Scheme dependence of the next-to-next-to-leading QCD corrections to $\Gamma_{tot}(H^0 \to \text{hadrons})$ and the spurious QCD infrared fixed point

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The analytical result of reevaluation of the massless three-loop next-to-next-to-leading-order QCD corrections to $\Gamma_{tot}(H^0 \to \text{hadrons})$ is presented. The analysis of the scheme-dependence property is considered. The value of the scheme-invariant quantity ρ_2 is determined. It turns out to be negative and large. Therefore, the truncated effective β function has an infrared zero. It is argued that this "fixed point" is a spurious one. The problem of getting reliable information from the perturbative series with large coefficients is discussed.

I. INTRODUCTION

It is known that the perturbation-theory (PT) predictions are not unique because of the existence of the scheme-dependence problem. In QCD its consideration is of a special interest. Indeed, since the QCD running coupling constant in the experimentally available region of energies is not small, in the course of taking into account the higher-order PT corrections it may lead to additional theoretical ambiguities in the values of the fundamental parameters of the theory (say, the scale parameter Λ) which are extracted from the fits of experimental data (see, e.g., Ref. 1).

During the past years, several methods of dealing with scheme-dependence ambiguity have been discussed. First, the idea of minimization of higher-order PT corrections [usually calculated in the minimal-subraction (MS) scheme[2] or the modified $\overline{\text{MS}}$ scheme[3]] was proposed[3] and developed.[4–7] Such kinds of schemes include the G scheme,[5] a modification of the MS scheme convenient from the calculational point of view, and variously defined momentum-(MOM) subtraction schemes.[4,6,7] It should be stressed that in QCD the MOM schemes are gauge dependent. At the three-loop level this special feature of the MOM schemes has been extensively investigated in the works of Ref. 8.

Another method of dealing with scheme-dependence ambiguity has been considered in Ref. 9, where the principle of minimal sensitivity was proposed and certain scheme-invariant quantities were introduced. The importance of the scheme-invariant quantities has been deeply understood in the process of formulation of the massless scheme-invariant PT approach.[10] The generalization of this method to the massive case with a quark-mass pole has been proposed in Ref. 11. It is worth remembering that in the massless case the scheme-invariant PT (Ref. 10) is equivalent *a posteriori* to the effective-scheme method,[12,13] which is also referred to in the literature as the fastest-apparent-convergence (FAC) method.

In this work we discuss the result of reevaluation of the massless next-to-next-to-leading-order (NNL) three-loop corrections to Γ_{tot} $(H^0 \to \text{hadrons})$ (presented previously in Ref. 14) and compare it with the results of other works on the subject (see Refs. 15–19). Following the lines of Ref. 15, we analyze the scheme dependence of the results obtained on the choice of the renormalization parameter of the MS-like schemes. We also present the massless scheme-invariant analysis (see Ref. 20) and determine the numerical value of the scheme-invariant quantity ρ_2. The value of ρ_2 turns out to be negative and large.[21] We stress that this fact cannot be considered as a manifestation of the QCD fixed-point regime, previously discussed from various points of view in certain works on the subject.[22–25] We argue that the perturbative series for the effective β function of this channel explodes at the level of the NNL corrections. The problem of getting reliable information from PT series with large coefficients is discussed. The importance of obtaining the values of the scheme-invariant quantities ρ_2 for other physical quantities, in particular, for $\sigma_{tot}(e^+e^- \to \text{hadrons})$, is emphasized.

II. DESCRIPTION OF CALCULATIONS

Consider the standard SU(3)×SU(2)×U(1) model. The total hadronic decay width of a scalar Higgs boson is determined by the imaginary part of the two-point function of the quark scalar currents (see, e.g., Ref. 17):

$$\Gamma_{tot}(H^0 \to \text{hadrons}) = \frac{\sqrt{2}G_F}{M_H} \sum_{j=1}^{f} \text{Im}\Pi_j(q^2 = M_H^2) ,$$
$$\Pi_j(Q^2 = -q^2) = i \int e^{iqx} \langle 0 | T(J_j(x) J_j(0)) | 0 \rangle d^4x , \quad (2.1)$$

where $J_j(x) = m_j \bar{q}_j q_j$, $q_j = u,d,s,c,b,t$ quarks, and f is the number of quark flavors to be taken into account in the decay of a Higgs boson with a given mass M_H.

In the leading order of perturbation theory, the expression for this physical quantity is

$$\Gamma_{tot}(H^0 \to \text{hadrons}) = \frac{3\sqrt{2}}{8\pi} G_F M_H \sum_{j=1}^{f} m_j^2 \left[1 - \frac{4m_j^2}{M_H^2} \right]^{3/2} , \quad (2.2)$$

where m_j are the renormalized quark masses and M_H is the Higgs-boson mass. To sum up the nonleading $\ln(m_j^2/M_H^2)$ contributions to $\Gamma_{\rm tot}$, it was proposed to use the running quark masses.[16]

Let us note that far above the thresholds of production of the quark-antiquark pairs higher-order $O(\alpha_s)$ corrections can be more important than the $O(m_j^2/M_H^2)$ effects. Indeed, in the $\overline{\rm MS}$ scheme the two-loop massless corrections to $\Gamma_{\rm tot}$ are positive and sufficiently large.[17]

To simplify the calculation, it is convenient to introduce the function

$$\mathcal{D}_j(Q^2,m_j,\alpha_s)=Q^2\frac{d}{dQ^2}\left[\frac{\Pi_j(Q^2)}{Q^2}\right], \tag{2.3}$$

which satisfies the renormalization-group equation

$$\left[\mu^2\frac{\partial}{\partial\mu^2}+\beta(\alpha_s)\frac{\partial}{\partial\alpha_s}-\gamma_m(\alpha_s)m_j\frac{\partial}{\partial m_j}\right]\times\mathcal{D}_j\left[\frac{Q^2}{\mu^2},m_j,\alpha_s\right]=0\ . \tag{2.4}$$

The corresponding renormalization-group functions are defined as

$$\frac{1}{\pi}\beta(\alpha_s)=\frac{1}{\pi}\mu^2\frac{\partial\alpha_s}{\partial\mu^2}=-\sum_{i\geq 0}\beta_i\left[\frac{\alpha_s}{\pi}\right]^{i+2}, \tag{2.5}$$

$$m_j\gamma_m(\alpha_s)=-\mu^2\frac{\partial m_j}{\partial\mu^2}=m_j\sum_{i\geq 0}\gamma_i\left[\frac{\alpha_s}{\pi}\right]^{i+1}. \tag{2.6}$$

In the MS-like schemes the coefficients of the β function are known at the three-loop level:[26]

$$\begin{aligned}
\beta_0&=(11-\tfrac{2}{3}f)\tfrac{1}{4}\approx 2.75-0.1667f\ ,\\
\beta_1&=(102-\tfrac{38}{3}f)\tfrac{1}{16}\approx 6.375-0.7917f\ ,\\
\beta_2&=(\tfrac{2857}{2}-\tfrac{5033}{18}f+\tfrac{325}{54}f^2)\tfrac{1}{64}\\
&\approx 22.3203-4.3689f+0.0940f^2\ .
\end{aligned} \tag{2.7}$$

The three-loop coefficients of the γ_m function have been calculated in Ref. 27 and have the form

$$\begin{aligned}
\gamma_0&=1\ ,\\
\gamma_1&=(\tfrac{202}{3}-\tfrac{20}{9}f)\tfrac{1}{16}\approx 4.2083-0.1389f\ ,\\
\gamma_2&=\{1249-[\tfrac{2216}{27}+\tfrac{160}{3}\zeta(3)]f-\tfrac{140}{81}f^2\}\tfrac{1}{64}\\
&\approx 19.5156-2.2841f-0.0270f^2\ .
\end{aligned} \tag{2.8}$$

The reevaluation of the three-loop massless correction to the $\mathcal{D}$ function (2.3) has been independently made on an ES-1060 computer with the help of the SCHOONSCHIP program[28] and on a CDC-6500 computer with the help of the corrected SCHOONSCHIP program.[29] Note that the program of Ref. 28 has been written on the basis of the program of Ref. 29. These programs implement the integration-by-parts algorithm.[30] The results of both calculations are in agreement.

The general expression obtained for the $\mathcal{D}$ function has the form

$$\begin{aligned}
\mathcal{D}_j&\left[\frac{Q^2}{\mu^2},m_j,\alpha_s\right]\\
&=\frac{d(R)}{8\pi^2}m_j^2\left[1+\left[\frac{\alpha_s}{\pi}\right](d_{10}+d_{11}l)+\left[\frac{\alpha_s}{\pi}\right]^2(d_{20}+d_{21}l+d_{22}l^2)+O(\alpha_s^3)\right],
\end{aligned} \tag{2.9}$$

where $l=\ln(Q^2/\mu^2)$. In the $\overline{\rm MS}$ scheme $\mu^2=\mu^2_{\overline{\rm MS}}$ and the analytical expressions for the coefficients d_i read

$$\begin{aligned}
d_{10}&=\tfrac{17}{4}C_F,\quad d_{11}=-\tfrac{3}{2}C_F\ ,\\
d_{20}&=\{[\tfrac{893}{4}-62\zeta(3)]C_A-[65-163\zeta(3)]Tf\\
&\quad+[\tfrac{691}{4}-36\zeta(3)]C_F\}(C_F/16)\ ,\\
d_{21}&=(-\tfrac{284}{3}C_A+\tfrac{88}{3}Tf-105C_F)(C_F/16)\ ,\\
d_{22}&=(11C_A-4Tf+18C_F)(C_F/16)\ .
\end{aligned} \tag{2.10}$$

In Eqs. (2.10) the quadratic Casimir operators for the adjoint and defining representations of the group SU(N) are $C_A=N$ and $C_F=[(N^2-1)/2]/N$, $d(R)$ is the dimension of the representation, $T=\frac{1}{2}$ corresponds to the usual choice of the normalization condition $\mathrm{tr}(T^aT^b)=T\delta^{ab}$ for the generators of SU(N), and f is the number of flavors.

In the case of the standard representation of the group SU(3), one has $d(R)=3$, $C_A=3$, and $C_F=\frac{4}{3}$. Then the coefficients (2.10) take the form

$$\begin{aligned}
d_{10}&=\tfrac{17}{3},\quad d_{11}=-2\ ,\\
d_{20}&=\tfrac{10\,801}{144}-\tfrac{39}{2}\zeta(3)-[\tfrac{65}{24}-\tfrac{2}{3}\zeta(3)]f\ ,\\
d_{21}&=-\tfrac{106}{3}+\tfrac{11}{9}f,\quad d_{22}=\tfrac{19}{4}-\tfrac{1}{6}f\ .
\end{aligned} \tag{2.11}$$

The correction of $O(\alpha_s)$ reproduces the previously known result.[17] The coefficients of the logarithmic terms coincide with the ones obtained in Ref. 15 and are connected with the coefficients of the renormalization-group functions $\beta(\alpha_s)$ and $\gamma_m(\alpha_s)$ by the relations

$$\begin{aligned}
d_{11}&=-2\gamma_0,\quad d_{21}=-2\gamma_1-2\gamma_0d_{10}-\beta_0d_{10}\ ,\\
d_{22}&=\gamma_0\beta_0+2\gamma_0^2\ ,
\end{aligned} \tag{2.12}$$

which provide a useful check of the calculations. As regards the expression for the coefficient d_{20}, it differs from the one reported in Ref. 15. The difference found is due to the misprints at the inputs of the previously used SCHOONSCHIP packages.

III. SCHEME DEPENDENCE OF THE NEXT-TO-NEXT-TO-LEADING-ORDER QCD CORRECTIONS TO $\Gamma_{\rm tot}(H^0\to\text{hadrons})$

The final confirmation of the standard SU(2)×U(1) electroweak model implies the discovery of the scalar Higgs boson. The analysis of the experimental data from the CERN e^+e^- collider LEP excludes the existence of the standard Higgs boson with the mass in the ranges 80

MeV $\leq M_H \leq 12$ GeV (Ref. 31) and 3.0 GeV $\leq M_H \leq 19$ GeV.[32] It is expected that these bounds will be significantly improved.[33] In spite of the nonexistence of the standard Higgs bosons in these regions, it is of interest to present the expression for $\Gamma_{tot}(H^0 \to \text{hadrons})$ in the NNL level of perturbative QCD and to consider the problems related to the scheme dependence of the results obtained.

All calculations, discussed in the previous section, were done in the Euclidean region of momentum transfer. However, to obtain the theoretical expression for Γ_{tot} we are interested in, it is necessary to take into account the π^2 effects which appear after analytical continuation to the timelike region. After applying the renormalization-group method, the expression for Γ_{tot} acquires the form

$$\Gamma_{tot}(M_H)=\sqrt{2}G_F M_H \pi \sum_{j=1}^{f}\left[\mathcal{D}_j(M_H,\overline{m}_j,\overline{\alpha}_s)-\frac{3}{8\pi^2}\overline{m}_j^{\,2}\pi^2\frac{d_{22}}{3}\left[\frac{\overline{\alpha}_s}{\pi}\right]^2\right]\Bigg|_{s=M_H^2}, \tag{3.1}$$

where $\overline{m}_j$ and $\overline{\alpha}_s$ are the running masses and the running coupling constant, defined by Eqs. (2.5) and (2.6). The origin of the π^2 contribution can be understood if one keeps in mind that it results from the analytical continuation of the last $\ln^2(Q^2/\mu^2)$ term in Eq. (2.9).

Note that the question of taking into account the effects of analytical continuation has been discussed previously in the case of the $e^+e^- \to$ hadrons total cross section[34,35] where the different definitions of the expansion parameters in the timelike region were proposed. In these expansion parameters the π^2 factors were partly absorbed. However, in our case it makes sense to use the standard expansion.

Indeed, in the leading order of quark-mass expansion, the expression for Γ_{tot} can be written as

$$\Gamma_{tot}(M_H)=\frac{3\sqrt{2}}{8\pi}G_F M_H\sum_{j=1}^{f}\overline{m}_j^{\,2}\left[C(\overline{\alpha}_s)+O\left[\frac{\overline{m}_j^{\,2}}{M_H^2}\right]\right], \tag{3.2}$$

where f is the number of flavors of quarks to be taken into account in the decay of a Higgs boson with a given mass M_H. The coefficient function $C(\overline{\alpha}_s)$ is defined as

$$C(\overline{\alpha}_s)=1+k_1\left[\frac{\overline{\alpha}_s}{\pi}\right]+k_2\left[\frac{\overline{\alpha}_s}{\pi}\right]^2. \tag{3.3}$$

In the $\overline{\text{MS}}$ scheme the coefficients k_i obtained from Eqs. (2.11) and (3.1) read

$$\begin{aligned}k_1^{\overline{\text{MS}}}&=\tfrac{17}{3}\approx 5.6667\,,\\ k_2^{\overline{\text{MS}}}&=\tfrac{10\,801}{144}-\tfrac{39}{2}\zeta(3)-[\tfrac{65}{24}-\tfrac{2}{3}\zeta(3)]f-\pi^2(\tfrac{19}{12}-\tfrac{1}{18}f)\\ &\approx 51.6668-1.9070f-\pi^2(1.5833-0.0556f)\\ &\approx 35.9399-1.3586f\,.\end{aligned} \tag{3.4}$$

Note that as in the case of the model physical quantity in the $g\phi^4$ theory, calculated up to next-to-next-to-next-to-leading order,[36] the correction proportional to π^2 decreases the numerical values of the analyzed perturbative coefficients. Thus, in this case, it is not necessary to redefine the expansion parameter $\overline{\alpha}_s$ in the timelike region in contrast with the proposals of Refs. 34 and 35.

The considered expression for Γ_{tot} [Eq. (3.2)] is expressed in terms of the running scheme-dependent parameters $\overline{\alpha}_s$ and $\overline{m}_j$, which depend on the scale parameter Λ. The expression for $\overline{\alpha}_s$ is rather compact:

$$\frac{\overline{\alpha}_s}{\pi}=\frac{1}{\beta_0 L}-\frac{\beta_1\ln L}{\beta_0^3 L^2}+\frac{1}{\beta_0^5 L^3}(\beta_1^2\ln^2 L-\beta_1^2\ln L+\beta_2\beta_0-\beta_1^2)+O\left[\frac{1}{L^4}\right], \tag{3.5}$$

where $L=\ln(M_H^2/\Lambda^2)$. The analogous relations for the running masses $\overline{m}_j$ are more cumbersome. Solving Eq. (2.6), one gets

$$\begin{aligned}\overline{m}_j&=\hat{m}_j\exp\left[-\pi\int^{\overline{\alpha}_s}\frac{\gamma_m(x)}{\beta(x)}dx+\frac{\gamma_0}{\beta_0}\ln\left[\frac{2\beta_0}{\pi}\right]\right]\\ &=\hat{m}_j\left[\frac{2\beta_0\overline{\alpha}_s}{\pi}\right]^{\gamma_0/\beta_0}\Bigg\{1+\left[\frac{\gamma_1}{\beta_0}-\frac{\beta_1\gamma_0}{\beta_0^2}\right]\left[\frac{\overline{\alpha}_s}{\pi}\right]\\ &\quad+\frac{1}{2}\left[\left[\frac{\gamma_1}{\beta_0}-\frac{\beta_1\gamma_0}{\beta_0^2}\right]^2+\left[\frac{\gamma_2}{\beta_0}+\frac{\beta_1^2\gamma_0}{\beta_0^3}-\frac{\beta_1\gamma_1}{\beta_0^2}-\frac{\beta_2\gamma_0}{\beta_0^2}\right]\right]\left[\frac{\overline{\alpha}_s}{\pi}\right]^2+O(\overline{\alpha}_s^{\,3})\Bigg\}\end{aligned} \tag{3.6}$$

where

$$\hat{m}_j=m_j(\mu)\exp\left[\pi\int^{\alpha_s(\mu)}\frac{\gamma_m(x)}{\beta(x)}dx-\frac{\gamma_0}{\beta_0}\ln\left[\frac{2\beta_0}{\pi}\right]\right] \tag{3.7}$$

are the scheme-dependent, renormalization-group-invariant quark masses. The form of the additive constant in the exponent corresponds to the generally accepted choice of the arbitrary integration constants. For the cases $f=j$ and s,c,b,t quarks ($\hat{m}_s \approx 0.3$ GeV, $\hat{m}_c \approx 1.9$ GeV, $\hat{m}_b \approx 7.9$ GeV, $\hat{m}_t > ?$), the numerical expressions of Eqs. (3.6) read

$$\overline{m}_s = \hat{m}_s \left[\frac{9}{2}\frac{\overline{\alpha}_s}{\pi} \right]^{4/9} \left[1 + 0.89 \left[\frac{\overline{\alpha}_s}{\pi} \right] + 1.37 \left[\frac{\overline{\alpha}_s}{\pi} \right]^2 \right] ,$$

$$\overline{m}_c = \hat{m}_c \left[\frac{25\overline{\alpha}_s}{6\pi} \right]^{12/25} \left[1 + 1.01 \left[\frac{\overline{\alpha}_s}{\pi} \right] + 1.39 \left[\frac{\overline{\alpha}_s}{\pi} \right]^2 \right] , \tag{3.8}$$

$$\overline{m}_b = \hat{m}_b \left[\frac{23\overline{\alpha}_s}{6\pi} \right]^{12/23} \left[1 + 1.17 \left[\frac{\overline{\alpha}_s}{\pi} \right] + 1.50 \left[\frac{\overline{\alpha}_s}{\pi} \right]^2 \right] ,$$

$$\overline{m}_t = \hat{m}_t \left[\frac{7\overline{\alpha}_s}{2\pi} \right]^{4/7} \left[1 + 1.40 \left[\frac{\overline{\alpha}_s}{\pi} \right] + 1.79 \left[\frac{\overline{\alpha}_s}{\pi} \right]^2 \right] .$$

Comparing now Eqs. (3.8) with Eqs. (3.2)–(3.4), one can observe that in the $\overline{\text{MS}}$ scheme the corrections to the coefficient function $C(\overline{\alpha}_s)$ are larger than the corrections which come from the expansion of the anomalous dimension terms contained in Eqs. (3.8). Thus one might conclude that the analysis of the effects of the higher-order corrections to other quantities, say, characteristics of deep-inelastic scattering, should be started from the calculations of the $O(\alpha_s^2)$ corrections to the coefficient functions. Note that the analysis of the perturbative series to the model physical quantity considered in Ref. 36 resulted in a similar observation.

The expression for the running quark masses can also be expressed in terms of $L = \ln(s/\Lambda^2)$. Substituting Eq. (3.5) into Eq. (3.6), we have

$$\overline{m}_j = \hat{m}_j \left[\frac{2}{L} \right]^{\gamma_0/\beta_0} \left[1 + \frac{1}{\beta_0^3 L}(\beta_0\gamma_1 - \beta_1\gamma_0) - \frac{\ln L}{\beta_0^3 L}\beta_1\gamma_0 + \frac{1}{2\beta_0^6 L^2}(\beta_0^3\gamma_2 + \beta_0^2\gamma_1^2 - \beta_0^2\beta_1\gamma_1 - \beta_0\beta_1^2\gamma_0 + \beta_0^2\beta_2\gamma_0 - 2\beta_0\beta_1\gamma_0\gamma_1 + \beta_1^2\gamma_0^2) + \frac{\ln L}{\beta_0^6 L^2}(-\beta_0\beta_1\gamma_1 - \beta_0\beta_1\gamma_0\gamma_1 + \beta_1^2\gamma_0^2) + \frac{\ln^2 L}{2\beta_0^6 L^2}(\beta_0\beta_1^2\gamma_0 + \beta_1^2\gamma_0^2) + O\left[\frac{1}{L^3}\right] \right] . \tag{3.9}$$

One of the points of view of the analysis of the higher-order PT effects[37] presumes the expansion of measurable quantities in powers of $1/L$ and $\ln L$ terms with the help of the series (3.5) and (3.9), which are, generally speaking, asymptotic ones.[38] However, it seems to us more convenient to express the final results in terms of the running coupling constant $\overline{\alpha}_s$ and the running masses $\overline{m}_j$, which can further be reexpanded in $\overline{\alpha}_s$ with the help of Eq. (3.6).

Let us now discuss the scheme dependence of the results. It is known that the transformations within the class of the MS-like schemes do not affect the values of the scheme-dependent coefficients of the renormalization-group functions: namely, $\beta_i, i \geq 2$, and $\gamma_i, i \geq 1$, in Eqs. (2.5) and (2.6). Therefore, in these schemes the scheme-dependence ambiguity comes from the choice of the parameter μ only. The transformation from the $\overline{\text{MS}}$ to the MS scheme can be done by the replacement $\ln(M_H^2/\mu_{\overline{\text{MS}}}^2) = \ln(M_H^2/\mu_{\text{MS}}^2) + \gamma - \ln(4\pi)$ in Eq. (2.9). This leads to an increase of the numerical values of both scheme-dependent coefficients of Eqs. (3.2) and (3.3): namely,

$$k_1^{\text{MS}} \simeq 9.5428 ,$$
$$k_2^{\text{MS}} \simeq 137.8873 - 4.9017f - \pi^2(1.5833 - 0.0556f) \simeq 122.2608 - 4.3534f . \tag{3.10}$$

Since our final aim is to obtain the precise value of the scheme-invariant quantity ρ_2, introduced in Refs. 9 and 10, we will keep in the numerical results obtained from the analytical ones of Eq. (3.4) at least five significant digits.

The μ parameter of the G scheme[5] is connected with the $\overline{\text{MS}}$ scheme parameter by the relation $\mu_{\overline{\text{MS}}}^2 = \mu_G^2[\exp(-2) + O(\epsilon)]$ or $\ln(M_H^2/\mu_{\overline{\text{MS}}}^2) = \ln(M_H^2/\mu_G^2) + 2$ (see also Ref. 36). One can use this relation in Eq. (2.9) and obtain the numerical values of the coefficients k_1 and k_2 in the G scheme:

$$k_1^G \simeq 1.6667 ,$$
$$k_2^G \simeq -0.0999 - 0.1292f - \pi^2(1.5833 - 0.0556f) \approx -15.7264 + 0.4191f . \tag{3.11}$$

Thus the G scheme seems to be singled out not only from the point of view of a simplification of calculations (see, e.g., Refs. 5 and 30). Indeed, in the G scheme the values of the scheme-dependent coefficients (3.12) are smaller than those in the MS and $\overline{\text{MS}}$ schemes. A similar observation has been made in the course of the analysis of the PT corrections to the model physical quantity in the $g\phi^4$ theory in the next-to-next-to-next-to-leading order.[36] Note also that numerically the transformation relation between the μ parameters of the $\overline{\text{MS}}$ and G schemes is almost identical to the one between the μ parameters of $\overline{\text{MS}}$ and asymmetric MOM (AMOM) schemes of Refs. 6 and 7 obtained in the Feynman gauge, namely

$$\mu_{\overline{\text{MS}}}^2 = \mu_{\text{AMOM}}^2 \exp[-2 + (24 - 2f)/(99 - 6f)] \approx \mu_{\text{AMOM}}^2 \exp[-1.8]^7$$

(Ref. 7). Thus the G scheme preserves the attractive feature of the AMOM scheme, namely, the minimization of the next-to-leading-order (NL) PT corrections to certain physical quantities.

IV. COMPARISON WITH OTHER RESULTS FOR $\Gamma_{\rm tot}$

At the two-loop level the massless $\overline{\rm MS}$-scheme results of Eqs. (2.10), (2.11), and (3.1) are in agreement with those of Ref. 18. In Refs. 16, 18, and 19 the explicit mass dependence of the $O(\alpha_s)$ coefficient has been calculated in the case of the pole masses $\tilde{m}_j$. The results of Refs. 16 and 19 are in agreement with each other. As regards the results of Ref. 18, they do not agree with Eqs. (3.2)–(3.4) of the present work. Indeed, let us use the following relation between the pole and running masses (see, e.g., Ref. 39):

$$\overline{m}_j(p^2)=\tilde{m}_j\left\{1-\left[\frac{4}{3}-\ln\left[\frac{p^2}{\tilde{m}_j^2}\right]\right]\left[\frac{\overline{\alpha}_s}{\pi}\right]+O(\overline{\alpha}_s^2)\right\} . \quad (4.1)$$

The two-loop corrections to this relation are also known at present,[39] but will not be needed in further discussions. Using Eq. (4.1), one can get the following expression for $\Gamma_{\rm tot}$:

$$\Gamma_{\rm tot}(M_H)=\frac{3\sqrt{2}}{8\pi}G_F M_H \times\sum_{j=1}^{f}\tilde{m}_j^2\left\{1+\left[3+2\ln\left[\frac{M_H^2}{\tilde{m}_j^2}\right]\right]\left[\frac{\overline{\alpha}_s}{\pi}\right] +O\left[\frac{\tilde{m}_j^2}{M_H^2}\right]\right\} . \quad (4.2)$$

It is in agreement with the asymptotic expressions obtained from the results of Refs. 16 and 19, but is not consistent with the results of Ref. 18. Moreover, the explicit mass dependence of the $O(\alpha_s)$ correction to $\Gamma_{\rm tot}$ obtained in Ref. 18 disagrees with the recent corrected result of Ref. 19. Therefore, the discussed difference disfavors the findings of Ref. 18.

V. SCHEME-INVARIANT ANALYSIS

Let us consider the scheme-invariant analysis of the results previously discussed fixing the numbers of flavors to be taken into account in Eqs. (3.2) and (3.10). Since a scalar Higgs boson has not yet been discovered,[33] we shall consider the value of its mass as an arbitrary scale parameter $M_H^2=s$ and concentrate on the theoretical questions.

Instead of the quantity (3.2), it is convenient to introduce the function

$$R(a)=-\frac{s}{2}\frac{d\ln(\Gamma/M_H)}{ds}=\gamma_m(a)-\beta(a)\frac{\partial C(a)/\partial a}{2C(a)} , \quad (5.1)$$

where $a=\alpha_s/\pi$. It obeys the renormalization-group equation without the anomalous-dimension term:

$$\left[\mu^2\frac{\partial}{\partial\mu^2}+\beta(a)\frac{\partial}{\partial a}\right]R(a)=0 \quad (5.2)$$

[compare with Eq. (2.4)]. The coefficients of the β function

$$\beta(a)=\mu^2\frac{\partial a}{\partial\mu^2}=-\beta_0\beta_a^2[1+c_1 a+c_2 a^2+O(c_3 a^3)] \quad (5.3)$$

are determined by Eqs. (2.5) and (2.7), i.e., $c_1=\beta_1/\beta_0$, $c_2=\beta_2/\beta_0$. The solution of Eqs. (5.2) and (5.3) is

$$R(\overline{a})=r_0\overline{a}[1+r_1\overline{a}+r_2\overline{a}^2+O(r_3\overline{a}^3)] , \quad (5.4)$$

where

$$r_0=\gamma_0=1, \quad r_0 r_1=\gamma_1+\frac{\beta_0 k_1}{2} , \qquad r_0 r_2=\gamma_2+\beta_0 k_2-\frac{\beta_0 k_1^2}{2}+\frac{\beta_1 k_1}{2} . \quad (5.5)$$

The coefficients β_0,β_1,β_2 and $\gamma_0,\gamma_1,\gamma_2$ are known from the results of Refs. 27 and 28 [see Eqs. (2.7) and (2.8)], and the coefficients k_1 and k_2 in the MS-like schemes are determined by Eqs. (3.4), (3.11), and (3.12).

For simplicity let us consider the case of $f=3$ numbers of flavors. Then the numerical values of the coefficients c_1,c_2 read $c_1=1.7778$, $c_2=4.4711$, and the corresponding $\overline{\rm MS}$-scheme series for the quantity R has the form

$$R^{\overline{\rm MS}}(a)=\overline{a}(1+10.167\overline{a}+59.322\overline{a}^2) . \quad (5.6)$$

Also, for the PT series for $\Gamma_{\rm tot}$, in the MS scheme the absolute numerical values of the coefficients r_1 and r_2 become larger,

$$R^{\rm MS}(\overline{a})=\overline{a}(1+14.528\overline{a}+174.758\overline{a}^2) , \quad (5.7)$$

while in the G scheme they are significantly smaller:

$$R^{G}(\overline{a})=\overline{a}(1+5.667\overline{a}-19.926\overline{a}^2) . \quad (5.8)$$

Let us now transform this series to the effective scheme[12,13] which is *a posteriori* equivalent to the scheme-invariant PT.[10] In its framework the behavior of the physical quantity $R\equiv\overline{a}_{\rm eff}$ is governed by the effective β function with the scheme-invariant coefficients,[10] namely,[40]

$$\frac{\partial R}{\partial\ln|s|}=\beta_{\rm eff}(R) =-\beta_0 R^2[1+c_1 R+\rho_2 R^2+O(\rho_3 R^3)] . \quad (5.9)$$

Note that in the fixed-scheme approach one is dealing with *two* perturbative series for $R(\overline{a})$ and $\beta(\overline{a})$, which are, generally speaking, asymptotic ones. In the effective-scheme approach (or scheme-invariant PT) the behavior of the physical quantity R is governed by *one*, generally speaking, asymptotic and process-dependent series for the $\beta_{\rm eff}$ function with the scheme-invariant coefficients. Thus, in order to investigate the concrete behavior of this series, it is necessary to calculate the invariants ρ_2, ρ_3, etc.

In the considered case the value of the scheme-invariant quantity ρ_2 is negative and large. Indeed, for $f=3$ one has

$$\rho_2=c_2+r_2-c_1 r_1-r_1^2\approx-57.6 . \quad (5.10)$$

We checked that this value can be obtained starting from any of MS-like schemes, namely, from any PT series [Eqs. (5.6), (5.7), and (5.8)]. The errors of rounding off the decimals affect the nonpresented digits in Eq. (5.10) only. Thus, at the three-loop level, the corresponding $\beta_{\rm eff}$ function

$$\beta_{\rm eff}(R)=-2.25R^2(1+1.78R-57.6R^2) \qquad (5.11)$$

has positive zero at the point $R_+^{(0)}\simeq 0.15$ or $\bar{a}_{\rm eff}^{(0)}=R_+^{(0)}\pi\simeq 0.46$, where PT can be applicable in principle. At first glance this zero might be considered as a QCD infrared fixed point, e.g., in the spirit of the discussions of Ref. 24.

Indeed, the general solution of the renormalization-group equations (5.3) and (5.9) for the truncated NNL approximation of the β functions can be written as

$$\ln\frac{s}{\Lambda^2}=\int_{\bar a}^{\infty}\frac{dx}{\beta_0x^2(1+c_1x)} - \int_0^{\bar a}dx\left[\frac{1}{\beta_0x^2(1+c_1x+c_2x^2)} - \frac{1}{\beta_0x^2(1+c_1x)}\right] . \qquad (5.12)$$

In the case of the $\beta_{\rm eff}$ function with a large negative coefficient $\tilde{c}_2=\rho_2$,[40] the explicit expression of the solution of Eq. (5.12) for $\bar a=R$ has the form

$$\ln\frac{s}{\tilde\Lambda^2_{\rm eff}}=\frac{1}{\beta_0R}+\frac{c_1}{\beta_0}\ln\frac{c_1R}{1+c_1R} +\frac{c_1}{2\beta_0}\ln\frac{(1+c_1R)^2}{\rho_2(R-R_+^{(0)})(R+|R_-^{(0)}|)} +\frac{2\rho_2-c_1^2}{2\beta_0\rho_2(R_+^{(0)}+|R_-^{(0)}|)} \times\left[\ln\left|\frac{R-R_+^{(0)}}{R+|R_-^{(0)}|}\right|-\ln\left|\frac{R_+^{(0)}}{R_-^{(0)}}\right|\right] , \qquad (5.13)$$

where

$$\tilde\Lambda^2_{\rm eff}=\Lambda^2_{\rm eff}(\beta_0/c_1)^{c_1/\beta_0} =\Lambda^2_{\overline{\rm MS}}\exp(r_1^{\overline{\rm MS}}/\beta_0)(\beta_0/c_1)^{c_1/\beta_0} ,$$

and $R_+^{(0)}\simeq 0.15$ and $R_-^{(0)}\simeq -0.12$ are the positive ("physical") and the negative ("nonphysical") roots of the three-loop $\beta_{\rm eff}$ function (5.11). One can check that the values of the roots are weakly sensitive to the numbers of flavors taken into account. Thus one might conclude that at $s\ll\Lambda^2_{\rm eff}$ the infrared fixed point regime $R\to R_+^{(0)}$ can be realized.

However, one should be careful in making definite conclusions about the substantiation of asymptotic perturbative series predictions in certain regions of energies. First, since the value of $\Lambda_{\rm eff}$, say, for $\Lambda_{\overline{\rm MS}}\simeq 250$ MeV, is not small, namely,

$$\Lambda_{\rm eff}=\Lambda_{\overline{\rm MS}}\exp\left[\frac{r_1^{\overline{\rm MS}}}{2\beta_0}\right]=\Lambda_{\overline{\rm MS}}9.58\simeq 2\ \text{GeV} , \qquad (5.14)$$

one can conclude that the NL PT predictions are justified only in the region $\sqrt{s}\gg\Lambda_{\rm eff}=2$ GeV (see, e.g., Refs. 13 and 41, where similar problems have been discussed on the basis of the discovery of the large values of the ratios $\Lambda_{\rm eff}/\Lambda_{\overline{\rm MS}}$ in different channels).

Further, to consider the problem of taking into account the NNL perturbative contributions, it is necessary to analyze the important question of applicability of asymptotic PT series and the estimation of their errors. In the fixed renormalization schemes (MS-like, MOM schemes), these questions should be investigated for the PT series for both physical quantities and the renormalization-group β function. The characteristic feature of the methods[10,12,13] is that one should consider only one PT series for the $\beta_{\rm eff}$ function with the scheme-invariant coefficients.

It should be stressed that in the series (5.11) considered by us the NNL term becomes comparable with the leading term at $R=R^*\sim 0.13<R_+^{(0)}$ and that the NL term of the $\beta_{\rm eff}$ function is essentially smaller than both the leading and NNL ones. Let us assume that at $R=R^*$ the NL term is the smallest in the whole PT series for the $\beta_{\rm eff}$ function and that the corresponding series explodes at the NNL level. The scale of explosion S^* can be estimated from the numerical solution of Eq. (5.13) with the initial condition $R=R^*$. For the value $\Lambda_{\overline{\rm MS}}\simeq 250$ MeV this estimate reads

$$\sqrt{S^*}\simeq\Lambda_{\rm eff}0.38\simeq 0.9\ \text{GeV} , \qquad (5.15)$$

where the result of Eq. (5.15) where the result of Eq. (5.14) was taken into account.

It is known that for the sign-alternating asymptotic series the most exact value for its sum results from terminating the series with only half the least term.[42] In the case of the series (5.11), it is not even clear whether its sign-alternating character will survive in higher orders of PT. Therefore, for the values of the effective constants $R^*\sim 0.13$ or scales $S^*\sim 1\ \text{GeV}^2$, where according to our assumption the scheme-invariant NNL series (5.9) explodes, the expression for the $\beta_{\rm eff}$ function should be approximated in accordance with Ref. 42 as

$$\beta_{\rm eff}(R)=-\beta_0R^2(1+Tc_1R) , \qquad (5.16)$$

with $0<T\le 1$ for the sign-constant series, $T=\frac{1}{2}$ for the sign-alternating one, and an accuracy of the order of the least term β_1R^2.[42] Thus, the zero of the three-loop $\beta_{\rm eff}$ function, including the positive one $R_+^{(0)}>R^*$ and therefore the indication to the QCD infrared fixed point, turns out to be a spurious one and the NL-order criterion of applicability of the PT series $\sqrt{s}\gg\Lambda_{\rm eff}$ effectively survives.

Another argument in favor of the above-discussed point of view comes from the estimation of the value of the four-loop coefficient in Eq. (5.9). It is determined by the following scheme-invariant combination:[10]

$$\rho_3=c_3+2r_3+4r_1^3+c_1r_1^2-2c_2r_1-6r_1r_2 , \qquad (5.17)$$

where the coefficients r_i,c_i must be calculated in the same scheme. Using the $\overline{\rm MS}$-scheme results for c_1,c_2,r_1,r_2, one can obtain for the scheme-invariant quantity ρ_3 the following estimate (remember that $f=3$):

$$\rho_3=c_3+2r_3+677.86 . \qquad (5.18)$$

For the cases of $f=4$ and 5 numbers of flavors, the calculated numerical contributions are even larger (~ 750 and ~ 830 accordingly).

It is known that in the pure SU(2) non-Abelian gauge theory the asymptotic PT series for Green's functions are sign-constant ones.[43] Therefore, one can assume that a similar situation takes place in QCD and that in the MS-like scheme for $f \leq 5$ numbers of flavors one will get $c_3(f)>0$. Further, in view of the fact that in the $\overline{\rm MS}$ scheme $r_f(f)$ is quite sizable, let us also assume that $r_3(f)>0$ despite the considerable negative four-loop π^2 analytic-continuation contributions. In this case the values of ρ_3 are positive and large [see Eq. (5.18)]. This argument supports our point of view that near $R^* \sim 0.13 < R_+^{(0)}$ (or $\bar{a}^*_{\rm eff} \sim 0.4 < \bar{a}^{(0)}_{\rm eff} \sim 0.47$) the asymptotic explosion of the PT series (5.9) with the scheme-invariant coefficients manifests itself at the NNL level and that the more accurate PT approximation of the $\beta_{\rm eff}$ function is the two-loop (NL) equation (5.16). Further extension of the scheme-invariant predictions to the region of larger coupling constants must be based on the development of methods of resummation of the QCD asymptotic series (5.9).

Let us now present the comments about the behavior of the PT series (5.1) and (5.3) in the $\overline{\rm MS}$ scheme. First, solving Eq. (5.6) with respect to $\bar{a}_{\rm MS}$ for the value $R^* \simeq 0.13$, which in accordance to our point of view is relevant to the asymptotic explosion of the scheme-invariant PT series [see Eq. (5.11) and the discussions below], one gets the corresponding value of the $\overline{\rm MS}$-scheme constant $\bar{a}_{\rm MS} \simeq 0.067$ (or $\bar{\alpha}_s \simeq 0.21$). At this point neither the three-loop series for the MS-scheme QCD β function (5.3) nor the series (5.6) explode. In the $\overline{\rm MS}$ scheme the NL and NNL terms of Eq. (5.6) become comparable with the leading one at $\bar{a}^*_{\overline{\rm MS}} \simeq 0.1 (\bar{\alpha}^*_s \sim 0.31)$ and $\bar{a}^{**}_{\rm MS} \sim 0.13$ ($\bar{\alpha}^{**}_B \sim 0.41$) correspondingly. Since for the NNL approximation of the $\overline{\rm MS}$-scheme β function there are no problems of taking into account large PT corrections, the corresponding energy region of applicability of the PT series (5.6) can be estimated from the explicit solution of Eq. (5.12) in the $\overline{\rm MS}$ scheme, namely,

$$\ln\frac{s}{\tilde{\Lambda}^2_{\overline{\rm MS}}} = \frac{1}{\beta_0 \bar{a}} + \frac{c_1}{\beta_0}\ln\frac{c_1\bar{a}}{1+c_1\bar{a}} + \frac{c_1}{2\beta_0}\ln\frac{(1+c_1\bar{a})^2}{1+c_1\bar{a}+c_2\bar{a}^2} + \frac{2c_2-c_1^2}{\sqrt{\Delta}\beta_0}\left[\arctan\frac{c_1+2c_2\bar{a}}{\sqrt{\Delta}} - \arctan\frac{c_1}{\sqrt{\Delta}}\right], \quad (5.19)$$

where $\tilde{\Lambda}^2_{\overline{\rm MS}} = \Lambda^2_{\overline{\rm MS}}(\beta_0/c_1)^{c_1/\beta_0}$, $\Delta = 4c_2 - c_1^2$, and $\bar{a} \ll \bar{a}^*_{\overline{\rm MS}}$. Substituting the numerical values for β_0, c_1, c_2, and $\bar{a}^*_{\overline{\rm MS}}$ into Eq. (5.19), one gets the estimate

$$\sqrt{s} \gg \Lambda_{\overline{\rm MS}} 5.2 \simeq 1.3 \text{ GeV}, \quad (5.20)$$

for $\Lambda_{\overline{\rm MS}} \simeq 250$ MeV. Therefore, two conclusions are in order. First, contrary to the scheme-invariant analysis, in the $\overline{\rm MS}$ scheme the problem of the spurious NNL infrared fixed point does not appear. Second, the region of nonapplicability of the PT predictions for the quantity of Eq. (5.1) is $\sqrt{s} = M_H \lesssim 2$ GeV for the value $\Lambda_{\overline{\rm MS}} = 250$ MeV [compare estimates of Eqs. (5.14) and (5.20)]. Therefore, to estimate the values of the characteristics of the decays of the light Higgs particles to light hadrons, it is necessary to use the nonperturbative methods, e.g., those considered in Ref. 44.

VI. CONCLUSION

In this work we presented the analytical results for the three-loop NNL QCD correction to $\Gamma_{\rm tot}(H^0 \to \text{hadrons})$ and analyzed the scheme dependence of the results obtained. The case when the Higgs-boson mass lies far above the thresholds of production of the quark-antiquark pairs was considered ($M_H \gg 2m_f$), and the dependence of the analyzed perturbative coefficients from quark masses has been neglected. In this approximation the contributions to $\Gamma_{\rm tot}$ of the subprocess $H^0 \to gg$ (see Ref. 45) is not taken into account, since its contribution to $\Gamma_{\rm tot}$ is damped by the additional $O(m_j^2/m_H^2)$ factor, where m_j are the masses of the quarks propagating in the internal quark loops. The neglect of the massive dependence allows us to determine the value of the scheme-invariant quantity ρ_2. It turned out that this value is negative and large. The application of the scheme-invariant PT,[10] which is equivalent *a posteriori* to the effective scheme method,[12,13] allowed us to show that the indication to the QCD infrared fixed point in this channel is spurious one. We concluded that near the zero of the effective β function the scheme-invariant series asymptotically explodes. It would be interesting to compare this conclusion with the concrete behavior of the scheme-invariant series in other cases. It is especially important to understand the behavior of the scheme-invariant series in the case of $\sigma_{\rm tot}(e^+e^- \to \text{hadrons})$. Note that the recent result of reevaluation of the four-loop corrections to the QED β function[46] demonstrates that the corrected value of the four-loop contribution to $\sigma_{\rm tot}(e^+e^- \to \text{hadrons})$ will be significantly smaller than the one reported in Ref. 47.

Note added in proof. Prior to the acceptance of this work for publication, it was found that the reevaluation of the next-to-next-to-leading-order QCD corrections to $\sigma_{\rm tot}(e^+e^- \to \text{hadrons})$ and to $\Gamma(\tau^- \to \nu_\tau + \text{hadrons})$, with the help of the SCHOONSCHIP program[29] previously corrected by two of us (A.L.K. and S.A.L.), resulted in the negative values of the corresponding scheme-invariant quantities ρ_2.[28] Two comments are now in order. First, the origin of the effect $\rho_2^{e^+e^-} < 0$ (Ref. 48) is not identical to the similar results of the effective scheme-invariant analysis of the quantity discussed above, $\Gamma_{\rm tot}(H^0 \to \text{hadrons})$ and $\Gamma(\tau^- \to \nu_\tau + \text{hadrons})$[48] in the next-to-next-to-leading order of perturbative QCD. Second, the detailed consideration of the physical conclusions of the investigations discussed in our present work can be of interest for further studies.

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