Rare kaon decays $K^+ \to \pi^+ l^+ l^-$: A reappraisal

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We reexamine the various contributions to $K^+ \to \pi^+ l^+ l^-$ using recent experimental and theoretical advances as input. The decay is shown to be dominated by long-distance contributions which are now calculable. We present the expected spectra and predict $\Gamma(K^+ \to \pi^+ \mu^+ \mu^-)/\Gamma(K^+ \to \pi^+ e^+ e^-)=0.24$.

I. INTRODUCTION

The decay of charged K mesons to a pion and a lepton Dalitz pair[1] has been the subject of intense theoretical scrutiny[2–14] during the last two decades (see Ref. 4 for a comprehensive list of early papers). The initial attractiveness of this decay, emphasized in several of the pioneering papers of the new wave,[2–8] was due to the possibility of it being caused by the basic $s \to d\gamma$ transition. As such, it would constitute a precious direct test for a standard-model loop calculation of a flavor-changing radiative transition, with gluonic corrections. However, it soon became apparent[4,7,9] that contributions to the decay from long-distance photon emission are very important and might even dominate the transition amplitude.

We emphasize from the outset that at the present stage of theoretical development the calculation of a weak radiative process in the (s,d,u) sector requires the consideration of two distinct types of contributions: (a) photon emission from short distances ($x \sim 1/M_W$), which is treated by the use of the electroweak standard-model $\Delta S=1$ nonleptonic Hamiltonian, expressed in terms of local quark operators; and (b) photon emission from intermediate-state hadrons, for which the distance scale is the confinement radius (so-called "long-distance" emission). The effect of the QCD interactions at short distances is included in the coefficients of the quark operators in the leading-log approximation by use of renormalization-group equations. The strong interactions come into effect also through the hadronization process of intermediate states which is responsible for mechanism (b), as well as in the evaluation of the matrix elements of the quark operators between hadronic states. Now, since some of the contributions to the $K^+ \to \pi^+ l^+ l^-$ amplitude enter with opposite signs,[4] doubts have been raised[6] as to whether a calculation with reasonable precision for this decay can be at all performed.

In this paper we show that by taking into account recent developments of both experimental and theoretical nature, there is now the possibility of a much-improved estimation of the various contributions to $K^+ \to \pi^+ l^+ l^-$ decay. The developments of consequence are as follows. Firstly, there are the two independent measurements[15,16] of the form factor in $K_L \to e^+ e^- \gamma$ decays, which establish for the first time the size of the weak nonleptonic matrix element between vector states. The knowledge of this matrix element is essential[10] for the long-distance treatment of $K^+ \to \pi^+ l^+ l^-$ and now it has been determined[15,16] that it is suitably described[17] by the use of the QCD-corrected weak nonleptonic Hamiltonian of the standard model.[18] Secondly, there are now fairly accurate measurements[19] of the charged-pion and kaon radii, which are also needed for the calculation of the long-distance emission. On the theoretical side, there are several calculations available[12,13,20–22] of the short-distance $s \to d l^+ l^-$ transition including QCD corrections, also for a prospective very heavy t quark.

II. CALCULATION OF CONTRIBUTIONS TO THE DECAY AMPLITUDE

We proceed to calculate the $K^+ \to \pi^+ l^+ l^-$ amplitude along the lines developed in Refs. 4, 6, and 10. The general form of the decay amplitude is

$$M(K^+ \to \pi^+ l^+ l^-)=\frac{ie^2 G_F}{\sqrt{2}} s_1 c_1 c_3 \frac{a(\xi)s}{4\pi^2}[\sin\xi(1-s/m_\rho^2)^{-1}+\cos\xi(1-s/m_\rho^2)^{-1}(1-s/m_{K^*}^2)^{-1}] \times (p_K+p_\pi)^\mu \frac{\bar{u}(p_-)\gamma_\mu v(p_+)}{s}, \tag{1}$$

where $s=(p_-+p_+)^2=(p_K-p_\pi)^2$, $s_1=\sin\theta_1$, etc. [Cabibbo-Kobayashi-Maskawa (CKM) angles]. This representation[10,23] of the amplitude exhibits the general property that all contributions (short and long distance) can be grouped onto two classes, whose relative weight is conveniently parametrized by the angle ξ: (a) flavor-diagonal photon transitions, followed or preceded by a weak transition between pseudoscalar states; and (b) direct flavor-changing radiative transitions, involving a weak matrix element between vector states (e.g., $K^*\to\rho\to\gamma^*$). The two types of transitions entertain different form factors and their relative phase and magnitude affect the ratio $R=\Gamma(K^+\to\pi^+\mu^+\mu^-)/\Gamma(K^+\to\pi^+e^+e^-)$. As it was shown in Ref. 10, for specific values of $\tan\xi$ this ratio may differ considerably from the phase-space value which is $R=0.196$.

The existing experiment[24] on this decay reports a branching ratio of $B(K^+\to\pi^+e^+e^-)=(2.7\pm0.5)\times10^{-7}$. To connect (1) with this figure we note that if the bracketed form factor is taken to be 1, then $a(\xi)^{\text{expt}}=1.11$.

The totality of contributions to the $K^+\to\pi^+l^+l^-$ amplitude has been diagrammatically organized by Vainshtein *et al.*,[4] and we follow here this approach as well as some of their results. We consider the following four groups of contributions (exhibited in Figs. 1–5 of Ref. 4).

A_1, short-distance contributions, usually denoted as "electromagnetic penguin."

A_2, long-distance contributions due to (strong) transitions $K\to S\pi$ followed by $S\to\gamma^*$, where S is a strangeness = 1, intermediate hadronic state.

A_3, contributions involving the pseudoscalar "K-π" transition with pole, nonpole, and contact terms, dominated by the long-distance component.

A_4, contributions from the "strong penguin" diagrams.

The A_i are normalized to correspond to Eq. (1). Thus A_3 and A_4 are essentially type-(a) contributions adding to $a(\xi)\sin\xi$ while A_1, A_2 are type-(b) contributions adding to $a(\xi)\cos\xi$. Now to be detailed expressions for A_i.

The A_1 amplitude generated by the quark-level $s\to dl^+l^-$ transition derives from the electromagnetic penguin $s\to d\gamma^*\to dl^+l^-$, the Z^0 penguin $s\to dZ^{0*}\to dl^+l^-$, and the "W box" diagram.[12,13,20] The latter two are significant for a heavy top quark. Using the unitarity of the CKM matrix one can reexpress A_1 as a sum of contributions from intermediate c and t quarks in the loop:

$$A_1=\pi\left[\mathrm{Re}\left[\frac{V_{cs}^*V_{cd}}{V_{us}^*V_{ud}}\right]F_c+\mathrm{Re}\left[\frac{V_{ts}^*V_{td}}{V_{us}^*V_{ud}}\right]F_t\right]. \tag{2}$$

F_c is generated by the electromagnetic penguin while F_t contains contributions from Z^0 penguin and W box as well. F_t also contains an axial part coupled to $\bar e\gamma_\mu\gamma_5 e$, which was not exhibited in (1) for the following reason: although F_t is larger than F_c by about one order of magnitude for $M_t>100$ GeV/c^2 (Refs. 12, 13, and 20), its contribution to A_1 is negligible since $|V_{ts}^*V_{td}|/|V_{cs}^*V_{cd}|\simeq s_{23}^2\simeq2\times10^{-3}$. Hence, the second term in (2) is merely a few percent correction and may be safely neglected. Then A_1 is given by the c-quark term only:[12]

$$A_1=\pi\,\mathrm{Re}\left[\frac{V_{cs}^*V_{cd}}{V_{u^*s}V_{ud}}\right]\left[\frac{-16}{99\alpha_s(m_c^2)}(1-K_{\mu/c}^{-33/27})K_{c/b}^{-6/25}K_{b/W}^{-6/23}+\frac{8}{45\alpha_s(m_c^2)}(1-K_{\mu/c}^{-15/27}K_{c/b}^{15/27}K_{b/W}^{12/23}\right]=0.10\ , \tag{3}$$

where $K_{\alpha/\beta}$ are ratios of the strong coupling $\alpha_s(m_\alpha^2)/\alpha_s(m_\beta^2)$. Although there is a delicate cancellation in (3), the order of magnitude is consistent with alternative estimates and one concludes that the short-distance contribution to $K^+\to\pi^+l^+l^-$ is negligible.

For the A_2 contribution we assume that the intermediate hadronic state in $K\to\pi S\to\pi l^+l^-$ is dominated by the K^* meson.[17] The recent experiments[15,16] on the $K_L\to\gamma e^+e^-$ form factor have confirmed that the $\langle K^*|H_W|V\rangle$ matrix element ($V=\rho,\dots$) is of the size obtained[17] from the weak Hamiltonian

$$H_W=H_{\mathrm{NL}}^{\mathrm{QCD}}=\sqrt{2}G_F\sum_{i=1}^{6}c_iO_i$$

as given in Ref. 18, whose definitions of operators we follow. The gauge-invariant $\langle K^*|\gamma\rangle$ transition which is treated in the vector-dominance formalism[25] contains a contact $K^*-\gamma$ term as well as a transition via vector mesons which combine so that is vanishes on mass shell[26,17] as required by gauge invariance.

It is given by

$$\langle K^*|\gamma\rangle=\sqrt{2}G_Fes_1c_1c_3\frac{m_{K^*}^2m_\rho^2}{2f_{K^*}f_\rho^2(s-m_{K^*}^2)}\left[T_\rho\left[1+\frac{m_\rho^2}{s-m_\rho^2}\right]+\frac{m_\omega^2T_\omega}{9m_\rho^2}\left[1+\frac{m_\omega^2}{s-m_\omega^2}\right]\right.$$
$$\left.+\frac{2m_\phi^2T_\phi}{9m_\phi^2}\left[1+\frac{m_\phi^2}{s-m_\phi^2}\right]\right]\epsilon_\mu^{(K^*)}\epsilon_\mu^{(\gamma)}\ , \tag{4a}$$

$$\sqrt{2}G_{F_1}s_1c_1c_3m_{K^*}^2m_\rho^2T_\rho/2f_{K^*}f_\rho=\langle K^{*0}|H_{\mathrm{NL}}^{\mathrm{QCD}}|\rho^0\rangle,\ \text{etc.}\ , \tag{4b}$$

and we assumed nonet symmetry in relating the vector-meson couplings f_ρ, f_ω, f_ϕ. This leads to an A_2 amplitude

$$A_2=\sqrt{2}(4\pi^2)g_{K^{*0}K^+\pi^-}\frac{1}{2f_{K^*}f_\rho^2}\left\{\left[\left[\frac{4}{3\sqrt{2}}\left[-\frac{c_1}{2}+c_2+c_3\right]-\frac{4\sqrt{2}}{3}c_4\right]+\frac{m_\omega^2}{9m_\rho^2}\left[\sqrt{2}\left[\frac{c_1}{3}+2c_2+2c_3\right]\right]\right]+\frac{2m_\phi^2}{9m_\rho^2}\left[\frac{8c_2}{3}-4c_3\right]\right\} \tag{5a}$$

which numerically is

$$A_2=A_2(\rho)+A_2(\omega)+A_2(\phi)=0.77-0.10-0.02=0.65\ . \tag{5b}$$

In estimating (5) we used the experimental values for $g_{K^*K\pi}$, f_{K^*}, f_ρ and the calculated[18] ones for c_i. We remark that the O_5, O_6 strong-penguin operators do not contribute[17] to vector-vector transitions, unlike in the transitions between pseudoscalar states.

The type-(a) weak transitions connecting pseudoscalar states to which a flavour-diagonal photon transition is attached are separated into A_3+A_4 as follows: the part induced by operators O_1-O_4 is denoted by A_3 and the strong penguin contribution induced by operators O_5, O_6 is denoted by A_4. The latter vanishes in the absence of QCD corrections.

For A_3 we use the result of Ref. 4 to which we refer for details of derivation

$$A_3=\frac{4\pi^2 f_K f_\pi \eta_3}{m_K^2-m_\pi^2}\left[\frac{m_K^2 r_K^2}{6}-\frac{m_\pi^2 r_\pi^2}{6}\right]=1.46\pm0.25\ . \tag{6}$$

The figures used are $f_\pi=132$ MeV, $f_K=158.4$ MeV, and[19] $r_K^2=0.34\pm0.05$ fm^2, $r_\pi^2=0.439\pm0.008$ fm.^{2} $\eta_3=1.25$ is the correction factor[4] for gluon exchanges and the uncertainty specified is due solely to that originated from radii.

The contribution of the strong penguin diagrams to $a(\xi)\cos\xi$ is negligibly small[4,17] as we have already mentioned and their contribution to $a(\xi)\sin(\xi)$ is given by[4]

$$A_4=\frac{32\pi^2\beta f_\pi f_K m_\pi^2 m_K^2}{9(m_u+m_s)m_u(m_K^2-m_\pi^2)}\frac{r_\pi^2-r_K^2}{6}$$
$$=-0.74\pm0.37\ . \tag{7}$$

We have taken here the uncertainty[19] in $r_\pi^2-r_K^2=0.10\pm0.045$ fm^2 to cover also the theoretical uncertainty in the derivation of this controversial matrix element. Note that we used for the gluonic correction $\beta=-0.1$ as derived in Ref. 4, which is insufficient to explain the $\Delta I=\frac{1}{2}$ rule in K decays.

Using now $(A_1+A_2)=a\cos\xi$, $(A_3+A_4)=a\sin\xi$ in (1), we find for $K^+\to\pi^+e^+e^-$ a rate of $(5.6\pm2.3)\times10^{-7}$. The ξ parameter defined in Eq. (1) is given by $\tan\xi=(A_3+A_4)/(A_1+A_2)=0.96$, or $\xi=0.76$. The ratio $R=\Gamma(K^+\to\pi^+\mu^+\mu^-)\Gamma(K^+\to\pi^+e^+e^-)$ is expected to carry less uncertainty than the individual amplitudes and we predict for it $R=0.24$, some 20% higher than the phase-space value.

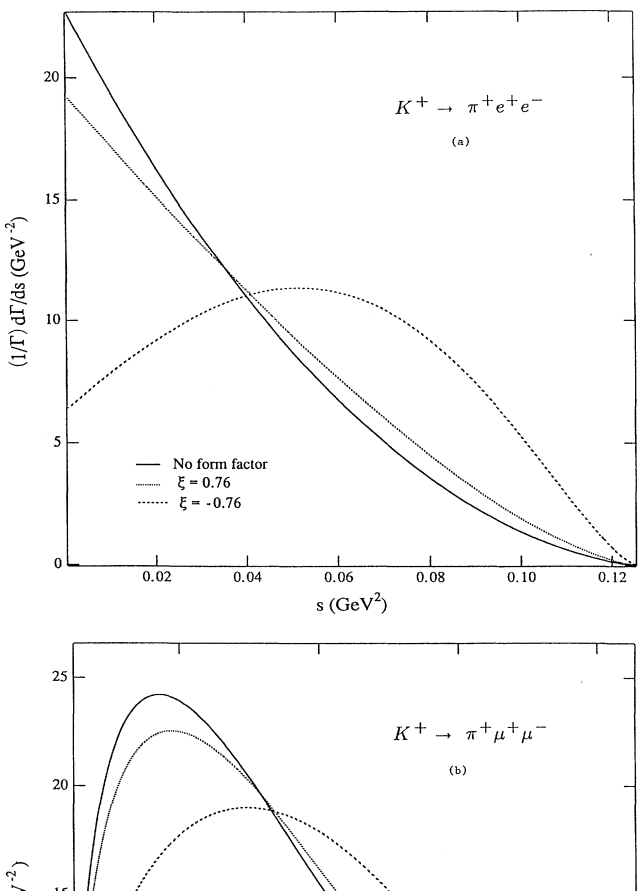

FIG. 1. The decay distributions for (a) $K^+\to\pi^+e^+e^-$ and (b) $K^+\to\pi^+\mu^+\mu^-$ as a function of the invariant mass of the lepton pair. The solid line is for constant form factor. The dotted line with $\xi=0.76$ is the prediction of the present calculation. The broken line is the spectrum of an hypothetical model with $\xi=-0.76$.

III. DISCUSSION

To summarize, we repeat that recent experimental and theoretical developments afford now a more reliable calculation of $K \to \pi^+ l^+ l^-$ than previously[4–10] possible. Clearly, the short-distance $s \to d l^+ l^-$ contribution given by A_1 is of minor import. The A_2, A_3 contributions are now better evaluated, owing to the recent experimental advances.[15,16,19] On the other hand, the estimation of A_4 is still fraught by a large uncertainty. Altogether, we believe that the amplitude is now calculable within the specified uncertainty. Needless to say, since the uncertainty involves theoretical input, the figure we offer for the uncertainty in the calculated branching ratio is based on both experimental input and best judgment. It is gratifying that without any artificial adjustment, one obtains a decay rate fairly close to that given by the only available experiment[24] so far.

It is interesting to note that the calculated amplitude has a non-negligible $I=\frac{3}{2}$ component, generated via the A_2 contribution in addition to that coming from A_3. The calculated rate is apparently on the higher side. However, it is remarkable to record the consistency of the emerging picture, whereby long-distance contributions verified in $K_L \to e^+ e^- \gamma$, are leading to a consistent rate for $K_L \to \mu^+ \mu^-$ (Ref. 27) and for the decay discussed here.

The lepton spectrum can be seriously affected by the relative phase and magnitude of the contributions of groups (a) and (b). For the value we obtained in the present calculation, $\xi = +0.76$, these spectra deviate only slightly from the phase-space distribution. Nevertheless, it is worth emphasizing that in a hypothetical model, with e.g., $\xi = -0.76$, the spectrum would be very different, as depicted in Figs. 1(a) and 1(b). This plot is given in order to emphasize the sensitivity of the spectrum to the relative size of the two classes of contributions, $a(\xi)\sin\xi$ and $a(\xi)\cos\xi$, especially for negative values of ξ. Of course, the measurement of these spectra is most desirable and, we hope, will be forthcoming. In concluding, we emphasize that all previous analyses (except Refs. 10 and 11) have worked with constant form factors, which is not justified as proved also in the recent[15,16] $K_L^0 \to \gamma e^+ e^-$ experiments. In the present analysis we have used the form-factor dependence specified in Eq. (1) which is based[10] on general theoretical arguments. In Ref. 11 an effective form factor arises as a result of the specific form of the chiral Lagrangian employed.

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