Centrifugal force in Ernst spacetime

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We show that in Ernst spacetime, which represents the gravitational field of a mass embedded in a magnetic field, the centrifugal force acting on a particle in circular orbit *reverses its sign twice,* once very close to $r=3m$, and again later at a distance far away from the center, depending upon the combined values of M and B, the mass and the strength of the magnetic field. The second reversal means that beyond a certain distance, depending upon Bm, in Ernst geometry no Keplerian motion is possible for a test particle.

Recently it has been shown by Abramowicz and Prasanna[1] (AP) that in static spacetimes the commonly understood "centrifugal force" reverses its sign across the circular photon orbit (which for the Schwarzschild solution is at $r=3m$) at which the geodesic curvature radius tends to infinity. Following this, Abramowicz and Miller[2] pointed out that the earlier obtained result of Chandrasekhar and Miller,[3] regarding the decrease of the ellipticity of quasistationarily contracting relativistic MacLaurin spheroids with a fixed mass and a total angular momentum for $r<5m$, can also be explained through this effect of centrifugal force reversal close to the black-hole radius. Prasanna and Chakrabarti,[4,5] using the notion of optical reference geometry invoked by Abramowicz, Carter, and Lasota[6] (ACL) in Kerr spacetime, showed that the centrifugal and Coriolis forces reverse sign at several different locations. More recently Abramowicz and Bicak,[7] considering the effect in Reissner-Nordström spacetime, showed that some of the peculiar properties of the circular motion of charged ultrarelativistic particles close to the circular photon orbit can be better understood through the centrifugal force reversal. Abramowicz[8] has pointed out, through intuitive arguments and gedanken experiment, how one can appreciate this effect in a physically meaningful approach.

It is generally believed in the study of extended structures such as galaxies that the effects of general relativity are unimportant and further while considering the dynamical effects the role of the magnetic field is completely ignored. However, it may not be out of the way to try and understand qualitatively the effects of general relativity and of the weak magnetic field in the spacetime structure of massive extended objects. The only exact solution of Einstein's equations known to represent the spacetime of a massive body M embedded in an otherwise uniform magnetic field B_Γ is due to Ernst[9] and is given by the metric

$$ds^2=-\Lambda^2[(1-2m/r)dt^2-(1-2m/r)^{-1}dr^2 -r^2d\theta^2]+(r^2\sin^2\theta/\Lambda^2)d\phi^2 \ , \tag{1}$$

with

$$\Lambda=(1+B^2r^2\sin^2\theta) \ ,$$
$$m=MG/c^2, \quad B=B_\Gamma G^{1/2}/c^2 \ .$$

Now following the approach of ACL as described in AP we introduce the optical reference geometry through the 3+1 conformal splitting:

$$ds^2=\Phi(-dt^2+\tilde{d}l^2) \ , \quad \tilde{d}l^2=\tilde{g}_{ij}dx^idx^j \ \ (i,j=1,\ldots,3) \ . \tag{2}$$

We get thus

$$\Phi=\Lambda^2(1-2m/r) \ , \tag{3}$$

$$\tilde{g}_{rr}=(1-2m/r)^{-2} \ ,\quad \tilde{g}_{\theta\theta}=r^2(1-2m/r)^{-1} \ ,\quad \tilde{g}_{\varphi\varphi}=\frac{r^2\sin^2\theta}{\Lambda^4}\left[1-\frac{2m}{r}\right]^{-1} \ . \tag{4}$$

The circumferential radius $\tilde{r}$ of the projected circular orbit in this three-space is

$$\tilde{r}=(\tilde{g}_{\varphi\varphi})^{1/2}=\frac{r\sin\theta}{\Lambda^2}\left[1-\frac{2m}{r}\right]^{-1/2} \ . \tag{5}$$

The geodesic curvature radius $\mathcal{R}$, as given in AP [Eq. (2.17)],

$$\mathcal{R}=\tilde{r}/[\tilde{g}^{ik}\tilde{\nabla}_i(\tilde{r})\tilde{\nabla}_k(\tilde{r})]^{1/2} \ , \tag{6}$$

turns out in this case (for $\theta=\pi/2$) to be

$$\mathcal{R}=r\Lambda/[(1-3m/r)-3B^2r^2(1-5m/3r)] \ . \tag{7}$$

It is simple to recognize that $\mathcal{R}\to\infty$ exactly at the locations of the extrema of the effective potential for photons in Ernst spacetime:

$$V_{\text{eff}}=\frac{\Lambda^4l^2}{r^2}(1-2m/r) \ , \tag{8}$$

l being the specific angular momentum. Unlike in Schwarzschild spacetime where there is only one circular photon orbit at $r=3m$, in this case we find that there can be in principle two such orbits which change with different values of Bm.

If Ω is the angular velocity of a particle as measured by the stationary observer at infinity, then

$$l=\Omega\tilde{r}^2=\frac{\Omega r^2}{\Lambda^4}(1-2m/r)^{-1} \ . \tag{9}$$

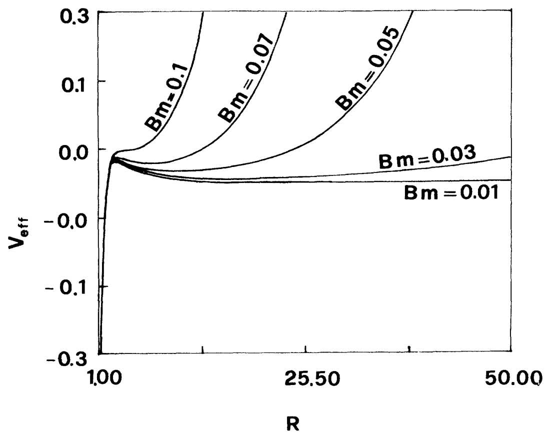

FIG. 1. The photon effective potential as a function of the radial distance from the central body depicting the maximum and the minimum.

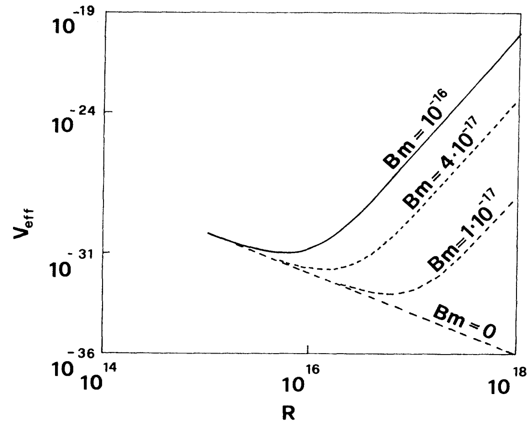

FIG. 2. The photon effective potential as a function of the radial distance from the central body depicting the maximum and the minimum.

Using the definition of the "orbit speed" $\tilde{v}$, as in AP, one gets the centrifugal force C_F acting on the particle,

$$C_F=\frac{m_0\tilde{v}^2}{\mathcal{R}}=\frac{m_0 l^2}{r^3}\left[\frac{r^2\Lambda^3(r-2m)}{r^3-l^2\Lambda^4(r-2m)}\right]\times\left[\left[1-\frac{3m}{r}\right]-B^2r^2\left[3-\frac{5m}{r}\right]\right], \quad (10)$$

and the gravitational force G_F:

$$G_F=-\partial_r\Phi=-2\Lambda\left[\frac{m}{r^2}+2B^2r\left[1-\frac{3m}{2r}\right]\right]. \quad (11)$$

It is important to notice that, as $r>2m$, G_F remains negative *irrespective* of the value of B whereas C_F changes sign at the roots of the cubic:

$$f(R)\equiv-3B^2m^2R^3+5B^2m^2R^2+R-3=0 \quad (12)$$

$(R=r/m)$.

The discriminant Δ of this cubic is given by

$$\Delta=-\frac{B^6m^6}{27}(13\,500B^4m^4-12\,168B^2m^2+108) \quad (13)$$

and thus, depending upon the values of Bm, Δ is either positive, zero, or negative. It is easy to see that for $Bm>0.095$, Δ is either zero or positive and for $Bm<0.095$, Δ is negative. Writing Bm in the c.g.s. units one finds that

$$Bm=2\times10^{-53}B_\Gamma M, \quad (14)$$

which shows that $B_\Gamma M$ has to be extremely large for $Bm>0.095$ (in fact for $Bm=0.01$ and $M=10^{10}M_\odot B_\Gamma$ has to be $\gtrsim 2.5\times10^9$ G), which is very unrealistic. Thus for any reasonable combination of Bm it is in fact very small and thus less than 0.095, ensuring that all the three roots of the cubic are *real*. From Eq. (12) it is clear that the cubic can have only two positive roots, the other being negative. Hence neglecting the negative root ($r\nless0$) one finds that the "centrifugal force" changes sign at two places—one very close to $r=3m$, and the other far away depending upon Bm.

It is not difficult to understand this behavior of centrifugal force reversal at two locations if one looks at the structure of the effective potential for photons as depicted in Figs. 1 and 2. Figure 1 shows the curves for small values of Bm wherein one can see the minimum of the potential occurring for values <0.1, and Fig. 2 for extremely small values of Bm, the minimum occurring very far away. As explained by Abramowicz[8] the centrifugal force attracts particles towards the stable circular photon orbit which occurs at the minimum of the effective potential. Just to illustrate we also give in Figs. 3 and 4 the location of the roots of $f(R)$ wherein again it is clear that for $Bm=0.095$ there appears a double root and for values less than that, two distinct positive roots.

It is clear that for spacetimes having combined gravitational and magnetic fields there exist two extrema for the photon effective potential: a maximum around $3m$ and a minimum quite far away. *As the centrifugal force reverses sign at both these points, the Keplerian law of motion is thus valid only for the region inbetween these two photon orbits.*

It may be argued that for large-scale astronomical structures such as galaxies, the mass distributions would be such that it may not be suitable to consider a Schwarzschild-like solution for the field. However, the location where the second reversal of centrifugal force occurs is really far away and the field gradient at this distance is very, very small. As can be seen from the numbers associated with the $V_{\rm eff}$ (Fig. 2) after the minimum the raise is extremely small and thus for practical purposes one can assume it to be almost flat. Though the magnetic field energy density is much smaller compared to the radiation and dust in the outer regions of a structure such as a galaxy, the overall qualitative effect of the magnetic field in reversing the force direction may have significant consequences.

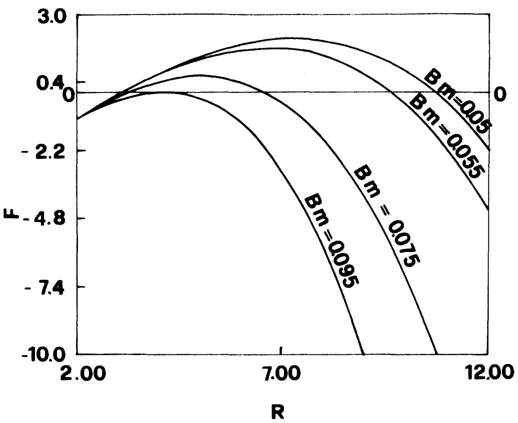

FIG. 3. The location of the roots of Eq. (12) for different values of *Bm*.

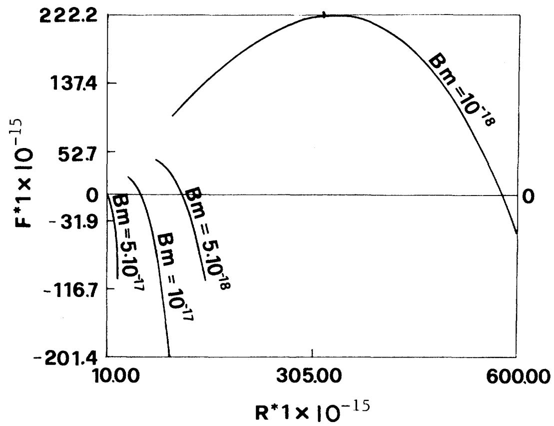

FIG. 4. The location of the roots of Eq. (12) for different values of *Bm*.

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