|0\rangle \ ,$$

$$|q\bar{q}g\rangle = \sum_{\substack{s,s',\lambda \\ a,i,j}} \int d^3p\, d^3q\, G(\mathbf{p},\mathbf{q},s,s',\lambda) b_i^\dagger(\mathbf{p},s) \times d_j^\dagger(\mathbf{q},s')\frac{\lambda^a}{2} ija_a^\dagger(-\mathbf{p}-\mathbf{q},\lambda)|0\rangle \ .$$

The operators $b^\dagger$, $d^\dagger$, and $a^\dagger$ are creation operators for quark, antiquark, and gluon fields with the momentum, color, and polarization indicated. The functions F and G are variational coefficients. For example, the form of $F(G)$ is

$$F(\mathbf{p},\delta,\delta) = f(p)\bar{u}(\mathbf{p},\delta)\gamma_5 v(-\mathbf{p},\delta) \ ,$$

$$G(\mathbf{p},\mathbf{q},s,s',\gamma) = g(|\mathbf{p}+\mathbf{q}|)\bar{u}(\mathbf{p},s)\gamma_5 \not{\epsilon}(-\mathbf{p}-\mathbf{q},\gamma) \times v(\mathbf{q},s')$$

if one wishes to construct a pseudoscalar state ($J^{PC}=0^{-+}$). The function $f(p)(g(|\mathbf{p}+\mathbf{q}|))$ depends only on the magnitude of $\mathbf{p}(\mathbf{p}+\mathbf{q})$ and it is this remaining dependence which is in principle optimized.

We next construct the quantity

$$\frac{\langle \text{meson}|H|\text{meson}\rangle}{\langle \text{meson}|\text{meson}\rangle} \ ,$$

which is a *functional* of F and G. Applying the variational principle and taking the functional derivatives,

$$\delta M = \delta \frac{\langle \text{meson}|H|\text{meson}\rangle}{\langle \text{meson}|\text{meson}\rangle} = 0 \ ,$$

leads to the coupled integral equations

$$\begin{aligned} MF(\mathbf{p}\sigma\delta) = {} & 2E_p F(\mathbf{p}\sigma\delta) + \frac{4\alpha_s}{3}\frac{m^2}{2\pi^2}\sum_{\sigma'\delta'}\int \frac{d^3q\, F(\mathbf{q}\sigma'\delta')}{E_pE_q|\mathbf{p}-\mathbf{q}|^2}\bar{u}(\mathbf{p}\sigma)u(-\mathbf{q}\sigma')\bar{v}(-\mathbf{q}\delta')v(\mathbf{p}\delta) \\ & + \left[\frac{4\alpha_s}{3}\right]^{1/2}\frac{m}{2\pi}\sum_{\sigma'\lambda}\int\frac{d^3q}{(E_pE_q|\mathbf{p}-\mathbf{q}|)^{1/2}}[G(\mathbf{p},-\mathbf{q}\sigma\sigma'\lambda)\bar{v}(-\mathbf{q}\sigma')\boldsymbol{\gamma}\cdot\boldsymbol{\varepsilon}(\mathbf{q}-\mathbf{p},\lambda)v(-\mathbf{p}\delta) \\ & \qquad - G(\mathbf{q},-\mathbf{p}\sigma'\delta\lambda)\bar{u}(\mathbf{p}\sigma)\boldsymbol{\gamma}\cdot\boldsymbol{\varepsilon}(\mathbf{p}-\mathbf{q},\lambda)u(\mathbf{q}\sigma')] \\ & + \frac{bm^2}{\pi^2}\sum_{\sigma'\delta'}\int\frac{d^3q\, F(\mathbf{q}\sigma'\delta')}{E_pE_q|\mathbf{p}-\mathbf{q}|^4}\bar{u}(\mathbf{p}\sigma)u(\mathbf{q}\sigma')\bar{v}(-\mathbf{q}\delta')v(-\mathbf{p}\delta) \ , \end{aligned} \tag{3}$$

$$\begin{aligned} MG(\mathbf{p},-\mathbf{q}ss'\lambda) = {} & (E_p+E_q+|\mathbf{p}-\mathbf{q}|)G(\mathbf{p},-\mathbf{q}ss'\lambda) \\ & + \frac{4\alpha_s}{3}\frac{m^2}{2\pi^2}\sum_{\sigma\sigma'}\int\frac{d^3k\, d^3k'}{(E_pE_qE_kE_{k'})^{1/2}}G(\mathbf{k},-\mathbf{k}'\sigma\sigma'\lambda)\delta^3(\mathbf{p}-\mathbf{q}-\mathbf{k}-\mathbf{k}') \\ & \times\left[\frac{\bar{u}(\mathbf{p}s)u(-\mathbf{k}\sigma)\bar{v}(-\mathbf{k}'\sigma')v(\mathbf{q}s')}{|\mathbf{p}-\mathbf{k}|^2}\right] \\ & + \left[\frac{4\alpha_s}{3}\right]^{1/2}\frac{m}{2\pi}\sum_\sigma\frac{1}{(E_pE_q|\mathbf{p}-\mathbf{q}|)^{1/2}}[F(\mathbf{p}s\sigma)\bar{v}(-\mathbf{p}\sigma)\boldsymbol{\gamma}\cdot\boldsymbol{\varepsilon}(\mathbf{q}-\mathbf{p},\lambda)v(-\mathbf{q}s') \\ & \qquad - F(\mathbf{q}\sigma s')\bar{u}(\mathbf{p}s)\boldsymbol{\gamma}\cdot\boldsymbol{\varepsilon}(\mathbf{q}-\mathbf{p},\lambda)u(\mathbf{q},\sigma)] \ . \end{aligned} \tag{4}$$

Solving the coupled equations would give a true variational upper bound to the quark-antiquark bound-state mass. Unfortunately this is an extremely difficult problem to solve. Therefore as in previous work[9] we drop the second term of the second equation. This amounts to dropping some terms of order α_s^5 and higher in the context of perturbation theory. From a physical point of view we are neglecting the Coulomb effects on the $|q\bar{q}g\rangle$ state and of course the structure of this "intermediate state" effects the structure of the $|q\bar{q}\rangle$ state in turn.

After making this approximation and substituting equation

$$\begin{aligned} G(\mathbf{p},-\mathbf{q}\sigma\delta\lambda) = {} & -\left[\frac{4\alpha_s}{3}\right]^{1/2}\frac{1}{2\pi|\mathbf{p}-\mathbf{q}|}\sum_{\sigma'}\frac{m}{(E_pE_q|\mathbf{p}-\mathbf{q})|^{1/2}}[F(\mathbf{p}\sigma\sigma')\bar{v}(-\mathbf{p}\sigma')\boldsymbol{\gamma}\cdot\boldsymbol{\varepsilon}(\mathbf{q}-\mathbf{p},\lambda)v(-\mathbf{q}\delta) \\ & \qquad - F(\mathbf{q}\sigma'\delta)\bar{u}(\mathbf{p}\sigma)\boldsymbol{\gamma}\cdot\boldsymbol{\varepsilon}(\mathbf{q}-\mathbf{p},\lambda)u(\mathbf{q}\sigma')] \end{aligned} \tag{5}$$

into Eq. (3) we obtain

$$MF(\mathbf{p}\sigma\delta)=2E_pF(\mathbf{p}\sigma\delta)-\frac{4\alpha_s}{3}\frac{m^2}{2\pi^2}\sum_{\sigma'\delta'}\int\frac{d^3q}{E_pE_q|\mathbf{p}-\mathbf{q}|^2}\Bigg[F(\mathbf{q}\sigma'\delta')u^\dagger(\mathbf{p}\sigma)u(\mathbf{q}\sigma')v^\dagger(\mathbf{q}\delta')v(\mathbf{p}\delta)$$
$$+\sum_\lambda F(\mathbf{q}\sigma'\delta')\bar{u}(\mathbf{p}\sigma)\boldsymbol{\gamma}\cdot\boldsymbol{\varepsilon}(\mathbf{q}-\mathbf{p},\lambda)u(\mathbf{q}\sigma')$$
$$\times\bar{v}(\mathbf{q}\delta')\boldsymbol{\gamma}\cdot\boldsymbol{\varepsilon}(\mathbf{q}-\mathbf{p},\lambda)v(\mathbf{p}\delta)\Bigg]$$
$$+\frac{bm^2}{\pi^2}\sum_{\sigma'\delta'}\int\frac{d^3q}{E_pE_q|\mathbf{p}-\mathbf{q}|^4}F(\mathbf{q}\sigma'\delta')\bar{u}(\mathbf{p}\sigma)u(\mathbf{q}\sigma')\bar{v}(-\mathbf{q}\delta')v(-\mathbf{p}\delta)\;, \tag{6}$$

where we can clearly identify the origin of the four terms. They are, respectively, fermion kinetic energy, instantaneous Coulomb, transverse photon and linear potential contributions. We note as in the previous equation we have performed a mass renormalization which is trivial at this stage as its amounts to identifying the self-energy contributions of a bare quark with the energy (E_p) of a physical quark. In practice, we performed an exactly analogous calculation of the one-particle energy, where in a relativistic theory the result can only be $(m_R^2+p^2)^{1/2}$. We identified the counterterm necessary to give a finite expression and as expected, the bound-state calculations with this counterterm gave the above finite expressions.

We choose a particular state with given J^{PC} quantum numbers by selecting a particular $\Gamma_Y=\Gamma\times Y_{lm}(\theta,\phi)$. Thus,

$$F(\mathbf{p},\sigma,\delta)=\bar{u}(\mathbf{p},\sigma)\Gamma_Y v(-\mathbf{p},\delta)\;,$$

where Γ is one of the sixteen Dirac matrices $\underline{I},\gamma_5,\gamma_\mu,\gamma_\mu\gamma_5,\sigma_{\mu\nu}$. The angular dependence is then completely specified and in all cases considered here the angular integration can be performed explicitly, leading finally to integral equations of the form

$$Mf(p)=2E_pf(p)-\frac{1}{4\pi}\int\frac{q}{p}dq\,f(q)\left[\frac{4\alpha_s}{3}K_g(p,q)-bK_s(p,q)\right]\,, \tag{7}$$

where $K_g(p,q)$ is the kernel which arises from all the effects of gluon exchange and $K_s(p,q)$ arises from the confining term.

In fact all nonexotic states can be accessed from one of three general sets

pseudoscalar: $\gamma_5\times Y_{lm}$ $\quad J^{PC}=0^{-+},1^{+-},2^{-+},\ldots,$

scalar: $I\times Y_{lm}$ $\quad J^{PC}=0^{++},1^{--},2^{++},\ldots,$

pseudovector: $\gamma_5\boldsymbol{\gamma}\times Y_{lm}$ $\quad J^{PC}=1^{++},2^{--},3^{++},\ldots,$

In coupling representation the *Ansätze* are

$$|q\bar{q}\rangle=2\sqrt{2}\int d^3p\,f(p)|L=l,S=0,J=l,m_j=m\rangle\quad(\text{pseudoscalar set, }\Gamma_Y=\gamma_5\times Y_{lm})\;, \tag{8}$$

$$|q\bar{q}\rangle=2\left[\frac{2l}{2l+1}\right]^{1/2}\int d^3p\,f(p)\frac{p}{E}\left[|L=l-1,S=1,J=l,m_j=m\rangle-\left[\frac{l+1}{l}\right]^{1/2}|L=l+1,S=1,J=1,m_j=m\rangle\right]$$
$$(\text{scalar set, }\Gamma_Y=I\times Y_{lm})\;, \tag{9}$$

$$|q\bar{q}\rangle=4(2l+1)\left[\frac{(l+1)(l+2)}{(2l+1)(2l+3)}\right]^{1/2}\int d^3p\,f(p)\frac{p}{E}|L=l+1,S=1,J=l+1,m_j=m\rangle$$
$$(\text{pseudovector set, }\Gamma_Y=\gamma^5\boldsymbol{\gamma}\times Y_{lm})\;. \tag{10}$$

Note the pseudoscalar set is a spin-singlet set. The scalar and pseudovector sets are spin triplets. Furthermore, the scalar set is a mixture of states with $L=J-1$ and $L=J+1$.

III. KERNELS

We have obtained general expressions for the kernels of all these sets. It is sometimes necessary to combine kernels either to obtain kernels for pure unmixed L states or for calculational ease. We will present the ${}^3P_2$ as a special example.

In the following, we present all the kernels in the more compact three-dimensional momentum space form. The corresponding radial kernels are given in the Appendix. In three-dimensional momentum space, the integral eigenvalue equation is

$$Mf(p)=2E_pf(p)-\frac{1}{2\pi^2}\int\frac{d^3qf(q)}{|\mathbf{p}-\mathbf{q}|^2}\left[\frac{4\alpha_s}{3}K_g(\mathbf{p},\mathbf{q})-\frac{b}{|\mathbf{p}-\mathbf{q}|^2}K_s(\mathbf{p},\mathbf{q})\right]\,, \tag{11}$$

where $K_g(\mathbf{p},\mathbf{q})$ and $K_s(\mathbf{p},\mathbf{q})$ are the gluon exchange and confining kernels, respectively.

(1) Pseudoscalar set.

All the kernels for the pseudoscalar set can be written, in general, as

$$K_g(\mathbf{p},\mathbf{q})=\frac{1}{2E_pE_q}\left[3E_pE_q-m^2-\mathbf{p}\cdot\mathbf{q}+2\frac{p^2q^2-(\mathbf{p}\cdot\mathbf{q})^2}{|\mathbf{p}-\mathbf{q}|^2}\right]P_l\left[\frac{\mathbf{p}\cdot\mathbf{q}}{p,q}\right], \tag{12}$$

$$K_s(\mathbf{p},\mathbf{q})=-\frac{1}{E_pE_q}(E_pE_q+m^2-\mathbf{p}\cdot\mathbf{q})P_l\left[\frac{\mathbf{p}\cdot\mathbf{q}}{pq}\right], \tag{13}$$

where the P_l are the Legendre functions.

(2) Scalar set.

This is a mixed state set except for the 0^{++} sector. We present our kernels in both the mixed state form and the pure state case.

(a) In a fully mixed form, the $L=J-1$ are mixed with the $L=J+1$ states even in the nonrelativistic limit. The kernels are

$$K_g(\mathbf{p},\mathbf{q})=\frac{1}{2E_pE_qpq}\left[3p^2q^2-(E_pE_q-3m^2)\mathbf{p}\cdot\mathbf{q}+2(E_pE_q-m^2)\frac{p^2q^2-(\mathbf{p}\cdot\mathbf{q})^2}{|\mathbf{p}-\mathbf{q}|^2}\right]P_l\left[\frac{\mathbf{p}\cdot\mathbf{q}}{pq}\right], \tag{14}$$

$$K_s(\mathbf{p},\mathbf{q})=\frac{1}{E_pE_qpq}[p^2q^2-(E_pE_q+m^2)\mathbf{p}\cdot\mathbf{q}]P_l\left[\frac{\mathbf{p}\cdot\mathbf{q}}{pq}\right]. \tag{15}$$

(b) For the $L=J-1$ states, the *Ansatz* used here is specified by

$$\Gamma_Y=(\gamma^i-i\sigma^{0i})\times Y_{lm}\ .$$

The kernels are

$$K_g(\mathbf{p},\mathbf{q})=\frac{1}{6E_pE_q}\left[-E_pE_q+3m^2+2m(E_p+E_q)+5\mathbf{p}\cdot\mathbf{q}+2(E_p-m)(E_q-m)\left[\frac{\mathbf{p}\cdot\mathbf{q}}{pq}\right]^2\right.$$
$$\left.+2[E_pE_q+E_p^2+E_q^2+3m^2+3m(E_p+E_q)](E_p-m)(E_q-m)\frac{p^2q^2-(\mathbf{p}\cdot\mathbf{q})^2}{p^2q^2|\mathbf{p}-\mathbf{q}|^2}\right]P_l\left[\frac{\mathbf{p}\cdot\mathbf{q}}{pq}\right], \tag{16}$$

$$K_s(\mathbf{p},\mathbf{q})=\frac{1}{3E_pE_q}\left[-E_pE_q-m^2-2m(E_p+E_q)+3\mathbf{p}\cdot\mathbf{q}-2(E_p-m)(E_q-m)\left[\frac{\mathbf{p}\cdot\mathbf{q}}{pq}\right]^2\right]P_l\left[\frac{\mathbf{p}\cdot\mathbf{q}}{pq}\right]. \tag{17}$$

The only kernels, which can be used directly to obtain the eigenvalues for some given J^{PC} numbers, are those of the $^3S_1$ states. All the other kernels need to be combined with others to calculate the eigenvalues for given J^{PC} numbers. We shall take the $^3P_2$ sector as an example to demonstrate this.

(c) For the $L=J+1$ states, we use an *Ansatz* specified by

$$\Gamma_Y=(\gamma^i-i\sigma^{0i}-Y_{1i})\times Y_{lm}\ .$$

The kernels become

$$K_g(\mathbf{p},\mathbf{q})=\frac{1}{12E_pE_q}P_l\left[\frac{\mathbf{p}\cdot\mathbf{q}}{pq}\right]\left[10m^2-4(E_p+m)(E_q+m)+17\mathbf{p}\cdot\mathbf{q}+\frac{2}{p^2q^2}[2m^2-E_pE_q+2(E_p+m)(E_q+m)](\mathbf{p}\cdot\mathbf{q})^2\right.$$
$$+\frac{1}{p^2q^2}[6p^2q^2+3(E_pE_q-m^2)(p^2+q^2)+2m(E_p+E_q)(E_p-E_q)^2$$
$$\left.+2(E_p+m)(E_q+m)(E_p-E_q)^2]\frac{p^2q^2-(\mathbf{p}\cdot\mathbf{q})^2}{|\mathbf{p}-\mathbf{q}|^2}\right], \tag{18}$$

$$K_s(\mathbf{p},\mathbf{q})=\frac{1}{6E_pE_q}P_l\left[\frac{\mathbf{p}\cdot\mathbf{q}}{pq}\right]\left[4m(E_p+E_q)-E_pE_q-m^2+6\mathbf{p}\cdot\mathbf{q}-\frac{1}{p^2q^2}[(E_pE_q+m^2)+4m(E_p+E_q)](\mathbf{p}\cdot\mathbf{q})^2\right]. \tag{19}$$

The kernels for the $^3D_1$ states can be used directly to obtain the eigenvalues of the $^3D_1$ sector. The other kernels again have to be combined with others to get eigenvalues of states with given J^{PC} numbers.

(3) Pseudovector set.

The kernels for the pseudovector set can be written as

$$K_g(\mathbf{p},\mathbf{q})=\frac{1}{4E_pE_qpq}\left[-p^2q^2+2\mathbf{p}\cdot\mathbf{q}(E_pE_q+m^2)+3(\mathbf{p}\cdot\mathbf{q})^2 +2(E_pE_q+E_p^2+E_q^2-3m^2-\mathbf{p}\cdot\mathbf{q})\frac{p^2q^2-(\mathbf{p}\cdot\mathbf{q})^2}{|\mathbf{p}-\mathbf{q}|^2}\right]P_l\left[\frac{\mathbf{p}\cdot\mathbf{q}}{pq}\right], \tag{20}$$

$$K_s(\mathbf{p},\mathbf{q})=\frac{1}{2E_pE_qpq}[p^2q^2-2(E_pE_q+m^2)\mathbf{p}\cdot\mathbf{q}+(\mathbf{p}\cdot\mathbf{q})^2]P_l\left[\frac{\mathbf{p}\cdot\mathbf{q}}{pq}\right]. \tag{21}$$

Only the kernels for the 1^{++} sector can be used to evaluate the eigenvalues directly. All the others have to be combined with other kernels to obtain the eigenvalues of states with given J^{PC} numbers.

(4) 1^{--} sector.

There are three different ways to formulate the *Ansatz* of the 1^{--} sector. The states are specified by $\Gamma_Y=\gamma^i,\sigma^{0i}$ and $I\times Y_{1i}$. The first two are the $^3S_1$ dominant states. The third one is a fully mixed form of $^3S_1$ and $^3D_1$ states. However we can obtain the *Ansätze* for both pure $^3S_1$ states and the $^3D_1$ states by combining the generators.

(a) $\Gamma_Y=\gamma$.

The kernels for the $^3S_1$ dominant states with $\Gamma_Y=\gamma$ are

$$K_g(\mathbf{p},\mathbf{q})=\frac{1}{2E_pE_q(2E_p^2+m^2)^{1/2}(2E_q^2+m^2)^{1/2}} \times\left[[3m^2(E_p^2+E_q^2)-E_pE_q(E_pE_q-m^2)+\mathbf{p}\cdot\mathbf{q}(2E_pE_q+3m^2)+3(\mathbf{p}\cdot\mathbf{q})^2] +2(E_pE_q+E_p^2+E_q^2-\mathbf{p}\cdot\mathbf{q})\frac{p^2q^2-(\mathbf{p}\cdot\mathbf{q})^2}{|\mathbf{p}-\mathbf{q}|^2}\right], \tag{22}$$

$$K_s(\mathbf{p},\mathbf{q})=\frac{1}{E_pE_q(2E_p^2+m^2)^{1/2}(2E_q^2+m^2)^{1/2}} \times[-E_pE_q(E_pE_q+m^2)-m^2(E_p+E_q)^2+(2E_pE_q+m^2)\mathbf{p}\cdot\mathbf{q}-(\mathbf{p}\cdot\mathbf{q})^2]. \tag{23}$$

(b) $\Gamma_Y=\sigma^{0i}$.

The kernels for the $^3S_1$ dominant states with $\Gamma_Y-\sigma^{0i}$ are

$$K_g(\mathbf{p},\mathbf{q})=\frac{1}{2E_pE_q(E_p^2+2m^2)^{1/2}(E_q^2+2m^2)^{1/2}} \times\left[[5m^2E_pE_q-m^2(E_p^2+E_q^2)+3m^4+\mathbf{p}\cdot\mathbf{q}(3E_pE_q+2m^2)-(\mathbf{p}\cdot\mathbf{q})^2]+2(3m^2+\mathbf{p}\cdot\mathbf{q})\frac{p^2q^2-(\mathbf{p}\cdot\mathbf{q})^2}{|\mathbf{p}-\mathbf{q}|^2}\right], \tag{24}$$

$$K_s(\mathbf{p},\mathbf{q})=\frac{1}{E_pE_q(E_p^2+2m^2)^{1/2}(E_q^2+2m^2)^{1/2}}\{-m^2[(E_pE_q+m^2)+(E_p^2+E_q^2)]+\mathbf{p}\cdot\mathbf{q}(E_pE_q+2m^2-\mathbf{p}\cdot\mathbf{q})\}. \tag{25}$$

(c) $^3S_1$ states.

Sandwiching the Hamiltonian between the $^3S_1$ *Ansatz* given by $\Gamma_Y=\gamma^i-i\sigma^{0i}$ leads to the $^3S_1$ kernel equation. The kernels for the pure $^3S_1$ sector are given by Eqs. (16) and (17) in the case of $J=1$ and $L=0$.

(d) ^3D states.

The $^3D_1$ kernel equation is obtained by sandwiching the Hamiltonian between the $^3D_1$ *Ansatz* generated by $\Gamma_Y=\gamma^i-i\sigma^{0i}-Y_{1i}$. The kernels for the pure $^3D_1$ sector are given by Eqs. (18) and (19) in the case of $J=1$ and $L=2$.

(5) $^3P_2(2^{++})$ states.

As an example of how to evaluate the eigenvalues of the states in the scalar set (other than 1^{--}) with given J^{PC} numbers, we take the $^3P_2$ sector.

Sandwiching the Hamiltonian between the *Ansatz* Eq. (2) with $l=1$ would give the eigenvalues of the $^3P_2$ states. Practically this is difficult as we are unable to reduce the kernel equation from three dimensions to one dimension analytically. Instead we sandwich the Hamiltonian between the following *Ansatz*

$$|q\bar{q}\rangle=|m=-2\rangle+|m=2\rangle+\sqrt{2}|m=-1\rangle+\sqrt{2}|m=1\rangle+(\tfrac{8}{3})^{1/2}|m=0\rangle+(\tfrac{1}{3})^{1/2}|0^{++}\rangle, \tag{26}$$

where

$$|m=0\rangle=\frac{1}{\sqrt{6}}[|\Gamma_{^3S_1}\times Y_{1,-1}\rangle_+ +2|\Gamma_{^3S_1}\times Y_{1,0}\rangle_0+|\Gamma_{^3S_1}\times Y_{1,1}\rangle_-] ,$$

$$|m=1\rangle=\frac{1}{\sqrt{2}}[|\Gamma_{^3S_1}\times Y_{1,0}\rangle_+ +|\Gamma_{^3S_1}\times Y_{1,1}\rangle_0] ,$$

$$|m=-1\rangle=\frac{1}{\sqrt{2}}[|\Gamma_{^3S_1}\times Y_{1,-1}\rangle_0+|\Gamma_{^3S_1}\times Y_{1,0}\rangle_-] ,$$

$$|m=2\rangle=|\Gamma_{^3S_1}\times Y_{1,1}\rangle_+ ,$$

$$|m=-2\rangle=|\Gamma_{^3S_1}\times Y_{1,-1}\rangle_- ,$$

$$|0^{++}\rangle=\frac{1}{\sqrt{3}}[|\Gamma_{^3S_1}\times Y_{1,-1}\rangle_+ -|\Gamma_{^3S_1}\times Y_{1,0}\rangle_0+|\Gamma_{^3S_1}\times Y_{1,1}\rangle_-] ,$$

where

$$\Gamma_{^3S_1}=\gamma^i-i\sigma^{0i}$$

which leads to the eigenvalue equation (11) with the kernels

$$K_g(\mathbf{p},\mathbf{q})=\frac{27K_{11}(\mathbf{p},\mathbf{q})-K_{21}(\mathbf{p},\mathbf{q})}{26} \tag{27}$$

and

$$K_s(\mathbf{p},\mathbf{q})=\frac{27K_{12}(\mathbf{p},\mathbf{q})-K_{22}(\mathbf{p},\mathbf{q})}{26} , \tag{28}$$

where $K_{11}(\mathbf{p},\mathbf{q})$ and $K_{12}(\mathbf{p},\mathbf{q})$ are given by Eqs. (16) and (17) with $l=1$, and $K_{21}(\mathbf{p},\mathbf{q})$ and $K_{22}(\mathbf{p},\mathbf{q})$ are given by Eqs. (14) and (15) with $l=0$. The contribution of the $|0^{++}\rangle$ admixture can then be removed.

IV. NUMERICAL METHODS AND RESULTS

The kernels were checked in the following way. A partial nonrelativistic reduction of the K_g kernel was performed and the resulting kernels were sandwiched between hydrogenic wave functions. The energy eigenvalues could be obtained up to order α_s^4 and were found to agree with standard perturbation theory.[10] The K_s kernels were also reduced and the eigenvalues that were numerically generated agreed with the eigenvalues (corresponding to zeros of the Airy function) of a bound system in a linear potential.

We used both the variational method and the basis function expansion method to obtain the eigenvalues of the full problem.

We found that a two parameter hydrogenic type function was adequate to obtain four to five figure accuracy for the ground states of each J^{PC} family. Since this method was inconvenient for excited states we repeated the numerical calculations for all the states using the basis expansion method. In this approach the radial eigenfunction $f(p)$ is expanded in a complete set of known functions:

$$f_l(p)=\sum_{n=1}^{N} C_n^l R_{nl}(p) , \tag{29}$$

where in our case the $R_{nl}(p)$ were the radial hydrogenic functions. This leads to a matrix diagonalization problem

$$\sum_{n=1}^{N} A_{mn} C_n^l = M C_m^l , \tag{30}$$

where M is the diagonal dimension mass matrix and

TABLE I. Parameters used in fit to $c\bar{c}$ and $b\bar{b}$ mesons.

Parameters	Ψ Family	Υ Family
m_q	1.492 GeV	4.784 GeV
b	0.18 GeV^2	0.18 GeV^2
α_s	0.4200	0.3525

TABLE II. Comparison of theoretical model predictions to experimental data for $c\bar{c}$ systems. Note that the experimental masses and their uncertainties normally taken from the Particle Data Group (1988) unless otherwise indicated.

JPC	nL	Ψ Family Theory	Experiment
0−+	1s	2.980	2.9796±0.0017
	2s	3.60	3.594±0.005[a]
1−−	1s	3.114	3.0969±0.0001
	2s	3.69	3.6860±0.0001
	3s	4.04	4.040±0.010
	4s	4.32	4.415±0.006
	1d	3.757	3.7699±0.0025
	2d	4.28	4.159±0.020
0++	1p	3.398	3.4151±0.0010
1+−	1p	3.520	3.5254±0.0008[b]
1++	1p	3.532	3.5106±0.0005
2++	1p	3.551	3.5563±0.0004
2−+	1d	3.799	

[a]Reference 13.
[b]Reference 14.

TABLE III. Comparison of theoretical model predictions to experimental data for $b\bar{b}$ systems. Note that the experimental masses and their uncertainties normally taken from the Particle Data Group (1988) unless otherwise indicated.

JPC	nL	ϒ Family Theory	Experiment
0−+	1s	9.331	
1−−	1s	9.460	9.4603±0.0002
	2s	10.00	10.0233±0.0003
	3s	10.32	10.3553±0.0005
	4s	10.57	10.5800±0.0035
	5s	10.79	10.865±0.008
	6s	10.99	11.019±0.008
	1d	10.141	
0++	1p	9.853	9.8598±0.0013
	2p	10.207	10.2353±0.0011
1+−	1p	9.899	9.8948±0.0015[a]
1++	1p	9.897	9.8919±0.0007
	2p	10.245	10.2552±0.0004
2++	1p	9.907	9.9132±0.0006
	2p	10.257	10.2690±0.0007
2−+	1d	10.142	

$$A_{mn}=K_{mn}-Q_{mn} ,$$

where

$$K_{mn}=2\int_0^\infty \sqrt{1+x^2}x^2R_{ml}(x)R_{nl}(x)dx$$

and

$$Q_{mn}=\frac{1}{4\pi}\int_0^\infty x\,dx\,R_{nl}(x)\times\int_0^\infty y\,dy\,R_{nl}(g)[\tfrac{4}{3}K_g(x,y)-bK_s(x,y)] .$$

Typically we would truncate our expression at $N=15$ and perform the integration using Gauss-Legendre quadrature with 200 points to obtain four-figure accuracy.

Finally to compare with experiment we must fix our model parameters. We applied our model to the $c\bar{c}$ and $b\bar{b}$ systems.[11,12] We took a standard value for the string tension b and optimized the value of the quark masses and coupling constant α_s. The parameter choices are listed in Table I. The change of the value of α_s from the Ψ to the ϒ family is consistent with the running coupling constant behavior of QCD.

Our results are presented in Tables II and III and Figs. 1 and 2. The tables speak for themselves. The agreement between the model and experiment is quite good. The only significant disagreement is in the values above threshold where open channel effects are expected. One can certainly expect mass shifts on order of the typical (50 MeV) widths of these states.

In conclusion we have constructed a relativistic quark model for mesons. Bound-state kernels obtained from an underlying effective, relativistic field theory. This effective theory is Coulomb-gauge QCD (where covariance is exact but not manifest) augmented by a Lorentz scalar confining term whose strength and structure is dic-

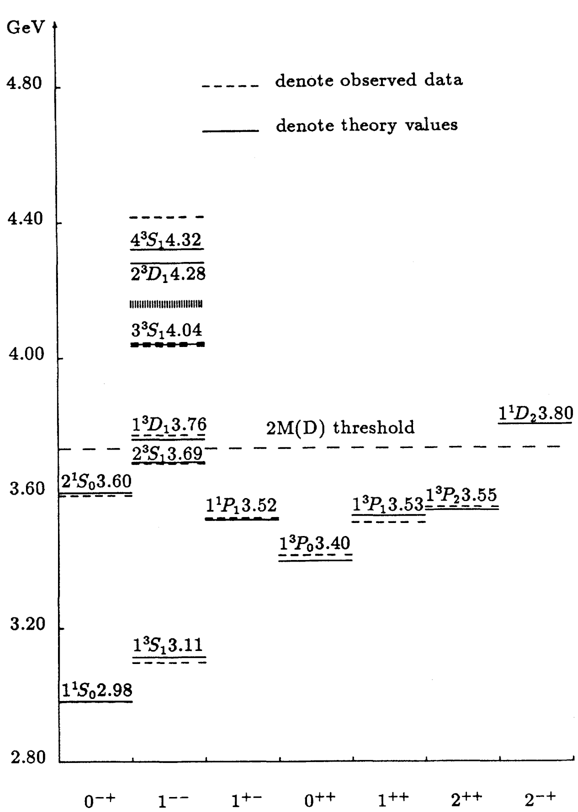

FIG. 1. Charmonium ($c\bar{c}$) spectrum.

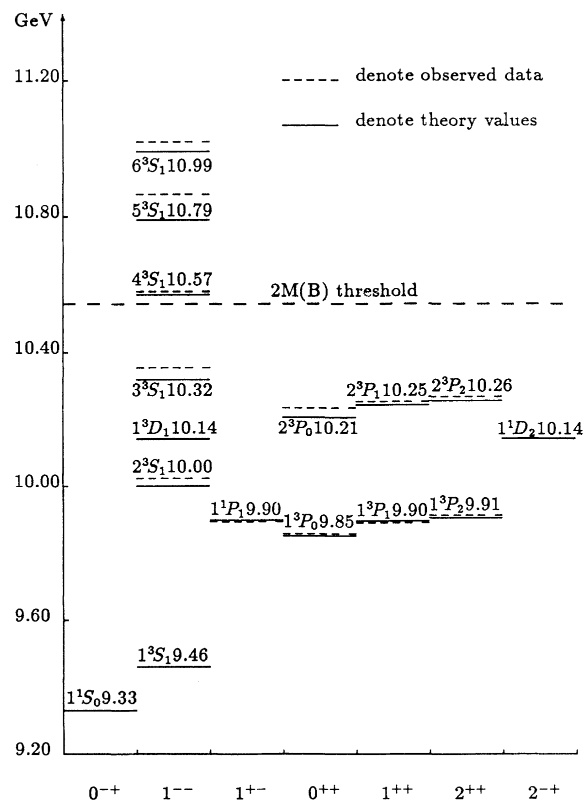

FIG. 2. b-quarkonium ($b\bar{b}$) spectrum.

tated by the lattice studies. We feel that the success of this model with its relativistic nature, economy of parameters and sound basis is a useful tool for the further understanding of heavy-quark (and possibly lighter) systems.

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APPENDIX

Performing an angular integration on Eq. (11) leads to the radial integral eigenvalue equation (7). We obtained all the following radial kernels.

(1) Pseudoscalar set:

(a) Even states:

$$J^{PC}=l^{-+} \quad \text{or } 0^{-+},2^{-+},4^{-+},\ldots, \text{ states ,}$$

where $l=2i$, $i=0,1,2,\ldots,$

$$K_1(p,q)=\frac{2pq}{E_pE_q}\sum_{k=0}^{i}\frac{(-1)^k(4i-2k)!}{2^{2i}k!(2i-k)!(2i-2k)!}\times\left\{-1+\left[2i-2k+1-8(i-k)\frac{p^2q^2}{(p^2+q^2)^2}+\frac{2}{p^2+q^2}(3E_pE_q-m^2)\right]\times\left[\left[\frac{p^2+q^2}{2pq}\right]^{2i-2k+1}\ln\left|\frac{p+q}{p-q}\right|-\sum_{j=1}^{ik-}\frac{1}{2i-2k-2j+1}\left[\frac{p^2+q^2}{2pq}\right]^{2j}\right]\right\}, \tag{A1}$$

$$K_2(p,q)=-\frac{4pq}{E_pE_q}\sum_{k=0}^{i}\frac{(-1)^k(4i-2k)!}{2^{2i}k!(2i-k)!(2i-2k)!}\times\left\{\frac{2(E_pE_q+m^2)-(p^2+q^2)}{(p^2-q^2)^2}-\frac{4(i-k)(E_pE_q+m^2)-(2i-2k+1)(p^2+q^2)}{4p^2q^2}\times\left[\left[\frac{p^2+q^2}{2pq}\right]^{2i-2k-1}\ln\left|\frac{p+q}{p-q}\right|-\sum_{j=0}^{i-k-1}\frac{1}{2i-2k-2j-1}\left[\frac{p^2+q^2}{2pq}\right]^{2j}\right]\right\}. \tag{A2}$$

(b) Odd states:

$$J^{PC}=l^{+-} \quad \text{or } 1^{+-},3^{+-},5^{+-},\ldots, \text{ states ,}$$

where $l=2i+1$, $i=1,3,5,\ldots,$

$$K_1(p,q)=\frac{2pq}{E_pE_q}\sum_{k=0}^{i}\frac{(-1)^k(4i-2k+2)!}{2^{2i+1}k!(2i-k+1)!(2i-2k+1)!}\times\left\{-\frac{2pq}{p^2+q^2}+\frac{1}{pq}\left[3E_pE_q-m^2+\frac{p^4+q^4}{p^2+q^2}+(i-k)\frac{(p^2-q^2)^2}{p^2+q^2}\right]\times\left[\left[\frac{p^2+q^2}{2pq}\right]^{2i-2k+1}\ln\left|\frac{p+q}{p-q}\right|-\sum_{j=0}^{i-k}\frac{1}{2i-2k-2j+1}\left[\frac{p^2+q^2}{2pq}\right]^{2j}\right]\right\}, \tag{A3}$$

$$K_2(p,q)=-\frac{4pq}{E_pE_q}\sum_{k=0}^{i}\frac{(-1)^k(4i-2k+2)!}{2^{2i+1}k!(2i-k+1)!(2i-2k+1)!}\times\left\{\frac{(p^2+q^2)(E_pE_q+m^2)-2p^2q^2}{pq(p^2-q^2)^2}-\frac{E_pE_q+m^2}{pq(p^2+q^2)}-\frac{1}{pq}\left[\frac{(2i-2k+1)(E_pE_q+m^2)}{(p^2+q^2)}-(i-k+1)\right]\times\left[\left[\frac{p^2+q^2}{2pq}\right]^{2i-2k+1}\ln\left|\frac{p+q}{p-q}\right|-\sum_{j=1}^{i-k}\frac{1}{2i-2k-2j+1}\left[\frac{p^2+q^2}{2pq}\right]^{2j}\right]\right\}. \tag{A4}$$

(2) Scalar set:
(a) The mixed form:
(1) Even states:

$J^{PC}=l^{++}$ or $0^{++},2^{++},4^{++},\ldots,$ states ,

where $l=2i$, $i=0,1,2,\ldots,$

$$K_1(p,q)=\frac{2}{E_pE_q}\sum_{k=0}^{i}\frac{(-1)^k(4i-2k)!}{2^{2i}k!(2i-k)!(2i-2k)!}\times\Bigg\{-(E_pE_q-m^2)-\frac{2m^2}{2i-2k+1}+\left[E_pE_q+m^2+\frac{6p^2q^2}{p^2+q^2}+2(i-k)(E_pE_q-m^2)\left[\frac{p^2-q^2}{p^2+q^2}\right]^2\right]\times\left[\left[\frac{p^2+q^2}{2pq}\right]^{2i-2k+1}\ln\left|\frac{p+q}{p-q}\right|-\sum_{j=1}^{i-k}\frac{1}{2i-2k-2j+1}\left[\frac{p^2+q^2}{2pq}\right]^{2j}\right]\Bigg\}, \tag{A5}$$

$$K_2(p,q)=\frac{4}{E_pE_q}\sum_{k=0}^{i}\frac{(-1)^k(4i-2k)!}{2^{2i}k!(2i-k)!(2i-2k)!}\times\Bigg\{\frac{2p^2q^2-(E_pE_q+m^2)(p^2+q^2)}{(p^2-q^2)^2}-\left[(i-k)-(2i-2k+1)\frac{(E_pE_q+m^2)(p^2+q^2)}{4p^2q^2}\right]\times\left[\left[\frac{p^2+q^2}{2pq}\right]^{2i-2k-1}\ln\left|\frac{p+q}{p-q}\right|-\sum_{j=0}^{i-k-1}\frac{1}{2i-2k-2j-1}\left[\frac{p^2+q^2}{2pq}\right]^{2j}\right]\Bigg\}. \tag{A6}$$

(2) Odd states:

$J^{PC}=l^{--}$ or $1^{--},3^{--},5^{--},\ldots,$ states ,

where $l=2i+1$, $i=0,1,2,\ldots,$

$$K_1(p,q)=\frac{2}{E_pE_q}\sum_{k=0}^{i}\frac{(-1)^k(4i-2k+2)!}{2^{2i+1}k!(2i-k+1)!(2i-2k+1)!}\times\Bigg\{-\frac{2pq}{p^2+q^2}(E_pE_q-m^2)+\frac{1}{pq}\left[3p^2q^2+E_pE_q(p^2+q^2)-[2p^2q^2-(i-k)(p^2-q^2)^2]\frac{E_pE_q-m^2}{p^2+q^2}\right]\times\left[\left[\frac{p^2+q^2}{2pq}\right]^{2i-2k+1}\ln\left|\frac{p+q}{p-q}\right|-\sum_{j=0}^{i-k}\frac{1}{2i-2k-2j+1}\left[\frac{p^2+q^2}{2pq}\right]^{2j}\right]\Bigg\}, \tag{A7}$$

$$K_2(p,q)=\frac{4}{E_pE_q}\sum_{k=0}^{i}\frac{(-1)^k(4i-2k+2)!}{2^{2i+1}k!(2i-k+1)!(2i-2k+1)!}\times\Bigg\{\frac{pq(p^2+q^2-2E_pE_q-2m^2)}{(p^2-q^2)^2}-\frac{pq}{p^2+q^2}-\left[\frac{(2i-2k+1)pq}{p^2+q^2}-\frac{(i-k+1)(E_pE_q+m^2)}{pq}\right]\times\left[\left[\frac{p^2+q^2}{2pq}\right]^{2i-2k+1}\ln\left|\frac{p+q}{p-q}\right|-\sum_{j=0}^{i-k}\frac{1}{2i-2k-2j+1}\left[\frac{p^2+q^2}{2pq}\right]^{2j}\right]\Bigg\}. \tag{A8}$$

(b) The $L=J-1$ states:
(1) Odd states:

$J^{PC}=(l+1)^{--}$ or $1^{--},3^{--},5^{--},\dots,$ states ,

where $l=2i$, $i=0,1,2,\dots,$

$$K_1(p,q)=\frac{2pq}{3E_pE_q}\sum_{k=0}^{i}\frac{(-1)^k(4i-2k)!}{2^{2i}k!(2i-k)!(2i-2k)!}$$
$$\times\left\{-\frac{1}{2i-2k+1}\left[5+(E_p-m)(E_p-m)\frac{p^2+q^2}{p^2q^2}+2(i-k+1)\left[3+(E_p-m)(E_q-m)\frac{(E_p-E_q)^2}{p^2q^2}\right]\right]\right.$$
$$+\left[\frac{2}{p^2+q^2}[-E_pE_q+3m^2+2m(E_p+E_q)]+(E_p-m)(E_q-m)\frac{p^2+q^2}{p^2q^2}+5\right.$$
$$\left.+2\left[3+(E_p-m)(E_q-m)\frac{(E_p-E_q)^2}{p^2q^2}\right]\left[1+\frac{(i-k)(p^2-q^2)^2}{(p^2+q^2)^2}\right]\right]$$
$$\left.\times\left[\left[\frac{p^2+q^2}{2pq}\right]^{2i-2k+1}\ln\left|\frac{p+q}{p-q}\right|-\sum_{j=1}^{i-k}\frac{1}{2i-2k-2j+1}\left[\frac{p^2+q^2}{2pq}\right]^{2j}\right]\right\}, \qquad \text{(A9)}$$

$$K_2(p,q)=\frac{4pq}{3E_pE_q}\sum_{k=0}^{i}\frac{(-1)^k(4i-2k)!}{2^{2i}k!(2i-k)!(2i-2k)!}$$
$$\times\left\{\frac{1}{(p^2-q^2)^2}[3(p^2+q^2)-14(E_pE_q+m^2)+8m(E_p+E_q)]-\frac{6(i-k+1)}{2i-2k+1}\frac{(E_p-m)(E_q-m)}{p^2q^2}\right.$$
$$+\left[\frac{4(i-k)}{(p^2+q^2)^2}[E_pE_q+m^2+2m(E_p+E_q)]\right.$$
$$\left.-\frac{3(2i-2k+1)}{p^2+q^2}+6(i-k+1)\frac{(E_p-m)(E_q-m)}{p^2q^2}\right]$$
$$\left.\times\left[\left[\frac{p^2+q^2}{2pq}\right]^{2i-2k+1}\ln\left|\frac{p+q}{p-q}\right|-\sum_{j=1}^{i-k}\frac{1}{2i-2k-2j+1}\left[\frac{p^2+q^2}{2pq}\right]^{2j}\right]\right\}. \qquad \text{(A10)}$$

(2) Even states:

$J^{PC}=(l+1)^{++}$ or $2^{++},4^{++},6^{++},\dots,$ states ,

where $l=2i=1$, $i=0,1,2,\dots,$

$$K_1(p,q)=\frac{2pq}{3E_pE_q}\sum_{k=0}^{i}\frac{(-1)^k(4i-2k+2)!}{2^{2i+1}k!(2i-k+1)!(2i-2k+1)!}$$
$$\times\left\{-\frac{2(E_p-m)(E_q-m)}{(2i-2k+3)pq}-\frac{2pq}{p^2+q^2}\left[3+(E_p-m)(E_q-m)\frac{(E_p-E_q)^2}{p^2q^2}\right]\right.$$
$$+\left[\frac{1}{pq}[-E_pE_q+3m^2+2m(E_p+E_q)]+5\frac{p^2+q^2}{2pq}+(E_p-m)(E_q-m)\frac{(p^2+q^2)^2}{2p^3q^3}\right.$$
$$\left.-2\left[3+(E_p-m)(E_q-m)\frac{(E_p-E_q)^2}{p^2q^2}\right]\left[\frac{pq}{p^2+q^2}-\frac{3(p^2+q^2)}{4pq}-\frac{(i-k)}{2pq}\frac{(p^2-q^2)^2}{p^2+q^2}\right]\right]$$
$$\left.\times\left[\left[\frac{p^2+q^2}{2pq}\right]^{2i-2k+1}\ln\left|\frac{p+q}{p-q}\right|-\sum_{j=0}^{i-k}\frac{1}{2i-2k-2j+1}\left[\frac{p^2+q^2}{2pq}\right]^{2j}\right]\right\}, \qquad \text{(A11)}$$

$$K_2(p,q)=\frac{4pq}{3E_pE_q}\sum_{k=0}^{i}\frac{(-1)^k(4i-2k+2)!}{2^{2i+1}k!(2i-k+1)!(2i-2k+1)!}$$
$$\times\left\{\frac{1}{pq(p^2+q^2)}[E_pE_q+m^2+2m(E_p+E_q)]\right.$$
$$-\frac{6pq}{(p^2+q^2)^2}-[7(E_pE_q+m^2)-4m(E_p+E_q)]\frac{p^2+q^2}{pq(p^2q^2)^2}$$
$$+\left[\frac{2i-2k+1}{pq(p^2+q^2)}[E_pE_q+m^2+2m(E_p+E_q)]\right.$$
$$\left.-\frac{3(i-k+1)}{pq}+3(2i-2k+3)(E_p-m)(E_q-m)\left[\frac{p^2+q^2}{2p^3q^3}\right]\right]$$
$$\left.\times\left[\left[\frac{p^2+q^2}{2pq}\right]^{2i-2k+1}\ln\left|\frac{p+q}{p-q}\right|-\sum_{j=0}^{i-k}\frac{1}{2i-2k-2j+1}\left[\frac{p^2+q^2}{2pq}\right]^{2j}\right]\right\}. \quad \text{(A12)}$$

(c) The $L=J+1$ states:
(1) Odd states:

$J^{PC}=(l+1)^{--}$ or $1^{--},3^{--},5^{--},\ldots,$ states ,

where $l=2i$, $i=0,1,2,\ldots,$

$$K_1(p,q)=\frac{pq}{3E_pE_q}\sum_{k=0}^{i}\frac{(-1)^k(4i-2k)!}{2^{2i}k!(2i-k)!(2i-2k)!}$$
$$\times\left\{\frac{2}{2i-2k+1}\frac{1}{p^2+q^2}[6m^2-4E_pE_q-4m(E_p+E_q)]\right.$$
$$-\frac{4(i-k)}{2i-2k+1}\frac{1}{(p^2+q^2)^2}[2m(E_p+E_q)(E_p-E_q)^2+6p^2q^2+3(E_pE_q-m^2)(p^2+q^2)$$
$$+2(E_p+m)(E_q+m)(E_p-E_q)^2]$$
$$+\left[\frac{2}{p^2+q^2}[6m^2-4E_pE_q-4m(E_p+E_q)]+17+\frac{p^2+q^2}{p^2q^2}[E_pE_q+4m^2+2m(E_p+E_q)]\right.$$
$$+[6p^2q^2+2m(E_p+E_q)(E_p-E_q)^2+3(E_pE_q-m^2)(p^2+q^2)$$
$$\left.+2(E_p+m)(E_q+m)(E_p-E_q)^2]\left[\frac{i-k+1}{p^2q^2}-\frac{4(i-k)}{(p^2+q^2)^2}\right]\right]$$
$$\left.\times\left[\left[\frac{p^2+q^2}{2pq}\right]^{2i-2k+1}\ln\left|\frac{p+q}{p-q}\right|-\sum_{j=0}^{i-k}\frac{1}{2i-2k-2j+1}\left[\frac{p^2+q^2}{2pq}\right]^{2j}\right]\right\}, \quad \text{(A13)}$$

$$K_2(p,q)=\frac{2pq}{3E_pE_q}\sum_{k=0}^{i}\frac{(-1)^k(4i-2k)!}{2^{2i}k!(2i-k)!(2i-2k)!}$$
$$\times\left\{\frac{6}{(p^2-q^2)^2}[(p^2+q^2)-2(E_pE_q+m^2)]-\frac{i-k+1}{2i-2k+1}\frac{1}{p^2q^2}[5(E_pE_q+m^2)+4m(E_p+E_q)]\right.$$
$$-\left[\frac{4(i-k)}{(p^2+q^2)^2}[4m(E_p+E_q)-(E_pE_q+m^2)]+\frac{6(2i-2k+1)}{p^2+q^2}\right.$$
$$\left.-\frac{(i-k+1)}{p^2q^2}[5(E_pE_q+m^2)+4m(E_p+E_q)]\right]$$
$$\left.\times\left[\left[\frac{p^2+q^2}{2pq}\right]^{2i-2k+1}\ln\left|\frac{p+q}{p-q}\right|-\sum_{j=1}^{i-k}\frac{1}{2i-2k-2j+1}\left[\frac{p^2+q^2}{2pq}\right]^{2j}\right]\right\}. \quad \text{(A14)}$$

(2) Even states:

$$J^{PC}=(l+1)^{++} \quad \text{or } 2^{++},4^{++},6^{++},\ldots, \text{ states },$$

where $l=2i+1$, $i=0,1,2,\ldots,$

$$\begin{aligned}
K_1(p,q)=\frac{pq}{3E_pE_q}\sum_{k=0}^{i}\frac{(-1)^k(4i-2k+2)!}{2^{2i+1}k!(2i-k+1)!(2i-2k+1)!}
&\times\Bigg\{-\frac{1}{pq(p^2+q^2)}[2m(E_p+E_q)(E_p-E_q)^2+2(E_p+m)(E_q+m)(E_p-E_q)^2\\
&\quad+6p^2q^2+3(E_pE_q-m^2)(p^2+q^2)]-\frac{2[2m^2-E_pE_q+2(E_p+m)(E_q+m)]}{pq(2i-2k+3)}\\
&\quad+\frac{1}{pq}\Bigg[6m^2-4E_pE_q-4m(E_p+E_q)+\tfrac{17}{2}(p^2+q^2)\\
&\quad+\left[\frac{p^2+q^2}{2pq}\right]^2[E_pE_q+4m^2+2m(E_p+E_q)]\\
&\quad+\frac{p^2+q^2}{2}\left[\frac{2i-2k+3}{2p^2q^2}-\frac{2(2i-2k+1)}{(p^2+q^2)^2}\right]\\
&\quad\times[2m(E_p+E_q)(E_p-E_q)^2+2(E_p+m)(E_q+m)(E_p-E_q)^2\\
&\quad+6p^2q^2+3(E_pE_q-m^2)(p^2+q^2)]\Bigg]\\
&\quad\times\left[\left[\frac{p^2+q^2}{2pq}\right]^{2i-2k+1}\ln\left|\frac{p+q}{p-q}\right|-\sum_{j=0}^{i-k}\frac{1}{2i-2k-2j+1}\left[\frac{p^2+q^2}{2pq}\right]^{2j}\right]\Bigg\}, \qquad \text{(A15)}
\end{aligned}$$

$$\begin{aligned}
K_2(p,q)=\frac{2}{3E_pE_q}\sum_{k=0}^{i}\frac{(-1)^k(4i-2k+2)!}{2^{i+1}k!(2i-k+1)!(2i-2k+1)!}
&\times\Bigg\{\frac{1}{p^2+q^2}[E_pE_q+{}^2-4m(E_p+E_q)]+\frac{6}{(p^2-q^2)^2}[2p^2q^2-(E_pE_q+m^2)(p^2+q^2)]\\
&\quad-\Bigg[\frac{2i-2k+1}{p^2+q^2}[4m(E_p+E_q)-(E_pE_q+m^2)]\\
&\quad+6(i-k+1)-\frac{2i-2k+3}{4p^2q^2}(p^2+q^2)[5(E_pE_q+m^2)+4m(E_p+E_p)]\Bigg]\\
&\quad\times\left[\left[\frac{p^2+q^2}{2pq}\right]^{2i-2k+1}\ln\left|\frac{p+q}{p-q}\right|-\sum_{j=0}^{i-k}\frac{1}{2i-2k-2j+1}\left[\frac{p^2+q^2}{2pq}\right]^{2j}\right]\Bigg\}. \qquad \text{(A16)}
\end{aligned}$$

(3) Pseudovector set:

(a) Odd states:

$$J^{PC}=(l+1)^{++} \quad \text{or } 1^{++},3^{++},5^{++},\ldots, \text{ states },$$

where $l=2i$, $i=0,1,2,\ldots,$

$$\begin{aligned}
K_1(p,q)=\frac{1}{E_pE_q}\sum_{k=0}^{i}\frac{(-1)^k(4i-2k)!}{2^{2i}k!(2i-k)!(2i-2k)!}
&\times\Bigg\{-\frac{i-k}{2i-2k+1}\left[\frac{2pq}{p^2+q^2}\right]^2[(E_p+E_q)^2-4m^2]\\
&\quad+\left[(E_p+E_q)^2+[(E_p+E_q)^2-4m^2]\left[i-k+1-\frac{4p^2q^2(i-k)}{(p^2+q^2)^2}\right]\right]\\
&\quad\times\left[\left[\frac{p^2+q^2}{2pq}\right]^{2i-2k+1}\ln\left|\frac{p+q}{p-q}\right|-\sum_{j=0}^{i-k}\frac{1}{2i-2k-2j+1}\left[\frac{p^2+q^2}{2pq}\right]^{2j}\right]\Bigg\}, \qquad \text{(A17)}
\end{aligned}$$

$$K_2(p,q)=\frac{2pq}{E_pE_q}\sum_{k=0}^{i}\frac{(-1)^k(4i-2k)!}{2^{2i}k!(2i-k)!(2i-2k)!}$$
$$\times\left\{\frac{i-k+1}{(2i-2k+1)pq}+\frac{2}{(p^2-q^2)^2}\left[4pq-(E_pE_q+m^2)\frac{p^2+q^2}{pq}\right]\right.$$
$$+\left[\frac{2(2i-2k+1)(E_pE_q+m^2)}{pq(p^2+q^2)}-\frac{4pq(i-k)}{(p^2+q^2)^2}-\frac{i-k+1}{pq}\right]$$
$$\left.\times\left[\left[\frac{p^2+q^2}{2pq}\right]^{2i-2k+1}\ln\left|\frac{p+q}{p-q}\right|-\sum_{j=1}^{i-k}\frac{1}{2i-2k-2j+1}\left[\frac{p^2+q^2}{2pq}\right]^{2j}\right]\right\}. \quad (A18)$$

(b) Even states:

$J^{PC}=(l+1)^{--}$ or $2^{--},4^{--},6^{--},\ldots,$ states ,

where $l=2i+1$, $i=0,1,2,\ldots,$

$$K_1(p,q)=\frac{1}{E_pE_q}\sum_{k=0}^{i}\frac{(-1)^k(4i-2k+2)!}{2^{2i+1}k!(2i-k+1)!(2i-2k+1)!}$$
$$\times\left\{-\frac{2pq}{2i-2k+3}-\frac{pq[(E_p+E_q)^2-4m^2]}{p^2+q^2}\right.$$
$$+\left[\frac{p^2+q^2}{pq}\right]\left[\frac{(E_p+E_q)^2}{2}+\tfrac{1}{4}[(E_p+E_q)^2-4m^2]\left[2i-2k+3-\frac{4p^2q^2(2i-2k+1)}{(p^2+q^2)^2}\right]\right]$$
$$\left.\times\left[\left[\frac{p^2+q^2}{2pq}\right]^{2i-2k+1}\ln\left|\frac{p+q}{p-q}\right|-\sum_{j=0}^{ik-}\frac{1}{2i-2k-2j+1}\left[\frac{p^2+q^2}{2pq}\right]^{2j}\right]\right\}, \quad (A19)$$

$$K_2(p,q)=\frac{2pq}{E_pE_q}\sum_{k=0}^{i}\frac{(-1)^k(4i-2k+2)!}{2^{2i+1}k!(2i-k+1)!(2i-2k+1)!}$$
$$\times\left\{-\frac{1}{p^2+q^2}+\frac{2}{(p^2-q^2)^2}[p^2+q^2-2(E_pE_q+m^2)]\right.$$
$$+\left[2(i-k+1)\frac{E_pE_q+m^2}{p^2q^2}-\frac{2i-2k+1}{p^2+q^2}-(2i-2k+3)\frac{p^2+q^2}{4p^2q^2}\right]$$
$$\left.\times\left[\left[\frac{p^2+q^2}{2pq}\right]^{2i-2k+1}\ln\left|\frac{p+q}{p-q}\right|-\sum_{j=0}^{i-k}\frac{1}{2i-2k-2j+1}\left[\frac{p^2+q^2}{2pq}\right]^{2j}\right]\right\}. \quad (A20)$$

(4) 1^{--} sector:

(a) $\Gamma_Y=\gamma$:

$$K_1(p,q)=\frac{pq}{E_pE_q(2E_p^2+m^2)^{1/2}(2E_q^2+m^2)^{1/2}}$$
$$\times\left[[2(E_p^2+E_q^2)(E_p+E_q)^2+3m^2(E_p-E_q)^2-4m^4]\frac{1}{pq}\ln\left|\frac{p+q}{p-q}\right|-4(E_p+E_q)^2-6m^2\right], \quad (A21)$$

$$K_2(p,q)=\frac{2pq}{E_pE_q(2E_p^2+m^2)^{1/2}(2E_q^2+m^2)^{1/2}}$$
$$\times\left[-2+\frac{1}{(p^2-q^2)^2}[2(2E_pE_q+m^2)(p^2+q^2)-4E_pE_q(E_pE_q+m^2)-4(E_pE_q+m^2)^2]\right.$$
$$\left.+[p^2+q^2-(2E_pE_q+m^2)]\frac{1}{pq}\ln\left|\frac{p+q}{p-q}\right|\right]. \quad (A22)$$

(b) $\Gamma_Y=\sigma^{0i}$:

$$K_1(p,q)=\frac{pq}{E_pE_q(E_p^2+2m^2)^{1/2}(E_q^2+2m^2)^{1/2}}\times\left[[10m^2E_pE_q-2m^2(E_p^2+E_q^2)+6m^4+(3E_pE_q+8m^2)(p^2+q^2)+(p^4+q^4)]\frac{1}{pq}\ln\left|\frac{p+q}{p-q}\right|-6E_pE_q-2E_p^2-2E_q^2-12m^2\right], \quad (A23)$$

$$K_2(p,q)=\frac{2pq}{E_pE_q(E_p^2+2m^2)^{1/2}(E_q^2+2m^2)^{1/2}}\times\left[-2+\frac{1}{(p^2-q^2)^2}[2(2E_pE_q+m^2)(p^2+q^2)-4p^2q^2-4m^2(E_p+E_q)^2-4m^2(E_pE_q+^2)]+[p^2+q^2-(E_pE_q+2m^2)]\frac{1}{pq}\ln\left|\frac{p+q}{p-q}\right|\right]. \quad (A24)$$

(c) $^3S_1$ sector: The kernels K_g and K_s are given by Eqs. (A9) and (A10) with $J=1$, respectively.
(d) $^3D_1$ sector: The kernels K_g and K_s are given by Eqs. (A13) and (A14) with $J=1$, respectively.
(5) $^3P_2(2^{++})$ sector:

$$K_g(p,q)=\frac{27K_{11}(pq)-K_{21}(pq)}{26}, \quad (A25)$$

and

$$K_s(p,q)=\frac{27K_{12}(pq)-K_{22}(pq)}{26}, \quad (A26)$$

where $K_{11}(p,q)$ and $K_{12}(p,q)$ are given by Eqs. (A11) and (A12) with $i=0$, and $K_{21}(p,q)$ and $K_{22}(p,q)$ are given by Eqs. (A5) and (A6) with $i=0$.

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