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Relativistic quantum Hall effect

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Quantum electrodynamics in 2+1 dimensions (QED_{2+1}) at finite density and temperature is analyzed by reducing it to an effective (0+1)-dimensional theory. A realization of QED_{2+1} at finite density in the context of (planar) gapless semiconductors is suggested.

In this Brief Report we are concerned with (2+1)-dimensional quantum electrodynamics (QED_{2+1}) at finite density, and with a physical realization of the theory in the context of condensed-matter physics. Relativisticlike fermions arise naturally in condensed matter when the spectrum of nonrelativistic fermions is linearized around a Fermi surface that is pointlike. The analogy has found applications in spin chains and Peierls dielectrics (for reviews, see Refs. 1 and 2, respectively) in planar systems,[3] and in (3+1)-dimensional systems.[4,5] Here, a condensed-matter realization of relativistic planar electrons is pointed out, taking into account a finite density.

The consequences of a large chemical potential in QED_{2+1} have only recently been studied in the literature. The investigation was carried out by Lykken *et al.* in their work on anyon superconductivity.[6] Earlier accounts[7,8] dealt with a small chemical potential μ only, i.e., $\mu \ll |m|$, with m the mass gap. As will become clear in the following, this restriction amounts to taking into account only the lowest Landau level.

One of the salient features of (2+1)-dimensional QED[9,10] is that the vacuum current induced by an external gauge field is of abnormal parity.[11] There are several ways to calculate this current; see, for example, Refs. 12–14. Below, we develop an extremely simple method to obtain the induced vacuum current. Our approach is somewhat similar to the calculation of anomalies in even-dimensional gauge theories with the Landau level technique.[4,15] In addition to giving a physically intuitive understanding of the vacuum result, it allows for a straightforward generalization to finite density and temperature. The scheme is also easily adapted to the nonrelativistic case, as we will demonstrate.

Let us start by considering vacuum QED_{2+1}, with $\mu = T = 0$. We choose the three-dimensional Dirac matrices in such a way that the Dirac Hamiltonian H takes the form

$$H = \boldsymbol{\sigma} \cdot \mathbf{m} \tag{1}$$

with $\boldsymbol{\sigma}$ the Pauli matrices and[16]

$$\mathbf{m} = (i\partial_1, i\partial_2, m) \ . \tag{2}$$

The theory will be minimal coupled to an external electromagnetic field A^μ. We assume this vector potential to be given by

$$A^0 = A^1 = 0, \quad A^2 = Bx \ , \tag{3}$$

with B a constant magnetic field.

The key observation is the fact that the theory is invariant under (generalized) translations in the xy plane. Indeed, although a translation by a distance a in the x direction changes the vector potential $A^2 \to A^2 + Ba$, this change may be canceled by a gauge transformation $A_\mu \to A_\mu + \partial_\mu \lambda$, with $\lambda = -Bay$ and, hence, the theory is invariant under this combined symmetry. As a result, the (2+1)-dimensional theory effectively reduces to a (0+1)-dimensional theory, i.e., quantum mechanics, cf. Schwinger's proper-time method.[17] Also, each energy eigenvalue is infinitely degenerate. The number of degenerate states per unit area is given by $|eB|/2\pi$ for each level. In the vacuum case under consideration only the lowest Landau level with energy eigenvalue

$$\lambda_0 = \begin{cases} m & \text{if } eB > 0 \ , \\ -m & \text{if } eB < 0 \end{cases} \tag{4}$$

is relevant. The reason is that the contributions of the higher Landau levels to the induced current cancel. The effective (0+1)-dimensional theory then reads

$$L = \psi^* i\partial_0 \psi - \lambda_0 \psi^* \psi \ , \tag{5}$$

where ψ is an anticommuting field. The propagator S_F that follows from this Lagrangian is

$$S_F(k) = \frac{1}{k_0 - \lambda_0 + ik_0\delta} \ , \tag{6}$$

with δ an infinitesimal positive constant that is set to zero at the end of the calculation. In terms of this propagator the particle "density" J^0 is given by

$$J^0 = -i \int \frac{dk_0}{2\pi} S_F(k) \ , \tag{7}$$

remembering that the effective theory is 0+1 dimensional, so that there is no integration over momenta as there is in higher dimensions. The (2+1)-dimensional particle density j^0 is obtained from (7) by taking into account the degeneracy of the level: viz.,

$$j^0 = -i\frac{|eB|}{2\pi}\int\frac{dk_0}{2\pi}\frac{1}{k_0-\lambda_0+ik_0\delta} \; . \tag{8}$$

Using contour integration one easily obtains from (8) with (4) the standard expression[9] for the particle density induced into the vacuum by an external magnetic field:

$$j^0 = -\frac{eB}{4\pi}\frac{m}{|m|} \; . \tag{9}$$

The induced density is of abnormal parity, since B is odd under parity transformations (naively one would expect j^0 to be parity even). In fact, (9) reflects the fact that a mass term in QED_{2+1} is not invariant under parity transformations. By Lorentz covariance it follows that in an arbitrary electromagnetic field, with field strength $F^{\mu\nu}$, the induced particle current reads[9]

$$j^\lambda = -\frac{e}{8\pi}\frac{m}{|m|}\epsilon^{\lambda\mu\nu}F_{\mu\nu} \; . \tag{10}$$

This result is thus shown to be due to the lowest Landau level only, as is the case with anomalies in even-dimensional gauge theories.[4] The corresponding term in the effective Euler-Heisenberg Lagrangian is the Chern-Simons term first introduced by Deser *et al.*:[18]

$$\mathcal{L}_{\text{eff}} = \gamma\epsilon^{\mu\nu\lambda}F_{\mu\nu}A_\lambda \; , \tag{11}$$

with $\gamma = e^2\text{sgn}(m)/16\pi$. This term is of topological origin and imparts mass to the gauge field A_μ.

The representation (8) is very convenient when considering a finite density of electrons. In the presence of a (large) chemical potential μ also the higher Landau levels with energy eigenvalues

$$\lambda_{\pm n} = \pm\sqrt{2n|eB|+m^2} \tag{12}$$

($n = 1,2,\ldots$) may be occupied. In this case, Eq. (8) generalizes to

$$j^0 = -i\frac{|eB|}{2\pi}\int\frac{dk_0}{2\pi}\sum_n\frac{1}{k_0-(\lambda_n-\mu)+ik_0\delta} \; , \tag{13}$$

where the sum is over all eigenvalues. The shift $k_0 \to k_0+\mu$ is the usual rule to incorporate a finite density of particles.[19] (We assume $\mu > 0$ throughout.) Carrying out the k_0 integration, we obtain

$$j^0 = \frac{|eB|}{2\pi}\left(-\frac{1}{2}\text{sgn}(eB)\frac{m}{|m|}\theta(|m|-\mu) + \frac{1}{2}\theta(\mu-|m|) + \sum_{n=1}^{\infty}\theta(\mu-\lambda_n)\right) , \tag{14}$$

thereby recovering the result of Lykken *et al.*:[6]

$$j^0 = -\frac{eB}{4\pi}\frac{m}{|m|}\theta(|m|-\mu) + \frac{|eB|}{2\pi}\left(\left[\frac{\mu^2-m^2}{2|eB|}\right]+\frac{1}{2}\right)\theta(\mu-|m|) \; , \tag{15}$$

which they obtained by directly solving the corresponding Green's-function equation. In (15) θ stands for the Haeviside unit step function, and $[x]$ denotes the largest integer less than x. The latter function reflects the presence of (relativistic) Landau levels. When the argument $(\mu^2-m^2)/2|eB|$ of this function is an integer, the value of the function is ambiguous. In the limit $\mu \to 0$ (15) yields the density (9) of abnormal parity. The terms in (15) that are relevant for large chemical potential ($\mu > |m|$) are of normal parity; i.e., they are parity even. These terms are, consequently, not related to the fact that a mass term in QED_{2+1} violates parity. Indeed, they survive the limit $m \to 0$ as opposed to the anomalous term in (15) which vanishes in this limit. (The absence of the sign anomaly in the massless limit when $\mu \neq 0$ was first noted by Niemi.[7])

If, in addition to a magnetic field, there is also a uniform static electric field E present, we find, for the induced current,

$$j^2 = -\frac{eE}{4\pi}\frac{m}{|m|}\theta(|m|-\mu) + \text{sgn}(eB)\frac{eE}{2\pi}\left(\left[\frac{\mu^2-m^2}{2|eB|}\right]+\frac{1}{2}\right)\theta(\mu-|m|) \; , \tag{16}$$

where, without loss of generality, the electric field is chosen in the x direction: $A^0 = -Ex$. Equation (16) is obtained by simply multiplying the density (15) with the drift velocity E/B. The terms in (16) that survive the limit $m \to 0$ are, again, parity normal. These terms, being proportional to sgn(eB), constitute the relativistic Hall current. The effective Lagrangian related to this current is given by (11) with[20]

$$\gamma = \frac{e^2}{16\pi}\frac{m}{|m|}\theta(|m|-\mu) - \frac{e^2}{8\pi}\text{sgn}(eB)\left(\left[\frac{\mu^2-m^2}{2|eB|}\right]+\frac{1}{2}\right)\theta(\mu-|m|) \; , \tag{17}$$

provided $(\mu^2-m^2)/2|eB|$ is not an integer.[6] The first term in (17) corresponds to a genuine Chern-Simons term that breaks parity. The second term, on the other hand, having a factor sgn(eB), is invariant under parity transformations. This term is dubbed a "pseudo-Chern-Simons" term by Abouelsaood.[21] Despite its invariance, the term does contribute to the mass of the gauge field.

For comparison, we next consider the nonrelativistic case. In this instance, the energy eigenvalues of the Landau levels are given by

$$\lambda_n = (n+\tfrac{1}{2})\Omega \tag{18}$$

with $n = 0,1,2,\ldots$, and Ω the cyclotron frequency

$$\Omega = \frac{|eB|}{m} \; , \tag{19}$$

where now $m > 0$. Contrary to the relativistic case, we have here, in the absence of antiparticles, only positive-energy eigenvalues. As usual in nonrelativistic many-body theory one has to include a convergence factor $\exp(ik_0\delta)$ in the propagator.[22] In this way, we obtain, for the induced density at $\mu > 0$,

$$j_0 = -i\frac{|eB|}{2\pi}\int\frac{dk_0}{2\pi}\sum_n \frac{e^{ik_0\delta}}{k_0-(\lambda_n-\mu)+ik_0\delta}$$
$$= \frac{|eB|}{2\pi}\sum_{n=0}^{\infty}\theta(\mu-(n+\tfrac{1}{2})\Omega) = \frac{|eB|}{2\pi}\left[\frac{\mu m}{|eB|}+\frac{1}{2}\right] . \tag{20}$$

Because of the [ ] function here the filling factor ν, which is defined as

$$\nu := \frac{j_0}{|eB|/2\pi} , \tag{21}$$

takes on integer values only. This was to be expected for an ideal electron gas at zero temperature; given a value of the chemical potential a Landau level below the Fermi surface is filled, while a level above the Fermi surface is empty. (When the energy of the Fermi surface and that of a Landau level coincide, the value of the [ ] function is ambiguous.) The current carried by the filled Landau levels is obtained by multiplying the induced density (20) with the drift velocity E/B, which is the same for both relativistic and nonrelativistic electrons. In this way one finds

$$j_y = \mathrm{sgn}(eB)\frac{eE}{2\pi}\left[\frac{\mu m}{|eB|}+\frac{1}{2}\right] . \tag{22}$$

Equations (20) and (22) should be compared with the relativistic analogues (15) and (16). Since in nonrelativistic planar electrodynamics parity is not violated by massive electrons, there can be no currents of abnormal parity. And indeed, (20) and (22) are parity normal. The corresponding term in the effective Lagrangian has, consequently, solely a pseudo-Chern-Simons piece.

So far we considered the systems at zero temperature. Finite temperature is easily incorporated in this scheme by replacing the k_0 integration with a summation over Matsubara frequencies $\omega_m = 2\pi m\beta^{-1}$:[19]

$$\int\frac{dk_0}{2\pi}g(k_0) \to i\beta^{-1}\sum_m g(i\omega_m) , \tag{23}$$

where g is an arbitrary function, $\beta = T^{-1}$ is the inverse temperature, and the sum is over all integers m. In the relativistic case we obtain from (8) with (23), using the identity

$$\beta^{-1}\sum_m\frac{1}{i\omega_m-(\lambda_n-\mu)} = \frac{1}{2}\tanh\left(\frac{\beta}{2}(\mu-\lambda_n)\right) , \tag{24}$$

the induced density at finite temperature:

$$j^0 = \frac{eB}{4\pi}\left[\theta(eB)\tanh\left(\frac{\beta}{2}(\mu-m)\right) - \theta(-eB)\tanh\left(\frac{\beta}{2}(\mu+m)\right)\right] + \frac{|eB|}{4\pi}\sum_{n=1}^{\infty}\left[\tanh\left(\frac{\beta}{2}(\mu-\lambda_n)\right) + \tanh\left(\frac{\beta}{2}(\mu+\lambda_n)\right)\right] . \tag{25}$$

It is easily checked that this expression reproduces the zero-temperature result (15). The first two terms at the right-hand side of (25) pertain to the lowest Landau level, which was considered earlier by Niemi and Semenoff.[12] Their result differs from (25) in that the step functions $\theta(\pm eB)$ are absent. For the corresponding induced current we find the expression

$$j^2 = \frac{eE}{4\pi}\left[\theta(eB)\tanh\left(\frac{\beta}{2}(\mu-m)\right) - \theta(-eB)\tanh\left(\frac{\beta}{2}(\mu+m)\right)\right] + \mathrm{sgn}(eB)\frac{eE}{4\pi}\sum_{n=1}^{\infty}\left[\tanh\left(\frac{\beta}{2}(\mu-\lambda_n)\right) + \tanh\left(\frac{\beta}{2}(\mu+\lambda_n)\right)\right] , \tag{26}$$

with λ_n given by (12).

In the nonrelativistic case, where there is an extra convergence factor present in the propagator, we employ the identity

$$\beta^{-1}\sum_m\frac{e^{i\omega_m\delta}}{in\omega_m-(\lambda_n-\mu)} = f(\lambda_n) , \tag{27}$$

where $f(x)$ is the Fermi-Dirac distribution function

$$f(x) = \left(e^{\beta(x-\mu)}+1\right)^{-1} . \tag{28}$$

With the eigenvalues given by (18), this leads to the results

$$j_0 = \frac{|eB|}{2\pi}\sum_{n=0}^{\infty}f((n+\tfrac{1}{2})\Omega) ,$$
$$j_y = \mathrm{sgn}(eB)\frac{eE}{2\pi}\sum_{n=0}^{\infty}f((n+\tfrac{1}{2})\Omega) , \tag{29}$$

where Ω is the cyclotron frequency (19). The last equation is the (nonrelativistic) Hall current at finite temperature.

We next turn to a discussion of a possible realization of the relativistic quantum Hall effect in the context of condensed-matter physics. More specifically, we will consider two-dimensional gapless semiconductors, or semimetals. These materials have isolated points in which the valence and conduction bands intersect; i.e., the Fermi surface consists of points. A small number of electrons, typically 10^{-4} per atom, will move from the occupied valence band to the unoccupied conduction band, leaving behind an equal number of holes in the valence band.

The prime example of a planar semimetal is graphite. To a first approximation, graphite may be thought of as composed of independent layers of carbon atoms, the separation between lattice planes along the c axis being 2.8 times the nearest-neighbor distance within planes.[23] Each layer has a honeycomb lattice and there are two degeneracy points per (two-dimensional) Brillouin zone.[3] As observed by Nielsen and Ninomiya,[4] near such diabolic points the electrons have a linear spectrum and are described by a two-level Hamiltonian. Hence, sufficiently close to a degeneracy point the electrons are modeled by relativistic massless fermions. (In Ref. 3 this has been explicitly demonstrated for the case of graphite, starting from a tight-binding description.) The interaction with an electromagnetic field is assumed to be obtained via minimal coupling. Since there are an equal number of particles and holes, the effective theory has particle-hole symmetry which implies that the chemical potential is zero.

The fact that there is an even number of degeneracies per Brillouin zone in graphite is a manifestation of fermion doubling on a lattice.[24,25] More specifically, in the context of two-dimensional gapless semiconductors, the no-go theorem due to Nielsen and Ninomiya[25] asserts that diabolic points come in parity-invariant pairs. This then implies the absence of (anomalous) electromagnetic currents in these systems with particle-hole symmetry.[3]

A nonzero electromagnetic current is easily obtained by doping. This breaks the particle-hole symmetry and leads to a finite chemical potential. The ensuing electromagnetic current is a (relativistic) Hall current

$$j^2 = \sum_a \mathrm{sgn}(eB)\frac{eE}{2\pi}\left(\left[\frac{\mu^2}{2|eB|}\right]+\frac{1}{2}\right) , \tag{30}$$

using Eq. (16) with $m = 0$. In (30) the sum is over all degeneracy points labeled by the index a. The equation shows that the Hall current is nonzero no matter how small the chemical potential. Hence, the presence of diabolic points in doped planar semimetals would manifest itself through a relativistic quantum Hall current.

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$$\gamma = \frac{e^2}{16\pi}\frac{m}{|m|}\theta(|m|-|\mu|) - \frac{e^2}{8\pi}\mathrm{sgn}(eB)\,\mathrm{sgn}(\mu)\left[\left(\frac{\mu^2-m^2}{2|eB|}\right)+\frac{1}{2}\right]\theta(|\mu|-|m|) .$$

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