Tagging the two sources of CP violations in the decays $K_S \to \gamma\gamma$ and $K_L \to \gamma\gamma$

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The author describes how the parameters representing the two sources of CP violations, the CP impurity in the state functions of the neutral kaons K_S and K_L and the CP violation in the transition matrix, can be obtained from the measurements of the form factors in the decays $K_S \to \gamma\gamma$ and $K_L \to \gamma\gamma$.

I. INTRODUCTION

The manifestation of CP violation in the decay of the neutral long-lived kaon into two pions has remained an enigma for more than a quarter of a century.[1] In the decay mode $K_L \to 2\pi$, the CP-violating amplitudes are partitioned into a component emanating from the CP impurity of the K_L state (tagged by the parameter ϵ) and a component originating from the transition matrix element (tagged by the parameter ϵ').[2–7] Although experiments lead to a conclusion that the main source of CP violation comes from ϵ, there is experimental evidence that $\epsilon' \neq 0$. There is, however, a certain disagreement between the two most recent determinations of the real part of the ratio ϵ'/ϵ:

$$\mathrm{Re}(\epsilon'/\epsilon) = \begin{cases} (3.3 \pm 1.1)\times 10^{-3} , \\ -(0.5 \pm 1.5)\times 10^{-3} . \end{cases} \quad (1)$$

The first number is the result from the NA31 Collaboration at CERN,[8] and the second one is from the Chicago-Fermilab Collaboration.[9] The possible existence of CP violations in the other decay modes of the long-lived K_L and the short-lived K_S has become an intellectual stimulus for both the theorists and experimentalists.[2,3,10]

In an earlier paper,[11] we have discussed the decay modes $K_S \to \gamma\gamma \to e^- e^+ \mu^- \mu^+$ and $K_L \to \gamma\gamma \to e^- e^+ \mu^- \mu^+$. We delineated how the measurements of the angular decay distributions with respect to the angle between the decay planes of the lepton pairs produced by the double internal conversions of the two photons can be used to determine the CP-violating and CP-conserving form factors in the decays $K_S \to \gamma\gamma$ and $K_L \to \gamma\gamma$. The object of this paper is to disentangle from the measurable form factors the two different sources of CP violation: the CP impurity, measured by the parameter ϵ, in the wave functions of K_S and K_L, and the CP violation occurring via the transition matrix. We explain in the next section how this is done and analyze some special cases. Before we proceed, we would like to point out that two recent papers[12,13] have reported the measurements of the single Dalitz decay $K_L \to e^+ e^- \gamma$ and the observations of an enhancement in the distribution of the invariant electron-positron pair mass. This enhancement has been interpreted as an evidence for a $K_L \to e^+ e^- \gamma$ form factor arising from virtual vector mesons' contributions to the photon propagator.[14] This means that the momentum dependence of the form factors H_1, G_1, H_2, and G_2 (or h_1, g_1, h_2, and g_2) and hence of the amplitudes $A_S(\pm)$ and $A_L(\pm)$ (discussed later in the paper) will likely show up in the measurements of the angular decay distributions $\Sigma_1(\phi)$, $\Sigma_2(\phi)$, $\Delta_1(\phi)$, and $\Delta_2(\phi)$. However, as will be seen later in the paper, since the relationships between the set of amplitudes $[A_L(\pm), A_S(\pm)]$ and the set of form factors (H_1, G_1, H_2, G_2) or (h_1, g_1, h_2, g_2) are purely algebraic, the technique discussed will remain applicable. What is germane is to keep in mind that the parameters being discussed can all be momentum dependent, reflecting dynamics of origins other than QED.

II. EXTRACTING THE TWO SOURCES OF CP VIOLATIONS FROM THE FORM FACTORS G_1, H_1, G_2, and H_2

We assume the phenomenological Lagrangian

$$L = \frac{iH}{4M} \Phi \varepsilon_{\mu\nu\alpha\beta} F_{\mu\nu} F_{\alpha\beta} + \frac{iG}{4M} \Phi F_{\mu\nu} F_{\mu\nu} , \quad (2)$$

for the $K\gamma\gamma$ vertex.[15–17] The meson field is Φ and its mass is M. The tensor $F_{\mu\nu}$ is $\partial_\mu A_\nu - \partial_\nu A_\mu$, where A_μ is the photon field. H and G are dimensionless form factors that parametrize the dynamics of the $K\gamma\gamma$ vertex. In general, these form factors depend on the momenta of the two photons, but in our discussion their momentum dependence is neglected within the range of energy involved.[11]

From the above Lagrangian, the decay rates of K_S and K_L into two photons are

$$\Gamma(K_S \to \gamma\gamma) = \frac{1}{16\pi} M_S (|H_1|^2 + 2|G_1|^2) = \frac{1}{16\pi} M_S g_1^2 \left[\left| \frac{h_1}{g_1} \right|^2 + 2 \right] , \quad (3)$$

$$\Gamma(K_L \to \gamma\gamma) = \frac{1}{16\pi} M_L (|H_2|^2 + 2|G_2|^2) = \frac{1}{16\pi} M_L h_2^2 \left[1 + 2 \left| \frac{g_2}{h_2} \right|^2 \right] , \quad (4)$$

where

$$H_1=h_1\exp(i\psi_{h_1})\ ,\quad H_2=h_2\exp(i\psi_{h_2})\ , \tag{5}$$

$$G_1=g_1\exp(i\psi_{g_1})\ ,\quad G_2=g_2\exp(i\psi_{g_2})\ , \tag{6}$$

$$\delta_1=\psi_{g_1}-\psi_{h_1}\ ,\quad \delta_2=\psi_{g_2}-\psi_{h_2}\ , \tag{7}$$

and M_S and M_L are the masses of K_S and K_L, respectively. It was the measurements of h_1/g_1, g_2/h_2, δ_1, and δ_2 that were discussed in Ref. 11.

We assume *CPT* invariance and adopt the convention $CP|K^0\rangle=|\bar{K}^0\rangle$. Therefore, the short-lived K_S and long-lived K_L states are expressed, in terms of the *CP*-violating parameter ϵ, as mixtures of the two strangeness eigenstates,[2-7] $|K^0\rangle$, with strangeness $S=1$ and $|\bar{K}^0\rangle$, with strangeness $S=-1$:

$$|K_S\rangle=\frac{1}{[(1+|\epsilon|^2)]^{1/2}}[(1+\epsilon)|K^0\rangle+(1-\epsilon)|\bar{K}^0\rangle]\ , \tag{8}$$

$$|K_L\rangle=\frac{1}{[(1+|\epsilon|^2)]^{1/2}}[(1+\epsilon)|K^0\rangle-(1-\epsilon)|\bar{K}^0\rangle]\ . \tag{9}$$

The parameter ϵ characterizes the *CP* impurity in the K_S and K_L states and has been measured in the decay $K^0\to 2\pi$ to have a modulus[18]

$$|\epsilon|=(2.27\pm 0.02)\times 10^{-3}\ . \tag{10}$$

From *CPT* invariance and unitarity it can be shown that the phase or argument of ϵ has a value[19]

$$\mathrm{Arg}(\epsilon)=(43.67\pm 0.14)^0\ . \tag{11}$$

We shall consider now the direct *CP* violation via the transition of the K^0 and $\bar{K}^0$ components of K_S and K_L into two photons. For convenience, we shall use the following notation to indicate the amplitudes of decay:[16,20-23]

$$\begin{aligned} A(+)&\equiv A[K^0\to 2\gamma(+)]\ ,\\ \bar{A}(+)&\equiv A[\bar{K}^0\to 2\gamma(+)]\ , \end{aligned} \tag{12a}$$

$$\begin{aligned} A(-)&\equiv A[K^0\to 2\gamma(-)]\ ,\\ \bar{A}(-)&\equiv A[\bar{K}^0\to 2\gamma(-)]\ , \end{aligned} \tag{12b}$$

$$\begin{aligned} A_S(+)&\equiv A[K_S\to 2\gamma(+)]\ ,\\ A_S(-)&\equiv A[K_S\to 2\gamma(-)]\ , \end{aligned} \tag{13a}$$

$$\begin{aligned} A_L(+)&\equiv A[K_L\to 2\gamma(+)]\ ,\\ A_L(-)&\equiv A[K_L\to 2\gamma(-)]\ , \end{aligned} \tag{13b}$$

where $2\gamma(+)$ and $2\gamma(-)$ are the $CP=+1$ and $CP=-1$ two-photon states, respectively. Let us define the parameters r_1, ϕ_1, r_2, ϕ_2, r, and ϕ as

$$-r_1\exp(i\phi_1)=\frac{A(-)}{\bar{A}(-)},\quad r_2\exp(i\phi_2)=\frac{A(+)}{\bar{A}(+)}\ , \tag{14}$$

$$ir\exp(i\phi)=\frac{\bar{A}(-)}{\bar{A}(+)}\ . \tag{15}$$

When there is *CP* invariance via direct transition, $A(+)=\bar{A}(+)$ and $A(-)=-\bar{A}(-)$ and, thus, r_1, r_2, ϕ_1, and ϕ_2 have the special values $r_1=1$, $\phi_1=0$, $r_2=1$, and $\phi_2=0$. The occurrence of *CP* violation via direct transition can therefore be indicated by the deviation of any of the parameters r_1, ϕ_1, r_2, and ϕ_2 from these values. We also note that, since both $K_S\to\gamma\gamma$ and $K_L\to\gamma\gamma$ do occur, the parameter r cannot be zero.

The following ratios of amplitudes can be expressed in terms of the above parameters:

$$\frac{A_S(-)}{A_L(-)}=\frac{r_1e^{i\phi_1}-Ee^{i\gamma}}{r_1e^{i\phi_1}+Ee^{i\gamma}}\ , \tag{16}$$

$$\frac{A_L(+)}{A_S(+)}=\frac{r_2e^{i\phi_2}-Ee^{i\gamma}}{r_2e^{i\phi_2}+Ee^{i\gamma}}\ , \tag{17}$$

$$\frac{A_L(-)}{A_S(+)}=-ire^{i\phi}\left[\frac{r_1e^{i\phi_1}+Ee^{i\gamma}}{r_2e^{i\phi_2}+Ee^{i\gamma}}\right]\ , \tag{18}$$

where

$$Ee^{i\gamma}=\frac{1-\epsilon}{1+\epsilon}\ . \tag{19}$$

The values of E and γ can be deduced from Eqs. (10) and (11):

$$E=0.9967,\quad \gamma=-0.1796^\circ\ . \tag{20}$$

The decay rates of $\Gamma(K_S\to\gamma\gamma)$ and $\Gamma(K_L\to\gamma\gamma)$ in terms of these amplitudes are[20,21]

$$\begin{aligned}\Gamma(K_S\to\gamma\gamma)&=\frac{1}{32\pi M_S}[|A_S(+)|^2+|A_S(-)|^2]\\ &=\frac{1}{32\pi M_S}\left[|A_S(+)|^2\left[1+\left|\frac{A_S(-)}{A_L(-)}\right|^2\left|\frac{A_L(-)}{A_S(+)}\right|^2\right]\right]\ ,\end{aligned} \tag{21}$$

$$\begin{aligned}\Gamma(K_L\to\gamma\gamma)&=\frac{1}{32\pi M_L}[|A_L(+)|^2+|A_L(-)|^2]\\ &=\frac{1}{32\pi M_L}\left[|A_S(+)|^2\left[\left|\frac{A_L(+)}{A_S(+)}\right|^2+\left|\frac{A_L(-)}{A_S(+)}\right|^2\right]\right]\ .\end{aligned} \tag{22}$$

Comparing Eqs. (21) and (22) with Eqs. (3) and (4) rewritten as

$$\Gamma(K_S\to\gamma\gamma)=\frac{1}{32\pi M_S}|2M_SG_1|^2\left[1+\frac{1}{2}\left|\frac{2M_SH_1}{2M_SG_1}\right|^2\right], \tag{23}$$

$$\Gamma(K_L\to\gamma\gamma)=\frac{1}{32\pi M_L}|2M_SG_1|^2\left[\frac{1}{2}\left|\frac{2M_LH_2}{2M_SG_1}\right|^2+\left|\frac{2M_LG_2}{2M_SG_1}\right|^2\right], \tag{24}$$

we make the identifications

$$2M_SG_1=A_S(+)\ , \tag{25}$$

$$\frac{1}{\sqrt{2}}\left[\frac{2M_LH_2}{2M_SG_1}\right]=\frac{A_L(-)}{A_S(+)}\ , \tag{26}$$

$$\left[\frac{2M_LG_2}{2M_SG_1}\right]=\frac{A_L(+)}{A_S(+)}\ , \tag{27}$$

$$\frac{1}{\sqrt{2}}\left[\frac{2M_SH_1}{2M_SG_1}\right]=\left[\frac{A_L(-)}{A_S(+)}\right]\left[\frac{A_S(-)}{A_L(-)}\right], \tag{28}$$

from which we obtain

$$\frac{H_1}{\sqrt{2}G_1}=\left[\frac{A_L(-)}{A_S(+)}\right]\left[\frac{A_S(-)}{A_L(-)}\right], \tag{29}$$

$$\frac{\sqrt{2}G_2}{H_2}=\left[\frac{A_L(+)}{A_S(+)}\right]\left[\frac{A_L(-)}{A_S(+)}\right]^{-1}. \tag{30}$$

Substituting Eqs. (5)–(7) into the left-hand sides and Eqs. (16)–(18) into the right-hand sides of the above two equations, we get

$$\frac{1}{\sqrt{2}}\left[\frac{h_1}{g_1}\right]e^{-i\delta_1}=-ire^{i\phi}\left[\frac{r_1e^{i(\phi_1-\gamma)}-E}{r_2e^{i(\phi_2-\gamma)}+E}\right], \tag{31}$$

$$\sqrt{2}\left[\frac{g_2}{h_2}\right]e^{i\delta_2}=\frac{ie^{-i\phi}}{r}\left[\frac{r_2e^{i(\phi_2-\gamma)}-E}{r_1e^{i(\phi_1-\gamma)}+E}\right]. \tag{32}$$

These two equations relate the set of parameters $(h_1,g_1,h_2,g_2,\delta_1,\delta_2)$ to the set of parameters $(r_1,\phi_1,r_2,\phi_2,r,\phi)$. We would like to equate the moduli and phases of the left-hand sides to those of the right-hand sides in Eqs. (31) and (32). To do this, we will reduce the right-hand sides of Eqs. (31) and (32) into the exponential form $Ae^{i\theta}$. First, we write down the numerators and denominators in exponential forms separately and then recombine. Let

$$\begin{aligned}De^{i\delta}&=r_1e^{i(\phi_1-\gamma)}-E\ ,\\ Ce^{i\sigma}&=r_1e^{i(\phi_1-\gamma)}+E\ ,\\ Be^{i\beta}&=r_2e^{i(\phi_2-\gamma)}-E\ ,\\ Ae^{i\alpha}&=r_2e^{i(\phi_2-\gamma)}+E\ ,\end{aligned} \tag{33}$$

from which it follows that

$$\begin{aligned}D^2&=r_1^2+E^2-2Er_1\cos(\phi_1-\gamma)\ ,\\ C^2&=r_1^2+E^2+2Er_1\cos(\phi_1-\gamma)\ ,\\ B^2&=r_2^2+E^2-2Er_2\cos(\phi_2-\gamma)\ ,\\ A^2&=r_2^2+E^2+2Er_2\cos(\phi_2-\gamma)\ ,\end{aligned} \tag{34}$$

$$\tan\delta=\frac{r_1\sin(\phi_1-\gamma)}{r_1\cos(\phi_1-\gamma)-E}\ , \tag{35}$$

$$\tan\sigma=\frac{r_1\sin(\phi_1-\gamma)}{r_1\cos(\phi_1-\gamma)+E}\ , \tag{36}$$

$$\tan\beta=\frac{r_2\sin(\phi_2-\gamma)}{r_2\cos(\phi_2-\gamma)-E}\ , \tag{37}$$

$$\tan\alpha=\frac{r_2\sin(\phi_2-\gamma)}{r_2\cos(\phi_2-\gamma)+E}\ . \tag{38}$$

Using Eq. (33) in Eqs. (31) and (32), we obtain

$$\frac{1}{\sqrt{2}}\left[\frac{h_1}{g_1}\right]\exp(-i\delta_1)=\frac{rD}{A}\exp\left[i\left[-\frac{\pi}{2}+\phi+\delta-\alpha\right]\right], \tag{39}$$

$$\sqrt{2}\left[\frac{g_2}{h_2}\right]\exp(+i\delta_2)=\frac{B}{rC}\exp\left[i\left[\frac{\pi}{2}-\phi+\beta-\sigma\right]\right], \tag{40}$$

and, therefore,

$$\frac{1}{\sqrt{2}}\left[\frac{h_1}{g_1}\right]=\frac{rD}{A}\ , \tag{41}$$

$$\sqrt{2}\left[\frac{g_2}{h_2}\right]=\frac{B}{rC}\ , \tag{42}$$

$$\delta_1=\pi/2-\phi+\alpha-\delta\ , \tag{43}$$

$$\delta_2=\pi/2-\phi+\beta-\sigma\ . \tag{44}$$

From Eqs. (43) and (44), we further get

$$\delta_1-\delta_2=(\alpha-\beta)+(\sigma-\delta)\ . \tag{45}$$

After multiplying Eqs. (41) and (42), rearranging terms, squaring, and using Eqs. (34), we obtain

$$K \equiv \left[\frac{h_1}{g_1}\right]^2 \left[\frac{g_2}{h_2}\right]^2 = \left[\frac{B}{A}\right]^2 \left[\frac{D}{C}\right]^2 , \tag{46}$$

$$= \left[\frac{r_2^2+E^2-2Er_2\cos(\phi_2-\gamma)}{r_2^2+E^2+2Er_2\cos(\phi_2-\gamma)}\right] \times \left[\frac{r_1^2+E^2-2Er_1\cos(\phi_1-\gamma)}{r_1^2+E^2+2Er_1\cos(\phi_1-\gamma)}\right] , \tag{47}$$

where we have denoted the product $(h_1/g_1)^2(g_2/h_2)^2$ by K. Note that the multiplication of Eqs. (41) and (42) has canceled out r and ϕ so that they do not appear in Eqs. (46) and (47).

The quantities (h_1/g_1), (g_2/h_2), δ_1, and δ_2 can be experimentally determined from measuring the angular distributions $d\Gamma(K_S \to \gamma\gamma \to e^-e^+\mu^-\mu^+)/d\phi$ and $d\Gamma(K_L \to \gamma\gamma \to e^-e^-\mu^+\mu^+)/d\phi$ as described in Ref. 11; hence, $K \equiv (h_1/g_1)^2(g_2/h_2)^2$ of Eq. (46) and $(\delta_1-\delta_2)$ of Eq. (45) can be known experimentally. The values of r_1, ϕ_1, r_2, and ϕ_2 have to adjust to yield those empirical values of K and $(\delta_1-\delta_2)$. Conversely, one can also explore what are the expected values of K and $(\delta_1-\delta_2)$ for certain assumed values of r_1, r_2, ϕ_1, and ϕ_2. In the following, we will analyze several cases using Eqs. (35)–(38) and (41)–(47). In all of these analyses, we will assume that the value of E is 1.

Case 1: $r_1=1$ and $\phi_1=\gamma$; r_2 and ϕ_2 are arbitrary. From Eqs. (31) and (46), this case implies that $h_1=0$ and, therefore, $K=0$. Conversely, if $h_1=0$, then $r_1=1$ and $\phi_1=\gamma$, which means if there is any direct *CP* violation, it is monitored by r_2 or ϕ_2 or by both.

Case 2: $r_2=1$ and $\phi_1=\gamma$; r_1 and ϕ_1 are arbitrary. From Eqs. (32) and (46), this case implies that $h_2=0$ and, therefore, $K=0$. Conversely, if $h_2=0$, then $r_2=1$ and $\phi_2=\gamma$, which means if there is any direct *CP* violation, it is monitored by r_1 or ϕ_1 or by both.

Note that when both the indirect mass-mixing *CP* violation and the direct transition *CP* violation are turned off, that is, when $\gamma=0$, $r_1=r_2=1$, and $\phi_1=\phi_2=0$, K is also zero.

Case 3: $r_1=r_2=1$; $\phi_1=\phi_2=0$. This obtains when *CP* violation occurs via mass mixing only as tagged by the parameter ϵ or, equivalently, by γ. Equations (35)–(38), (45), and (47) yield

$$\delta_1-\delta_2=-\pi , \quad K = \left[\tan\left[-\frac{\gamma}{2}\right]\right]^4 = 6.076\times 10^{-12} . \tag{48}$$

A possibility for this to happen is when

$$\frac{h_1}{g_1} = \frac{g_2}{h_2} = 1.57\times 10^{-3} ,$$

which has the same order of magnitude as the mass-mixing parameter ϵ.

Case 4: $r_1=1$, $\phi_1=0$, $r_2 \ll 1$. Expanding Eq. (47) and retaining terms of order r_2, we get

$$K = \left[\frac{1-2r_2\cos(\phi_2-\gamma)}{1+2r_2\cos(\phi_2-\gamma)}\right]\left[\frac{1-\cos(-\gamma)}{1+\cos(-\gamma)}\right] = [1-4r_2\cos(\phi_2-\gamma)]\left[\tan\left[-\frac{\gamma}{2}\right]\right]^2 . \tag{49}$$

Solving for $r_2\cos(\phi_2-\gamma)$ yields

$$r_2\cos(\phi_2-\gamma) = \frac{[1-K\cot^2(-\gamma/2)]}{4} \equiv K_2 . \tag{50}$$

We have denoted by K_2 the term $[1-K\cot^2(-\gamma/2)]/4$ because, as will be seen later, it will reoccur in other cases. Since K_2 depends on K and γ, K_2 is also empirically determinable.

Equations (35)–(38) yield

$$\tan\delta = -\cot(-\gamma/2), \quad \tan\sigma = \tan(-\gamma/2) , \tag{51}$$

$$\tan\beta = \frac{r_2\sin(\phi_2-\gamma)}{r_2\cos(\phi_2-\gamma)-1} , \quad \tan\alpha = \frac{r_2\sin(\phi_2-\gamma)}{r_2\cos(\phi_2-\gamma)+1} , \tag{52}$$

which imply

$$\delta-\sigma = \frac{\pi}{2} . \tag{53}$$

Using the above result in Eq. (45), we get

$$\alpha-\beta = \delta_1-\delta_2+\frac{\pi}{2} , \quad \tan(\alpha-\beta) = \tan\left[\delta_1-\delta_2+\frac{\pi}{2}\right] . \tag{54}$$

Employing the trigonometric identities

$$\tan(x-y) = \frac{\tan(x)-\tan(y)}{1+\tan(x)\tan(y)} , \tag{55}$$

$$\cot(u-v) = -\tan\left[u-v+\frac{\pi}{2}\right] , \tag{56}$$

in Eq. (53), we obtain

$$\frac{\tan(\alpha)-\tan(\beta)}{1+\tan(\alpha)\tan(\beta)} = -\cot(\delta_1-\delta_2) . \tag{57}$$

Substituting Eq. (52) in Eq. (57), expanding and retaining terms of order r_2 only, we get

$$2r_2\sin(\phi_2-\gamma) = -\cot(\delta_1-\delta_2) . \tag{58}$$

Dividing Eq. (58) by Eq. (50), one obtains

$$\tan(\phi_2-\gamma) = -\frac{\cot(\delta_1-\delta_2)}{2K_2} . \tag{59}$$

Equations (50) and (59) constitute two equations in two unknowns r_2 and ϕ_2. For convenience, we rewrite the two equations together:

$$\tan(\phi_2-\gamma)=-\frac{\cot(\delta_1-\delta_2)}{2K_2}\,,\qquad r_2=\frac{K_2}{\cos(\phi_2-\gamma)}\,. \tag{60}$$

Case 5: $r_1=1$, $\phi_1=0$, $r_2\gg 1$. Following the steps in case 4, but this time retaining terms of order $1/r_2$ only, we get

$$\tan(\phi_2-\gamma)=+\frac{\cot(\delta_1-\delta_2)}{2K_2}\,,\qquad \frac{1}{r_2}=\frac{K_2}{\cos(\phi_2-\gamma)}\,. \tag{61}$$

Case 6: $r_2=1$, $\phi_2=0$, $r_1\ll 1$. Following the steps in case 4, but retaining terms of order r_1, one gets

$$\tan(\phi_1-\gamma)=-\frac{\cot(\delta_1-\delta_2)}{2K_2}\,,\qquad r_1=\frac{K_2}{\cos(\phi_1-\gamma)}\,. \tag{62}$$

Case 7: $r_2=1$, $\phi_2=0$, $r_1\gg 1$. Following the steps in case 4, but retaining terms of order $1/r_1$, one obtains

$$\tan(\phi_1-\gamma)=+\frac{\cot(\delta_1-\delta_2)}{2K_2}\,,\qquad \frac{1}{r_1}=\frac{K_2}{\cos(\phi_1-\gamma)}\,. \tag{63}$$

Case 8: $r_1\ll 1$, $r_2\ll 1$, $\phi_1\neq\phi_2\pm\pi$. When expanded up to first powers of r_1 and r_2, Eq. (47) yields

$$K=\left[\frac{1-2r_2\cos(\phi_2-\gamma)}{1+2r_2\cos(\phi_2-\gamma)}\right]\left[\frac{1-2r_1\cos(\phi_1-\gamma)}{1+2r_1\cos(\phi_1-\gamma)}\right]=1-4r_1\cos(\phi_1-\gamma)-4r_2\cos(\phi_2-\gamma)\,. \tag{64}$$

Taking the tangents of both sides of Eq. (45) and using the trigonometric identities in Eqs. (55) and (56), one gets

$$\tan(\delta_1-\delta_2)=\frac{\tan(\alpha-\beta)+\tan(\sigma-\delta)}{1-\tan(\alpha-\beta)\tan(\sigma-\delta)}\,. \tag{65}$$

Reusing the identities in Eqs. (55) and (56) in both $\tan(\alpha-\beta)$ and $\tan(\sigma-\delta)$ and then substituting Eqs. (35)–(38), one obtains

$$\tan(\alpha-\beta)=2r_2\sin(\phi_2-\gamma)\,,\qquad \tan(\sigma-\delta)=2r_1\sin(\phi_1-\gamma)\,. \tag{66}$$

Substituting Eq. (66) into the right-hand side of Eq. (65), we get

$$\tan(\delta_1-\delta_2)=2r_2\sin(\phi_2-\gamma)+2r_1\sin(\phi_1-\gamma)\,. \tag{67}$$

One can now solve for r_1 and r_2 from Eqs. (64) and (67) to get

$$r_1=\left[\frac{1-K}{4}\sin(\phi_2-\gamma)-\frac{1}{2}\tan(\delta_1-\delta_2)\cos(\phi_2-\gamma)\right][\sin(\phi_2-\phi_1)]^{-1}\,,$$
$$r_2=\left[-\frac{1-K}{4}\sin(\phi_1-\gamma)+\frac{1}{2}\tan(\delta_1-\delta_2)\cos(\phi_1-\gamma)\right][\sin(\phi_2-\phi_1)]^{-1}\,. \tag{68}$$

Case 9: $r_1\ll 1$, $r_2\ll 1$, $\phi_1=\phi_2=\rho\neq\gamma$. We have denoted the common value of ϕ_1 and ϕ_2 by ρ.

Applying the trigonometric identity of Eq. (55) to $\tan(\alpha-\beta)$ and $\tan(\sigma-\delta)$ and then using Eqs. (33)–(38) while retaining terms of order r_1 and r_2, we get

$$\tan(\alpha-\beta)=2r_2\sin(\phi_2-\gamma)\,,\qquad \tan(\sigma-\delta)=2r_1\sin(\phi_1-\gamma)\,. \tag{69}$$

One can now take the tangent of Eq. (45), use the identity in Eq. (55), and then substitute Eq. (69) in it to get

$$\tan(\delta_1-\delta_2)=2(r_1+r_2)\sin(\rho-\gamma)\,. \tag{70}$$

Meanwhile, Eq. (47) becomes

$$K=\left[\frac{1-2r_2\cos(\phi_2-\gamma)}{1+2r_2\cos(\phi_2-\gamma)}\right]\left[\frac{1-2r_1\cos(\phi_1-\gamma)}{1+2r_1\cos(\phi_2-\gamma)}\right]\,. \tag{71}$$

One can solve for r_1 from Eq. (70) to get

$$r_1=-r_2+\frac{\tan(\delta_1-\delta_2)}{2\sin(\rho-\gamma)} \tag{72}$$

and substitute it in Eq. (71), which is transformed into

$$K=\left[\frac{1-2r_2\cos(\rho-\gamma)}{1+2r_2\cos(\rho-\gamma)}\right]\left[\frac{2A'+2r_2\cos(\rho-\gamma)}{2B'-2r_2\cos(\rho-\gamma)}\right]\,, \tag{73}$$

where

$$2A'=1-\frac{\tan(\delta_1-\delta_2)}{\tan(\rho-\gamma)}\,,\qquad 2B'=1+\frac{\tan(\delta_1-\delta_2)}{\tan(\rho-\gamma)}\,. \tag{74}$$

Expanding the right-hand side of Eq. (73) and retaining only terms linear in r_2, we can solve for r_2 to get

$$r_2=\frac{2A'-2B'K}{2\cos(\rho-\gamma)[(2A'-1)+K(2B'-1)]}\,. \tag{75}$$

Substituting Eq. (74) into Eq. (75) and juxtaposing the result to Eq. (72) for clarity, we have

$$r_2=\frac{-(K-1)\tan(\rho-\gamma)-(K+1)\tan(\delta_1-\delta_2)}{2\cos(\rho-\gamma)(K-1)\tan(\delta_1-\delta_2)}\,, \tag{76}$$

$$r_1=-r_2+\frac{\tan(\delta_1-\delta_2)}{2\sin(\rho-\gamma)}\,. \tag{76'}$$

Case 10: $r_1 \gg 1$, $r_2 \gg 1$, $\phi_1 = \phi_2 \pm \pi$. Following the steps in case 8, one obtains

$$\frac{1}{r_1} = \left[\frac{1-K}{4}\sin(\phi_2-\gamma) + \frac{1}{2}\tan(\delta_1-\delta_2)\cos(\phi_2-\gamma) \right] [\sin(\phi_2-\phi_1)]^{-1} ,$$
$$\frac{1}{r_2} = -\left[\frac{1-K}{4}\sin(\phi_1-\gamma) + \frac{1}{2}\tan(\delta_1-\delta_2)\cos(\phi_1-\gamma) \right] [\sin(\phi_2-\phi_1)]^{-1} . \tag{77}$$

Case 11: $r_1 \gg 1$, $r_2 \gg 1$, $\phi_1 = \phi_2 = \rho \neq \gamma$. Following the steps in case 9, one gets

$$\frac{1}{r_2} = \frac{(1-K)\tan(\rho-\gamma) + (1+K)\tan(\delta_1-\delta_2)}{2(1-K)\cos(\rho-\gamma)\tan(\delta_1-\delta_2)} ,$$
$$\frac{1}{r_1} = -\frac{1}{r_2} - \frac{\tan(\delta_1-\delta_2)}{2\sin(\rho-\gamma)} . \tag{78}$$

Case 12: $r_1 \ll 1$, $r_2 \gg 1$, $\phi_1 + \phi_2 \neq 2\gamma \pm \pi$. Following the steps in case 9, we get

$$r_1 = \left[\frac{1-K}{4}\sin(\phi_2-\gamma) + \frac{1}{2}\tan(\delta_1-\delta_2)\cos(\phi_2-\gamma) \right] [\sin(\phi_1+\phi_2-2\gamma)]^{-1} ,$$
$$\frac{1}{r_2} = \left[\frac{1-K}{4}\sin(\phi_1-\gamma) - \frac{1}{2}\tan(\delta_1-\delta_2)\cos(\phi_1-\gamma) \right] [\sin(\phi_1+\phi_2-2\gamma)]^{-1} . \tag{79}$$

Case 13: $r_1 \gg 1$, $r_2 \ll 1$, $\phi_1 + \phi_2 \neq 2\gamma \pm \pi$. Following the steps in case 9, we get

$$\frac{1}{r_1} = \left[\frac{1-K}{4}\sin(\phi_2-\gamma) - \frac{1}{2}\tan(\delta_1-\delta_2)\cos(\phi_2-\gamma) \right] [\sin(\phi_1+\phi_2-2\gamma)]^{-1} ,$$
$$r_2 = \left[\frac{1-K}{4}\sin(\phi_1-\gamma) + \frac{1}{2}\tan(\delta_1-\delta_2)\cos(\phi_1-\gamma) \right] [\sin(\phi_1+\phi_2-2\gamma)]^{-1} . \tag{80}$$

Case 14: $r_1 = r_2 = 1$, ϕ_1 and ϕ_2 arbitrary. From Eq. (47), we get

$$K = \left[\frac{1-\cos(\phi_2-\gamma)}{1+\cos(\phi_2+\gamma)} \right] \left[\frac{1-\cos(\phi_1-\gamma)}{1+\cos(\phi_1-\gamma)} \right]$$
$$= \left[\tan\left[\frac{\phi_2-\gamma}{2} \right] \right]^2 \left[\tan\left[\frac{\phi_1-\gamma}{2} \right] \right]^2 , \tag{81}$$

and if we let

$$\phi_2 = \phi_1 + \Delta\phi ,$$

Eq. (81) becomes

$$K = \left[\frac{\tan[(\phi_1-\gamma)/2] + \tan(\Delta\phi/2)}{\cot[(\phi_1-\gamma)/2] - \tan(\Delta\phi/2)} \right]^2 . \tag{82}$$

From Eqs. (35)–(38), we get

$$\tan\delta = \frac{\sin(\phi_1-\gamma)}{\cos(\phi_1-\gamma)-1} = -\cot\left[\frac{\phi_1-\gamma}{2} \right] ,$$
$$\tan\sigma = \frac{\sin(\phi_1-\gamma)}{\cos(\phi_1-\gamma)+1} = +\tan\left[\frac{\phi_1-\gamma}{2} \right] , \tag{83}$$

$$\tan\beta = \frac{\sin(\phi_2-\gamma)}{\cos(\phi_2-\gamma)-1} = -\cot\left[\frac{\phi_2-\gamma}{2} \right] ,$$
$$\tan\alpha = \frac{\sin(\phi_2-\gamma)}{\cos(\phi_2-\gamma)+1} = +\tan\left[\frac{\phi_2-\gamma}{2} \right] . \tag{84}$$

From Eq. (83),

$$\tan\sigma \tan\delta = -1 ,$$
$$\tan\sigma = -\cot(\delta) = \tan(\delta - \pi/2) ,$$
$$\delta - \sigma = \frac{\pi}{2} . \tag{85}$$

Similarly, from Eq. (84), we deduce that

$$\beta - \alpha = \frac{\pi}{2} . \tag{86}$$

Using Eqs. (85) and (86) in Eq. (45) produces the interesting result that

$$\delta_1 - \delta_2 = -\pi . \tag{87}$$

This result means that if $\delta_1 - \delta_2 \neq -\pi$, then one of r_1 and r_2 is not equal to one and there is *CP* violation via the transition matrix even if $\phi_1 = \phi_2 = 0$.

There are other interesting subcases which we can consider.

(a) $\phi_1 = 0$ and $\phi_2 = 0$, which is equivalent to $\Delta\phi = 0$. From Eq. (45),

$$K = \left[\tan\left[-\frac{\gamma}{2} \right] \right]^4 = 6.037 \times 10^{-12} . \tag{88}$$

This is identical to case 3 where *CP* violation occurs purely through mass mixing.

(b) $\phi_1 = 0$ and $\Delta\phi = \pm\pi$. From Eq. (45),

$$K=\left[\frac{\tan(-\gamma/2)+\tan(\pm\pi/2)}{\cot(-\gamma/2)-\tan(\pm\pi/2)}\right]^2=1 \ . \quad (89)$$

(c) $\phi_1=0$ and $\Delta\phi=+\pi/2$. From Eq. (45),

$$K=\left[\frac{\tan(-\gamma/2)+1}{\cot(-\gamma/2)-1}\right]^2=2.472\times10^{-6} \ . \quad (90)$$

(d) $\phi_1=0$ and $\Delta\phi=-\pi/2$. From Eq. (45),

$$K=\left[\frac{\tan(-\gamma/2)-1}{\cot(-\gamma/2)+1}\right]^2=2.442\times10^{-6} \ . \quad (91)$$

We note that for all these cases, once r_1, r_2, ϕ_1, and ϕ_2 are known, then r and ϕ can be computed from either Eq. (31) or (32). For example, in case 3, Eq. (31) gives

$$r=\frac{1}{\sqrt{2}}\left[\frac{h_1}{g_1}\right]\cot\left[-\frac{\gamma}{2}\right] ,$$
$$\phi=-\delta_1 \ . \quad (92)$$

III. CONCLUSION

We have extracted from the measurable form factors G_1, H_1, G_2, and H_2 of the decays of K_S and K_L into two photons the parameters describing the two possible sources of CP violation: the CP impurity in the state functions of K_S and K_L, and the CP violations in the transition matrix. The first source is tagged by ϵ or by γ, the phase of $(1-\epsilon)/(1+\epsilon)$, and the second source is traced by r_1, r_2, ϕ_1, and ϕ_2 in the ratios of amplitudes

$$\frac{A[K^0\to2\gamma(-)]}{A[\bar{K}^0\to2\gamma(-)]}=-r_1e^{i\phi_1} ,$$
$$\frac{A[K^0\to2\gamma(+)]}{A[\bar{K}^0\to2\gamma(+)]}=+r_2e^{i\phi_2} \ . \quad (93)$$

We have described in Ref. 11 how the magnitudes and relative phases of the form factors G_1, H_1, G_2, and H_2 can be obtained by measuring the angular distributions in the decay modes $K_S\to\gamma\gamma\to e^-e^+\mu^-\mu^+$ and $K_L\to\gamma\gamma\to e^-e^-\mu^+\mu^+$.

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