Universal multifractality in multiparticle production

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The G moments for the multifractal analysis of multiparticle production are investigated in a model-independent way. By successive bin splitting and assuming the existence of a multiplicity splitting function that depends on multiplicity, but applicable at all steps of the splittings, we study the ergodicity of horizontal and vertical averaging, and derive a universality relation for the G moments. It relates the G moments for different initial multiplicities to a common scaling function $\Gamma_q(\xi)$. The experimental verification of this scaling property would, on the one hand, signify self-similarity in the data, and, on the other, provide a convenient function for comparison not only among different experiments, but also between theory and experiment.

I. INTRODUCTION

In recent years the subject of intermittency[1] has gained significant attention in the theoretical and experimental investigation of multiparticle production.[2] Evidence has been found for power-law behavior of normalized factorial moments F_q, when the bin width of rapidity interval δ is decreased. The phenomenon therefore suggests self-similarity in multiplicity fluctuations in a range of resolution scale. It also suggests the existence of fractal properties in the problem.[3–6] Unlike the usual fractal types of behavior in geometrical and statistical systems,[7] the multiparticle-production processes have dynamical and kinematic features that pose special problems. The most obvious one among them is the finiteness of particle multiplicity in an event at finite energy. It means that self-similarity, if existent, cannot persist indefinitely to finer and finer scales of resolution. At a higher multiplicity, however, one expects the lower limit of that scale to be extended lower. The establishment of a mathematical description of that interrelationship is the purpose of this paper.

The formulation of our problem is quite general. We do not assume any specific dynamical model of particle production; any such model can be framed in our formalism. Thus we do not consider partons or strings. We dwell at the level of the observables and study their relationship upon successive partitioning of the phase space. A crucial quantity in the formalism is the multiplicity splitting function, which determines the universal properties of the multifractal structure. It will serve as the key link between dynamical models, on the one hand, and the universal multifractality of the observables, on the other. There is a basic assumption in the formulation, which accounts for the simplicity of our result. That assumption has to do with a form of factorization, as we shall describe. But in a Monte Carlo calculation such an assumption need not be made. Thus our formulation is basically quite general.

In order to emphasize the mathematical aspect of this problem, we shall not be concerned with detailed phenomenology in this paper. The comparison with data will be discussed in a future paper, especially after the data have been analyzed in a way suggested here. However, it does not mean that our concern now is largely abstract. On the contrary, this problem arose out of our awareness of certain trends in the data of e^+e^- annihilation [from the High Resolution Spectrometer (HRS) Collaboration[8]] and $\bar{p}p$ collision [from the UA1 Collaboration[9] and the Collider Detector at Fermilab (CDF) Collaboration[10]] on which multifractal analyses have been done. The aim of this paper is to understand the general features of those data.

One of the chief results of this investigation is an elucidation of the connection between the horizontal and vertical averaging, which amounts to ergodicity in the limit of infinite multiplicity. We discuss in what precise sense our problem at finite multiplicities has essential ergodicity. The universality property that we shall derive provides the rationale to do multifractal analysis with low numbers of bin splitting at low multiplicities: it is the same as with high numbers of bin splitting at high multiplicities.

In Sec. II we review the multifractal analysis and in Sec. II we consider the details of bin splitting. Section IV is where we derive our main result on universality, the implications of which are discussed in Sec. V. Concluding remarks are made in the last section.

II. MULTIFRACTAL ANALYSIS

To facilitate our discussion in the following, let us first summarize the procedure for multifractal analysis in high-energy collisions.[6,11] For simplicity, we discuss a one-dimensional analysis, but the procedure can clearly be generalized to the multidimensional phase space. Let the initial rapidity window be of width Y, and suppose that there are N particles detected in Y. Let Y be divided into M bins, each having a bin width $\delta = Y/M$. Using j to label the bins, let n_j be the multiplicity in the jth bin. Define $p_j \equiv n_j/N$. Then the G moments are

$$G_q(M,N) = \sum_{j=1}^{M}{}' p_j^q \,, \tag{2.1}$$

where the prime signifies that the sum is over nonempty bins only, and q is a real number, positive or negative. For fixed N and Y, a strict power-law behavior would demand

$$G_q(M) \propto \delta^{\tau(q)} . \tag{2.2}$$

However, such behavior is not realistic, since when M becomes large enough, p_j in (2.1) is either 0 or $1/N$, so $G_q = N^{1-q}$, independent of δ. In that region the strict form (2.2) cannot be valid. It is in this respect that our problem in multiparticle production is basically different from those in the geometrical and statistical problems.[7]

Since the $\delta \to 0$ limit is trivial, we clearly are interested in the region where M is not large, unless N is very large. If M is finite, one is not probing fractal behavior in the usual sense. It is our aim in this paper to illuminate the sense in which fractal behavior can arise in our problem. To set up the stage for that discussion, let us borrow the mathematics and terminology from the canonical multifractal analysis,[7,12,13] but not its concept; specifically, we leave out the limit $\delta \to 0$. Thus, following the procedure of Refs. 6 and 11, we select an M interval in which to determine the slope $\tau(q,M)$, i.e.,

$$\tau(q,M) = \frac{\Delta(\ln G_q)}{-\Delta(\ln M)} . \tag{2.3}$$

In the region where $\tau(q,M)$ does not depend on M we recover (2.2), but in our approach the local slope (2.3) plays the main role. From (2.3) α_q can be calculated as

$$\alpha_q = \frac{\partial}{\partial q}\tau(q,M) . \tag{2.4}$$

The spectral function $f(\alpha)$ is then obtained by the Legendre transform for each fixed M:

$$f(\alpha_q) = q\alpha_q - \tau(q,M) . \tag{2.5}$$

Note that the $\delta \to 0$ limit has not been taken above; nevertheless, we shall still refer to $f(\alpha)$ as the multifractal spectrum. Since there is an implicit dependence on M, $f(\alpha)$ is therefore not universal, but depends on where in M the analysis is done.

In Ref. 11 this procedure is followed for the determination of the multifractal structure of the ϕ^3 and gluon models. The M interval in which the analysis is performed is between $M=2$ and 8. It is the horizontal analysis for each event. Vertical averaging is done on the quantity $\ln G_q$, rather than G_q itself, so that from $\langle \ln G_q \rangle$ is determined $\langle \tau(q,M) \rangle$, which in turn yields $f(\alpha)$ in accordance to (2.4) and (2.5), the angular brackets being suppressed for brevity. In Refs. 8–10, analyses of the experimental data on e^+e^- annihilation and hadronic collisions, following the same procedure as in Ref. 11 but for $M=1$ to 2, have been done. The results yield smooth functions for $f(\alpha)$ that resemble one another as well as being similar to the shapes obtained by model calculation. However, all these results are for different energy $\sqrt{s}$, rapidity window Y, and M values. Thus the results cannot be directly compared. That is clearly a defect, the elimination of which at a fundamental level was the motivation that initiated the present investigation.

III. BIN SPLITTING

A consequence of our procedure of determining $f(\alpha)$ from the multiparticle data is that there exists an $f(\alpha)$ for one event and one partition of a rapidity interval, no matter how wide. This can easily be demonstrated analytically as follows.

Let there be N particles in a bin of unspecified width. Let the bin be split into two equal halves, and let N_1 particles be found in the left half, $N-N_1$ in the right. Define $x = N_1/N$. Then the G moment for this bin-splitting process is

$$G_q(x) = x^q + (1-x)^q . \tag{3.1}$$

Using (2.3) we obtain, after one splitting,

$$\tau(q) = -\frac{1}{\ln 2}\ln[x^q + (1-x)^q] . \tag{3.2}$$

It then follows from (2.4) that

$$\alpha_0 = -\frac{1}{2\ln 2}[\ln x + \ln(1-x)] , \tag{3.3}$$

$$\alpha_1 = -\frac{1}{\ln 2}[x\ln x + (1-x)\ln(1-x)] . \tag{3.4}$$

If $x \le \frac{1}{2}$, then

$$\alpha_\infty = -\frac{\ln(1-x)}{\ln 2}, \quad \alpha_{-\infty} = -\frac{\ln x}{\ln 2} . \tag{3.5}$$

Note that $\alpha_0 = (\alpha_\infty + \alpha_{-\infty})/2$ and

$$\Delta\alpha \equiv \alpha_{-\infty} - \alpha_\infty = \frac{1}{\ln 2}\ln\frac{1-x}{x} . \tag{3.6}$$

Thus, the width $\Delta\alpha$ is a measure of how much x deviates from $\frac{1}{2}$. At $q=0$, $f(\alpha_0) = -\tau(0) = 1$, so the fractal dimensional D_0, defined to be $f(\alpha_0)$, is the topological dimension 1, for all x satisfying $0 < x \le \frac{1}{2}$. Indeed, the spectrum $f(\alpha)$ can be determined from (2.5) for any fixed x. In Fig. 1 we illustrate the results for two values of x: $\frac{1}{4}$ and $\frac{1}{3}$. For $x = \frac{1}{2}$, we have $f(\alpha) = \alpha = 1$.

The $f(\alpha)$ curves in Fig. 1 appear to be complicated ways of representing two numbers, and there is no physics in them. There is also nothing fractal about them. However, if we perform this analysis for many events, the distribution in x has physical content, so the vertically averaged $f(\alpha)$ contains information about how the particles are distributed in the original y interval from event to

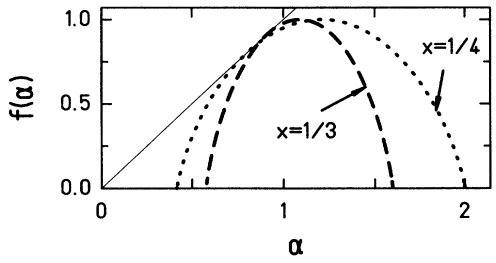

FIG. 1. $f(\alpha)$ for one event after one bin splitting with multiplicity fraction $x = \frac{1}{4}$ and $\frac{1}{3}$.

event, even if the average single-particle distribution in that interval is uniform. In particular, the frequency at which N_1 is zero will be revealed in the value of $f(\alpha_0)$, since, when $x=0$, the corresponding $f(\alpha)$ vanishes. Thus the fractal dimension gives a measure of the probability that N_1 or (N_2) is zero upon bin splitting.

More specifically, let $P(N_1,N)$ be the probability that one of the two halves has N_1 particles, when the initial multiplicity before bin splitting is N. It is normalized by

$$\sum_{N_1=0}^{N} P(N_1,N)=1 . \tag{3.7}$$

Let us use $x=N_1/N$ as a continuous variable, and define the multiplicity splitting function

$$\psi(x,N)=NP(N_1,N) \tag{3.8}$$

so that (3.7) takes the form

$$\int_0^1 dx\ \psi(x,N)=1 . \tag{3.9}$$

We shall use $\psi(x,N)$ in general, except in circumstances when the continuous approximation is invalid, at which point we revert to the discrete expression. Note that we retain the N dependence in $\psi(x,N)$. It is physically reasonable, and is the kernel of the universality relation that we shall derive, which need not obey strict self-similarity. Although we are not describing any model, we can note the distinction of our description at this level from the α model,[1] which conserves multiplicity only on the average at each bin splitting, and whose splitting function does not depend on the multiplicity.

The vertical average of $G_q(x)$ is

$$\overline{G}_q(1,N)=\int_0^1 dx\ \psi(x,N)[x^q+(1-x)^q] , \tag{3.10}$$

where the label 1 in $\overline{G}_q$ denotes one partition. This expression is well defined, if $\psi(x,N)$ vanishes sufficiently fast at the limits of integration to render the integral meaningful at all q. It is usually so, if N is large. However, if N is small, $\psi(x,N)$ may vanish at the limits too slowly, or may even be nonvanishing there. In that case (3.10) is divergent for $q<0$, if taken at face value. In such situations we have to use the well-defined discrete expression.

As an illustrative example we consider the purely statistical case of binomial distribution

$$P(N_1,N)=\begin{bmatrix} N \\ N_1 \end{bmatrix} 2^{-N} . \tag{3.11}$$

The resulting $f(\alpha)$ functions after vertical averaging at various fixed values of N are shown in Fig. 2. Note that the fractal dimension D_0, i.e., $f(\alpha_0)$, is visually less than 1 when N is small; in fact, it is

$$D_0=1+\frac{1}{\ln 2}\ln\left[1-\frac{P(0,N)+P(N,N)}{2}\right] . \tag{3.12}$$

As N increases, $f(\alpha)$ becomes wider with increasing peak until $N=8$; thereafter, $f(\alpha)$ becomes narrower with the peak saturating at 1. Clearly, when N becomes very large, $\psi(x,N)\to\delta(x-\frac{1}{2})$, and $f(\alpha)\to\delta_{\alpha,1}$. Thus it is only when N is not very large that $f(\alpha)$ is wide and contains fractal information.

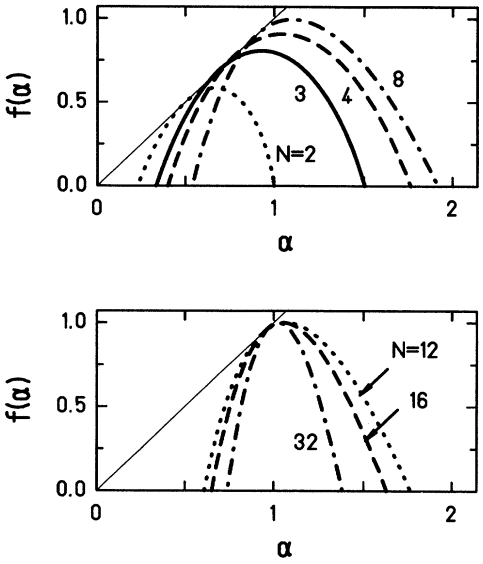

FIG. 2. $f(\alpha)$ after averaging over statistical distribution (3.11) for N ranging from 2 to 32.

IV. SELF-SIMILARITY, ERGODICITY, AND UNIVERSALITY

The consideration of the preceding section is a prelude to the investigation of the general problem of successive partitioning of the phase space. For, no matter how many cells the phase space has been divided into, a further step of partition of each one of those cells would involve the same type of analysis as considered above. In the case of one dimension the same multiplicity splitting function $\psi(x,n)$ will be relevant, if there is self-similarity at all scales of the bin sizes. The multiplicity n may fluctuate from bin to bin, and the fraction x will also fluctuate. If M is large enough, then there are enough bins to sample all values of x for each n. Thus in the horizontal analysis of one event, $\psi(x,n)$ is sampled in all x and n, subject to $\bar{n}=N/M$ (provided that M is sufficiently large), in presumably just the same way that $\psi(x,N)$ is sampled in the vertical analysis of one bin splitting per event, in which the event-averaged mean multiplicity $\langle N\rangle$ is set equal to $\bar{n}$ (provided that the total number of events is sufficiently large). If this relationship between the horizontal and vertical analyses is indeed true, then we have an ergodicity, similar to the relationship between time averaging and ensemble averaging in the usual statistical systems. In this section we shall see in what quantitative sense ergodicity follows from self-similarity in our problem of bin splitting, and then derive a remarkable universality relation.

We begin by first reorganizing the M bins according to their multiplicities; that is, denoting the number of bins having multiplicity n by $Q(n)$ we have

$$\sum_{j=1}^{M} = \sum_{n=0}^{N} \sum_{r=1}^{Q(n)} . \tag{4.1}$$

Clearly, $Q(n)$ satisfies the constraints for each event

$$\sum_{n=0}^{N} Q(n) = M, \quad \sum_{n=0}^{N} nQ(n) = N . \tag{4.2}$$

Now in further splitting each of the M bins into two halves, let the number of subbins containing n' particles arising from splitting the $Q(n)$ bins be denoted by $Q(n',n)$. Then, we have

$$\sum_{r=1}^{Q(n)} = \sum_{n'=0}^{n} \sum_{s=1}^{Q(n',n)} \tag{4.3}$$

and

$$Q(n) = \sum_{n'=0}^{n} Q(n',n) . \tag{4.4}$$

The multiplicity splitting function relevant at this level is therefore

$$\psi(x',n) = nQ(n',n)/Q(n) , \tag{4.5}$$

where

$$x' = n'/n . \tag{4.6}$$

Thus (4.4) implies the normalization condition

$$\int_0^1 dx' \psi(x',n) = 1 \tag{4.7}$$

when the continuous approximation is meaningful.

Since the M bins are obtained by bin doubling, we define μ to be the number of steps of the process:

$$M = 2^{\mu} . \tag{4.8}$$

Similarly, we define

$$N = 2^{\nu} . \tag{4.9}$$

Thus the average number of particles per bin after μ steps of bin splitting is

$$\bar{n} = N/M = 2^{\nu-\mu} . \tag{4.10}$$

Let us now denote the G moments after μ steps of bin splitting by $G_q(\mu,\nu)$, when the initial multiplicity is N. From (2.1) and (4.1), we have

$$G_q(\mu,\nu) = \sum_{j=1}^{M} {}'p_j^q = \sum_{n=1}^{N} \left[\frac{n}{N}\right]^q Q(n) . \tag{4.11}$$

At the next step of bin splitting, the multiplicity in bin j splits up into two fractions, x_j and $1-x_j$; consequently, we obtain

$$G_q(\mu+1,\nu) = \sum_{j=1}^{M} {}'p_j^q [x_j^q + (1-x_j)^q] . \tag{4.12}$$

Since $p_j = n/N$ implies $p_j x_j = n'/N$, with $n' = x_j n$, we may rewrite (4.12) as

$$G_q(\mu+1,\nu) = \sum_{n=1}^{N} \left[\frac{n}{N}\right]^q \sum_{n'=1}^{n-1} Q(n',n) \left[\left[\frac{n'}{n}\right]^q + \left[1-\frac{n'}{n}\right]^q \right] \tag{4.13}$$

which, by virtue of (4.5), becomes

$$G_q(\mu+1,\nu) = \sum_{n=1}^{N} \left[\frac{n}{N}\right]^q Q(n) \int_0^1 dx' \psi(x',n) [x'^q + (1-x')^q] \tag{4.14}$$

when M is large, i.e., when there are enough bins to sample all values of x', thereby rendering the integral expression sensible. The necessary care attending the integration in (4.14) near the limits of integration has already been discussed following (3.10).

So far we have not considered any model, nor made any approximation. Equation (4.14) is the exact expression for making one more bin splitting at the μth level. $\psi(x',n)$ is the multiplicity splitting function when j is swept over all M bins. When M is large, all x' and n are sampled, and this process of horizontal averaging is equivalent to the vertical averaging carried out in (3.10), except for the difference between N and n. If the ψ function did not depend on the multiplicity, we would have had exact self-similarity at all levels of bin splitting, and the result would be simple. With n dependence, we still have self-similarity, if the same $\psi(x,n)$ is used at all levels, but with n allowed to vary. This gives rise to a generalized form of self-similarity that is characteristic of multiparticle production. For a given n in (4.14) the integral over x' is the same as that in (3.10), if $N=n$. It is in this sense that we have ergodicity, i.e., the same integral over $\psi(x,n)$ is used in both horizontal and vertical averaging. Exact ergodicity is possible only in the limit of $M \to \infty$. We shall, however, use this concept even when M is finite, and not necessarily very large.

Using $\bar{G}_q(1,n)$ to denote either horizontally or vertically averaged G moments, we have

$$G_q(\mu+1,\nu) = \sum_{n=1}^{N} \left[\frac{n}{N}\right]^q Q(n) \bar{G}_q(1,n) . \tag{4.15}$$

Since $Q(n)$ is peaked in $n = \bar{n} = N/M$, we make the simplifying assumption that (4.15) can be well approximated by an expression with $\bar{G}_q(1,n)$ replaced by $\bar{G}_q(1,\bar{n})$, which can then be taken outside the summation. On account of (4.11) we thus get

$$G_q(\mu+1,\nu) = G_q(\mu,\nu) \bar{G}_q(1,\bar{n}) \tag{4.16}$$

which is a recursion relation that epitomizes the self-similarity in our problem. The trivial case where $\bar{G}_q(1,\bar{n})$ is independent of $\bar{n}$ corresponds to intermittency with a strict power-law behavior at all scales. Our interest is to investigate the consequences when the $\bar{n}$ dependence cannot be ignored, which is the unavoidable reality in multiparticle production at finite energy.

The approximation made in obtaining (4.16) from (4.15) tacitly assumes that n fluctuates around a mean value $\bar{n}$, which is roughly uniform throughout the entire Y interval. It implies that (4.16) may be true in the central plateau region in rapidity, but is less likely to be true in the fragmentation region where dN/dy changes drastically in y.

In a model calculation one can compute $\psi(x,n)$ and therefore $\bar{G}_q(1,n)$. Thus one can work directly with (4.15), or even as far back as (4.12) in a Monte Carlo simulation, without the necessity of an approximation that leads to (4.16). However, for an analytical derivation of the universality relation, (4.16) is the simplest starting point.

So far we have considered only horizontal analysis of one event. To be more precise, we should label $G_q, Q(n)$, and $\psi(x',n)$ in (4.14)–(4.16) with an index i that denotes the ith event. $Q_i(n)$ and $\psi_i(x',n)$ may differ from event to event. When N is large, we expect $Q_i(n)$ to be roughly the same for all events, but $\psi_i(x',n)$ may not, since n can be small. However, if we average (4.14) over a number of events, the sum over i is then primarily focused on $\psi_i(x',n)$, resulting in an effective $\psi(x',n)$, which is the vertical average. Indeed, vertical averaging is necessary in order to enhance statistics. Our proposal here is that events with different multiplicities in the initial Y interval should be separated in accordance to their ν values, and averages of $G_q(\mu,\nu)$ should be determined for each ν separately. Such a vertical average should be applied to (4.16), giving

$$\langle G_q(\mu+1,\nu)\rangle = \langle G_q(\mu,\nu)\rangle \bar{G}_q(1,\bar{n}) , \tag{4.16'}$$

where $\bar{G}_q(1,\bar{n})$ is as defined in (3.10) with $\psi(x,\bar{n})$ being the multiplicity splitting function appropriate for vertical analysis at $\bar{n}$. Because of the similarity between (4.16) and (4.16') we shall, for brevity, omit the angular brackets in (4.16') in the following, when the use of vertical averages of the G moments is understood.

Since, accordings to (4.10), $\bar{n}$ depends only on $\mu-\nu$, let us define, for notational convenience,

$$A_q(\mu-\nu) \equiv \bar{G}_q(1,\bar{n}) . \tag{4.17}$$

Then it follows from (4.16) that

$$\ln G_q(\mu+1,\nu) - \ln G_q(\mu,\nu) = \ln A_q(\mu-\nu) . \tag{4.18}$$

When μ is large, we may write the left-hand side (LHS) as $\partial \ln G_q(\mu,\nu)/\partial\mu$, so that upon integration we obtain

$$\ln G_q(\mu,\nu) = \int_0^{\mu} \ln A_q(\mu'-\nu)\, d\mu' . \tag{4.19}$$

Since the integrand depends only on the difference $\mu'-\nu$, we have a translationally invariant form for the difference

$$\ln G_q(\mu,\nu) - \ln G_q(\nu,\nu) = \Gamma_q(\mu-\nu) , \tag{4.20}$$

where

$$\Gamma_q(\mu-\nu) = \int_0^{\mu-\nu} \ln A_q(\mu')\, d\mu' . \tag{4.21}$$

Using (4.20) again to reexpress $\ln G_q(\nu,\nu)$, while remembering that $G_q(0,\nu)=1$, we get finally

$$\ln G_q(\mu,\nu) = \Gamma_q(\mu-\nu) - \Gamma_q(-\nu) . \tag{4.22}$$

This is the universality relation that we have aimed to derive. It expresses the scaling behavior of a function of two variables in terms of a scaling function of one variable only. Thus one function $\Gamma_q(\xi)$ determines the G moments as functions of μ for all values of ν.

If no new physical process is involved by varying s and Y, one expects from (4.20) a universal scaling function $\Gamma_q(\xi)$ that is independent of s and Y, and of ν for fixed $\mu-\nu$. This has phenomenological consequences to be discussed in the next section. We emphasize that the $\Gamma_q(\xi)$ function is obtained from $\ln\langle G_q\rangle$ not $\langle \ln G_q\rangle$. This differs from an earlier treatment of the G moments,[11] where vertical averaging was done on $\ln G_q$ for all N, whereas here the vertical averaging is done on G_q for a specific N at a time. Once the scaling function $\Gamma_q(\xi)$ is determined, there is no ambiguity in the determination of the spectrum function $f(\alpha)$, as far as the s and Y dependences are concerned.

It is possible that the approximation made in going from (4.15) to (4.16) is not valid if the range of initial y interval covers a widely varying $d\sigma/dy$ region, such as in the fragmentation region. In that case the scaling behavior is not expected. We therefore suggest experimental investigation in the central region initially, where $d\sigma/dy$ for soft processes is rather flat.

V. IMPLICATIONS OF THE UNIVERSALITY RELATION

Although (4.22) is derived on the assumption that μ and ν are very large, it would be especially interesting if the scaling property is valid even when they are small. The precociousness of its validity is not what one can derive from theoretical consideration, but must be inferred from the experimental data. We therefore recommend concerted investigations of this relationship in different experiments for various collisional processes at all energies and with different kinematical cuts.

If (4.22) is known to be valid, then it is easy to determine $\ln G_q(\mu,\nu)$ graphically from a plot of $\Gamma_q(\xi)$ by placing the origin of the graph of $\ln G_q(\mu,\nu)$ vs μ on the $\Gamma_q(\xi)$ curve at the point whose ξ value is $-\nu$. The $\Gamma_q(\xi)$ curve then traces out the $\ln G_q(\mu,\nu)$ curve for $\mu \geq 0$. Eventally, this gives rise to a family of $\ln G_q(\mu,\nu)$ for each fixed q but for various ν values. An illustration of this procedure is shown in Fig. 3 for a negative q.

Conversely, one can extract the $\Gamma_q(\xi)$ function from a family of $\ln G_q(\mu,\nu)$ functions, if universality is indeed correct. This is of particular relevance to experimental checks of the universality relation. Vertical averaging of $G_q(\mu,\nu)$ should therefore be done for a matrix of values of μ and ν, which need not be integers, but must include the diagonal ($\mu=\nu$) terms. One should then determine

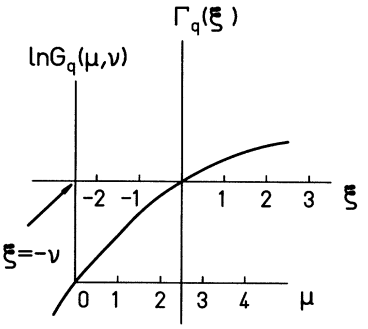

FIG. 3. Illustration of how $\ln G_q(\mu,\nu)$ is to be determined from $\Gamma_q(\xi)$: place the origin of the graph of $\ln G_q(\mu,\nu)$ vs μ on the $\Gamma_q(\xi)$ curve at the point where $\xi=-\nu$. Here, for example, $\nu=2.5$.

the LHS of (4.20) for each value of ν and plot it as a function of $\mu-\nu$. If the results of such various plots for different ν values agree, then universality will have been verified, and the resultant curve is, according to (4.20), just $\Gamma_q(\mu-\nu)$. To avoid the confusion arising from too many families of points, it may be advisable to begin the analysis with just integer values of μ and ν, ranging from 1 to 4 or 5 at present energies. These are not large values of μ and ν, yet $\nu=5$ may already be quite demanding on the statistics.

The validity of the universality would result in significant simplification in the interpretation of the data. Many features and advantages would emerge.

(1) Multifractal analysis based on $G_q(\mu,\nu)$ will yield the same $f(\alpha)$ for different ν values if the analysis is done for different μ intervals, corresponding to the same $\mu-\nu$ values. In other words, one need only work with the scaling function $\Gamma_q(\xi)$. For each region of ξ values, there is a corresponding $f(\alpha)$. Thus the vertical analysis done in Sec. III at $M=1$ and 2 yields the same result as one would obtain at high M at a correspondingly higher value of N. That is, at very high energy one can subdivide a Y interval into many bins, and the fractal structure in the horizontal analysis is the same as what one can infer at lower energy in a few bins by vertical analysis —a consequence of the self-similarity that results in the universality.

(2) Since ν is the only essential experimental characterization of the events under analysis, μ being a variable of the analysis, not of the events, many different features of high-energy collisions can now be unified. For each ν, one can consider different Y intervals and different s values (or dN/dy). If they all result in the same $\Gamma_q(\mu-\nu)$, then the underlying dynamics are the same. If, for example, the $\Gamma_q(\xi)$ curves in the fragmentation region (if it exists) differ from that in the central region, then the hadronization processes are basically different. Similarly, the results from different collisional processes can now be compared, independent of the experimental cuts. In e^+e^- annihilation processes, universality may not be valid, if the Y interval is wider than the narrow region of approximate constancy in dN/dy. If, for a small Y, a universal $\Gamma_q(\xi)$ can be extracted, it would be of great interest to compare it with that of hadronic collisions. The difference between hard and soft processes can in this approach be investigated with sharp focus.

(3) From (4.22) we get

$$G_q(\mu,\nu)=M^{(1/\mu\ln 2)}[\Gamma_q(\mu-\nu)-\Gamma_q(-\nu)] =M^{-\bar{\tau}(q,\mu,\nu)}, \tag{5.1}$$

where

$$\bar{\tau}(q,\mu,\nu)=-\frac{1}{\mu}\int_{-\nu}^{\mu-\nu}\frac{1}{\ln 2}\ln A_q(\xi)d\xi . \tag{5.2}$$

This last equation expresses a mean value obtained by averaging over a range μ. If $A_q(\mu-\nu)$ were independent of $\mu-\nu$, i.e., of $\bar{n}$, then we have strict self-similarity (as in the α model[1]) for which $\bar{\tau}(q,\mu,\nu)=\tau(q)$, and we get the scaling law $G_q=M^{-\tau(q)}$. In general, A_q depends on $\mu-\nu$; however, (5.2) suggests that an effective range in μ can be chosen such that $\bar{\tau}(q,\mu_{\rm eff},\nu)$ can characterize the problem. Starting from (5.1) we can then follow the procedure as described in Sec. II and perform the multifractal analysis for an effective μ approximation of the problem.

For a more detailed treatment of the problem the effective μ approach is not adequate. Universality implies that $f(\alpha)$ is invariant, whenever $\mu-\nu$ are common among different ν cases. A corollary to that is that if one performs the multifractal analysis in a fixed M interval as in (2.3)–(2.5) for all ν cases, then $f(\alpha)$ will depend on ν. From the general trend of the G moments,[8–11] we know that $f(\alpha)$ becomes more narrow as ν increases, corresponding to a decrease in multiplicity fluctuation at higher multiplicity densities. Since intermittency is weaker when $f(\alpha)$ is narrower,[14,15] it implies that the intermittency indices should decrease as dN/dy increases. That is just what has been observed in experiments.[16] Our conclusion here is that there is no need to invoke any new physical mechanisms to explain that dependence on dN/dy. Such properties are natural consequences of the universality and the specific form of the scaling function $\Gamma_q(\xi)$.

VI. CONCLUSION

We have shown that an interesting scaling property for the $G_q(\mu,\nu)$ moments follows, if the rapidity fluctuation can be characterized by a multiplicity splitting function $\psi(x,n)$, which specifies the probability that upon bin splitting one of the two subbins has fraction x of the initial multiplicity n. The function is allowed to depend on n, but is assumed not to depend on the bin size. This assumption implies a universality in the applicability of $\psi(x,n)$ at any step of the successive bin-splitting process. As a consequence, one can derive a recursion relation, which in turn yields the scaling property of $\ln G_q(\mu,\nu)$, as expressed in (4.22). The scaling function $\Gamma_q(\xi)$ therefore contains all the information about the multiplicity fluctuation at any initial N. The multifractal spectrum $f(\alpha)$ can be determined directly from $\Gamma_q(\xi)$ and will vary in shape in accordance to the region of ξ where the tangential slope is used to give $\tau(q)$. The variation in $f(\alpha)$ reflects the variation in the average multiplicity of the particles detected at the resolution scale, i.e., $\bar{n}=N/M$.

Since no model has been used, what is suggested is a way of presenting the data, if the universality relation is valid. Thus the complexity of multiplicity fluctuations is summarized in a set of $\Gamma_q(\xi)$ functions, which characterize the physical process. It should be independent of s and Y (in the central plateau region) so long as they vary in a range in which the basic dynamical process is invariant. Conversely, if $\Gamma_q(\xi)$ changes, it signifies the onset of some different dynamical process, such as minijet production or going off to the fragmentation region.

After this work was completed, Dibon and Markytan[17] analyzed the UA1 data in accordance to the procedure suggested here; their preliminary result remarkably verifies the validity of the universality relation (4.20).

The scaling functions $\Gamma_q(\xi)$ for different collision processes should be determined and compared. Such comparisons are better than comparing intermittency indices, since they involve the data themselves in the form of $\Gamma_q(\xi)$ without the ambiguity of deciding which region of $\ln M$ to extract the slopes for the intermittency indices.

Having seen the scaling properties of the G moments, it is natural to ask what the corresponding properties are for the factorial moments F_q.[1,2] Since F_q are not directly related to multiplicity fractions resulting from successive bin splittings, we have not been able to derive any simple recursion formula for them. Although we suspect some form of scaling for F_q, it is presumably not expressible in a simple form as in (4.20), except in the trivial case. That is, if F_q obeys a strict power law with the same exponent for each q but for any initial multiplicity N, then the straight line itself in the log-log plot is a scaling function. It has recently been found that the strict straight-line behavior occurs over an extended range in the two-dimensional $(\delta\eta\delta\phi)$ analyses,[18] although upon projection onto the one-dimensional subspace the linearity may be lost. This phenomenon of having weaker intermittency upon projection to lower-dimensional space has been found also in model calculations.[19,20] For the G moments, the linearity in the log-log plot in any dimension cannot be expected for reasons discussed following (2.2). Thus the extraction of the scaling curve $\Gamma_q(\xi)$ is an essential simplification in data presentation. The connection between $\Gamma_q(\xi)$ and a strict scaling law for the factorial moments remain to be investigated.

The q dependence of $\Gamma_q(\xi)$ can be transformed to the α dependence of $f(\alpha)$, according to the procedure described in Sec. II, provided that ξ is small or negative. When M is large compared to N, the problem loses multifractality. Indeed, $f(\alpha)$ has an implicit dependence on ξ. Yet, we may regard $f(\alpha)$ as scaling, i.e., independent of N, so long as it is determined at a fixed value of $\mu-\nu$. Since the dependence of $f(\alpha)$ on the value of $\mu-\nu$ is rather cumbersome to convey, it seems sensible to use $\Gamma_q(\xi)$ to express the ξ dependence for a fixed value of q, and use $f_\xi(\alpha)$ to express the α dependence for a fixed value of ξ.

Apart from being good functions to summarize the data on multiplicity fluctuations and allowing the results from different experiments to be compared directly, the scaling functions $\Gamma_q(\xi)$ and $f_\xi(\alpha)$ will also serve as the arena for confrontation between theory and experiment. Theoretical calculation of $\Gamma_q(\xi)$ should be done for various models. We shall consider phenomenology and do some model calculation in a separate paper.

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