Supercoherent states

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A general approach is presented for constructing coherent states for supersymmetric systems. It uses Rogers's supermanifold formulation of supergroups to extend the group-theoretic method. Supercoherent states are explicitly obtained for the supersymmetric harmonic oscillator. They are shown to be eigenstates of the supersymmetric annihilation operator and to be minimum-uncertainty states. Two more-complex situations with extended physical supersymmetries are also considered: an electron moving in a constant magnetic field, and the electron-monopole system. The supercoherent states for these systems are found using super Baker-Campbell-Hausdorff relations and their interpretation is elucidated.

I. INTRODUCTION

Coherent states[1] have widespread physical relevance. Originally discovered by Schrödinger,[2] modern interest developed in the context of path integrals[3] and quantum optics.[4,5] The ideas have evolved through several formulations and are now also an important group-theoretic concept.[6]

Many definitions of coherent states exist.[7] One describes coherent light beams generated, for example, by a laser. The requirement is that the annihilation operator a for each individual oscillator mode of the electromagnetic field satisfy $a|\alpha\rangle=\alpha|\alpha\rangle$. Here, $[a,a^\dagger]=1$ and α is a complex constant with conjugate $\bar{\alpha}$. The resulting unit-normalized states $|\alpha\rangle$ are given by $|\alpha\rangle=e^{-|\alpha|^2/2}\sum_n(\alpha^n/\sqrt{n!})|n\rangle$.

A second definition of a coherent state for oscillators assumes the existence of a unitary operator D "displacing" a according to the formula $D^{-1}(\alpha)aD(\alpha)=a+\alpha$. Explicitly, $D(\alpha)=\exp(\alpha a^\dagger-\bar{\alpha}a)$. A coherent state parametrized by α is given by the action of $D(\alpha)$ on the ground state $|0\rangle$. The unitarity of $D(\alpha)$ ensures the correct normalization of $|\alpha\rangle$. The Baker-Campbell-Hausdorff (BCH) relation[8–13] $e^Ae^B=e^{(A+B+[A,B]/2)}$, valid for any two operators A and B that both commute with the commutator $[A,B]$, implies the equivalence of this definition with the one above.

A third definition is based on the uncertainty relation. With the position q and momentum p given as usual by $q=(a^\dagger+a)/\sqrt{2\omega}$ and $p=i(a^\dagger-a)\sqrt{2/\omega}$, the coherent states defined above have the minimum-uncertainty value $\Delta q\Delta p=\frac{1}{2}$ and maintain this relationship in time. A coherent state can thus be viewed as a superposition of eigenstates that is "closest to classical." Note that, although this definition agrees with the other two in the simple oscillator case, the three definitions are generally inequivalent.[7]

Coherent states have two important properties. First, they are *not* orthogonal to each other: $\langle\alpha|\beta\rangle=\exp(\bar{\alpha}\beta-\frac{1}{2}|\alpha|^2-\frac{1}{2}|\beta|^2)$. Second, they provide a resolution of the identity; i.e., they form a complete (in fact, overcomplete) set of states, $\int|\alpha\rangle\langle\alpha|d^2\alpha=\pi\mathbb{1}$, and therefore can be used as a basis set.

A central goal of this paper is to extend the notion of coherent states to supersymmetric systems. Since these involve fermionic states, the physically relevant extension of any given definition of a coherent state is not immediately apparent. A key feature of our approach is the use of Grassmann-valued variables to define the supercoherent states. As shown here, one attractive consequence is the natural extension of the desirable properties of ordinary coherent states.

The method we adopt is an extension of the group-theoretic approach to coherent states,[6] which generalizes the displacement-operator definition using Lie groups. The essential observation of this approach is that the important aspects of coherent states follow from the properties of a symmetry group of the system. For example, in the case of the single harmonic oscillator, the symmetry is the Heisenberg-Weyl group, with algebra generated by the creation, annihilation, and identity operators. The group-theoretic approach is summarized in Sec. II A.

The basic idea is to extend the group-theoretic method to supergroups. In this paper, we use supergroups defined both as abstract groups and as superanalytic supermanifolds.[14–16] Some background material on Grassmann-valued analysis and supergroups is contained in Sec. II B. Section II C discusses the question of obtaining a unitary representation of a supergroup, which is necessary for the construction of supercoherent states.

In general, an important step in obtaining an explicit expression for a supercoherent state in the group-theoretic approach is finding a BCH relation for the supergroup. The BCH relations interrelate various supergroup coordinate schemes. A general algorithm for obtaining BCH formulas for supergroups has been developed.[17–19] It reduces to an initial-value problem for a system of ordinary nonlinear coupled differential equations containing dependent Grassman-valued variables.

The supersymmetric harmonic oscillator provides a suitable starting point for the explicit construction of supercoherent states. The relevant supergroup for this case is the super Heisenberg-Weyl group, with superalgebra generated by the identity and by bosonic and fermionic creation and annihilation operators. In Sec. III A, this system and its relationship to supersymmetric quantum mechanics[20–29] are described, and the BCH relations for the super Heisenberg-Weyl group are found. Section III B presents the associated supercoherent states and their features. The latter are natural extensions of the properties of coherent states of the ordinary Heisenberg-Weyl group.

The supersymmetric harmonic oscillator has relevance for the motion of a charged massive spin-$\frac{1}{2}$ particle in a constant magnetic field.[28,29] This permits exploration of the properties of supercoherent states and their interpretation in a physical context. The appropriate supercoherent states are constructed and these ideas are discussed in Sec. IV. The role of the Grassmann variables is clarified and insight is gained into the link between supercoherent states and the classical motion.

In Sec. V, we consider supercoherent states for the simple noncompact supergroup OSP(1/2). In particular, we treat the fermion-monopole system, which is known to have a dynamical OSP(1/2) supersymmetry.[30] The necessary BCH relations are available[17] and are used for explicit construction of the supercoherent states. Features of these states are analyzed.

Section VI concludes with a summary and a discussion of the relationship of this work to other constructions of supercoherent states.

II. FRAMEWORK

A. The group-theoretic approach

Coherent states can be treated as group-theoretic objects. This is readily seen in the context of the displacement-operator definition outlined in the Introduction. The standard creation, annihilation, and identity operators associated with the simple harmonic oscillator generate an abstract Lie algebra, the Heisenberg-Weyl algebra. The associated Lie group can be obtained, as usual, by exponentiation of the generators. One then recognizes the displacement operator $D(\alpha)$ as forming a unitary representation of the Heisenberg-Weyl group. The appearance of the BCH relation used in the Introduction is natural in this context, as in general a BCH formula links various group coordinates obtained by exponentiation.

A set of generalized coherent states for an arbitrary Lie group G can be defined in the following way.[3,6] Let $T(g)$ be a unitary irreducible representation of G acting in a Hilbert space H. Let $|\Psi_0\rangle$ be an arbitrary, fixed vector in H. Then, a set of states $\{|\Psi_g\rangle\}=\{T(g)|\Psi_0\rangle\}$ is called a coherent-state system. This extends the displacement-operator definition to arbitrary Lie groups. It also removes the restriction that the operators $T(g)$ act on an extremal state, which in general may not be well defined.

For the case of semisimple algebras, this definition permits an analogy to the minimum-uncertainty definition of coherent states given in the Introduction. The point is that one can define in purely group-theoretic terms a class of $|\Psi_0\rangle$ such that the corresponding coherent states are closest to classical in the sense that they minimize the dispersion

$$\Delta C_2=\langle \Psi_g|C_2|\Psi_g\rangle - g^{jk}\langle \Psi_g|X_j|\Psi_g\rangle\langle \Psi_g|X_k|\Psi_g\rangle \ . \tag{2.1}$$

Here, $C_2=g^{jk}X_jX_k$ is the quadratic Casimir operator, X_j are the generators of the Lie algebra, and g^{jk} is the Cartan-Killing metric tensor.

B. Background on superanalysis

This section summarizes results on Grassmann-valued analysis and supergroups useful in the subsequent discussion. See Refs. 14–16 and 31–33 for more details.

A Grassmann algebra B_L is an associative algebra generated by the identity 1 and by L elements β_a, $a=1,\ldots,L$, obeying the anticommutation relations $\{\beta_a,\beta_b\}=0$. The algebra is spanned by the basis 1, β_a, $\beta_a\beta_b,\ldots$ with $a<b<\cdots$, i.e., by the identity and all independent nonvanishing products of β_a. There are 2^L linearly independent basis elements. For our purposes, it suffices to consider finite L. The subset of basis elements consisting of the identity and all even products of the generators spans what is called the even part ${}^0B_L$ of B_L; the remaining basis elements span the odd part ${}^1B_L$.

Various definitions are possible for the complex conjugate $\overline{\beta_a}$ of the basis element β_a. We take[34] $\overline{\beta_a}$ as a generator distinct from β_a. Complex conjugation has the properties $\overline{(\overline{\beta}_a)}=\beta_a$, $\overline{(\beta_a\beta_b)}=\overline{\beta_b}\,\overline{\beta_a}$, and $\overline{(z\beta_a)}=\overline{z}\overline{\beta_a}$ where $z\in\mathbb{C}$.

A Grassmann-valued variable is one that takes values in a Grassmann algebra; i.e., it consists of a linear combination of the 2^L basis elements with complex coefficients. Even variables take values in ${}^0B_L$ and odd variables take values in ${}^1B_L$. A Grassmann-valued variable can also be split into the sum of a complex number called the body and a nilpotent piece called the soul, which is a linear combination of the generators β_a and all their distinct products.

Details of Grassmann-valued analysis may be found in Refs. 31–33. Here, we mention only that integration over Grassmann-valued variables is understood in the sense of Berezin: for even variables, integration is performed in complete analogy with integration over $\mathbb{C}$, while for odd variables θ integration is defined by $\int d\theta\equiv 0$, $\int \theta\, d\theta\equiv 1$.

This paper uses a definition[35] of supermanifolds and supergroups due to Rogers.[14–16] It mimics the standard definitions of analytic manifolds and Lie groups, but involves Grassmann-valued objects. For instance, Euclidean space $\mathbb{R}^m$, which is the Cartesian product of m copies of $\mathbb{R}$, is replaced by flat superspace $B_L^{m,n}$, which is the Cartesian product of m copies of ${}^0B_L$ with n copies of ${}^1B_L$. Similarly, an (m,n)-dimensional superanalytic supermanifold $S_L^{m,n}$ is defined as a Hausdorff space with an atlas such that $S_L^{m,n}$ is locally homeomorphic to $B_L^{m,n}$ and the transition functions are superanalytic. An (m,n)-dimensional supergroup H is taken as an abstract group that is also an (m,n)-dimensional superanalytic supermanifold $S_L^{m,n}$ with a superanalytic map $H\times H \to H$: $(h_1,h_2)\to h_1h_2^{-1}$.

In analogy with the case of Lie groups and Lie algebras, the set of left translations on a supergroup forms a supermodule W. This is the direct product of B_L with a superalgebra. We denote the superalgebra generators by $X_1,\ldots,X_m;X_{m+1},\ldots,X_{m+n}$. The supergroup can be reconstructed from the superalgebra as follows. The *even* part of W generates under commutation a $2^{L-1}(m+n)$-dimensional Lie algebra **h**. A Lie group H can then be obtained from the Lie algebra **h** in the usual way. This Lie group can be given a unique global superanalytic structure that makes it a supergroup. The important feature for the present work is that all elements of the Lie algebra **h** are even. This is essential in the definition of supercoherent states.

C. Unitary representations of supergroups

The group-theoretic approach to coherent states involves the use of unitary group representations. Evidently, unitary representations of supergroups are needed to extend the notions to supercoherent states.

We are mainly interested in physical applications. A set of operators X_i satisfying prescribed graded commutation relations is usually the information that can be directly extracted from the physical problem. This defines the underlying superalgebra, from which a Lie algebra **h** can be constructed as described in Sec. II B. The idea is to obtain unitary supergroup elements in terms of the generators of this Lie algebra.

A general element of **h** is a linear combination of the superalgebra generators X_i, where $X_1,\ldots,X_m$ and $X_{m+1},\ldots,X_n$ are multiplied by even and odd Grassmann-valued variables, respectively. The coordinates on the supergroup can then be found by exponentiation.[17,36] To obtain a unitary representation $T(g)$, we introduce a super-Hermitian basis[37] for the superalgebra by choosing the generators so that $X_j^\dagger = X_j$ for $j=1,\ldots,m$ and $X_j^\dagger=-X_j$ otherwise. Then,

$$T(g)=\exp\left[\sum_{j=1}^{m} iA_jX_j+\sum_{j=1}^{n} i\theta_jX_{m+j}\right] \tag{2.2}$$

is unitary when $A_i\in {}^0B_L$ and $\theta_i \in {}^1B_L$ are both real.

III. THE SUPERSYMMETRIC HARMONIC OSCILLATOR

A. The super Heisenberg-Weyl group

A supersymmetric quantum-mechanical system[20–23] has a Hamiltonian H that can be expressed in terms of supersymmetry generators Q_i, $i=1,2,\ldots,N$ as $\delta_{ij}H=\{Q_i,Q_j\}$ with $[Q_i,H]=0$. The superalgebra defined by these relations is denoted sqm(N). Here, we consider the special case sqm(2). For this case, the linear combinations $Q=(Q_i+iQ_2)/\sqrt{2}$ and $Q^\dagger=(Q_i-iQ_2)/\sqrt{2}$ are convenient. Then,

$$H=\{Q,Q^\dagger\},\quad [H,Q]=[H,Q^\dagger]=0\ . \tag{3.1}$$

This algebra has relevance for physical systems.[24–29]

In particular, we are interested in the supersymmetric oscillator. We introduce annihilation and creation operators a, $a^\dagger$; b, $b^\dagger$ satisfying the following nonvanishing graded commutation relations:

$$[a,a^\dagger]=\mathbb{1},\quad \{b,b^\dagger\}=\mathbb{1}\ . \tag{3.2}$$

This is a supersymmetric extension of the usual Heisenberg-Weyl algebra. States in the super Hilbert space are denoted by $|n,\nu\rangle$, where $n=0,1,2,\ldots$ and $\nu=0,1$. States with $\nu=0$ are called bosonic and states with $\nu=1$ are called fermionic. As usual,

$$\begin{aligned} a|n,\nu\rangle &= \sqrt{n}\,|n-1,\nu\rangle\ ,\\ a^\dagger|n,\nu\rangle &= \sqrt{n+1}\,|n+1,\nu\rangle\ ,\\ b|n,\nu\rangle &= \delta_{\nu 1}|n,0\rangle,\quad b^\dagger|n,\nu\rangle=\delta_{\nu 0}|n,1\rangle\ . \end{aligned} \tag{3.3}$$

The supersymmetric Hamiltonian and the supersymmetry generators have the form

$$H=a^\dagger a+b^\dagger b,\quad Q=ab^\dagger,\quad Q^\dagger=a^\dagger b\ . \tag{3.4}$$

Since $H|n,\nu\rangle=(n+\nu)|n,\nu\rangle$, the states $|n,0\rangle$ and $|n-1,1\rangle$ are degenerate except for the unique ground state $|0,0\rangle$. For unbroken supersymmetry $Q|0,0\rangle=Q^\dagger|0,0\rangle=0$, so the ground state has zero energy. The operator Q maps bosonic states into fermionic ones, while $Q^\dagger$ maps fermionic states into bosonic ones.

To construct the supercoherent states for this system, we need a unitary representation of the Heisenberg-Weyl algebra (3.2). A super-Hermitian basis for this algebra is

$$\begin{aligned} &X_1=a+a^\dagger,\quad X_2=i(a-a^\dagger),\quad X_3=\mathbb{1}\ ,\\ &X_4=i(b^\dagger+b),\quad X_5=b^\dagger-b\ . \end{aligned} \tag{3.5}$$

From Eq. (2.2), a unitary representation $T(g)$ of the supergroup is

$$\begin{aligned} T(g)&=\exp(iA_1X_1+iA_2X_2+iA_3X_3+i\theta_1X_4+i\theta_2X_5)\\ &=\exp(-\overline{A}a+Aa^\dagger+B\mathbb{1}+\theta b^\dagger+\overline{\theta}b)\ , \end{aligned} \tag{3.6}$$

where $A=A_2+iA_1$, $\theta=-\theta_1+i\theta_2$ and $B=iA_3$, i.e., $A\in {}^0B_L$ and $\theta\in {}^1B_L$ are complex and $B\in {}^0B_L$ is pure imaginary.

To obtain $T(g)$ as a conveniently ordered product of exponentials requires a suitable Baker-Campbell-

Hausdorff relation for the super Heisenberg-Weyl group. Using lemma 1 of Ref. 17, we find Eq (3.6) can be expressed as

$$T(g)=\exp\left[B+\frac{1}{2}\theta\bar{\theta}-\frac{|A|^2}{2}\right]\times\exp(Aa^{\dagger})\exp(\theta b^{\dagger})\exp(-\bar{A}a)\exp(\bar{\theta}b)\ . \qquad (3.7)$$

B. Supercoherent states

The action (3.7) of $T(g)$ on the ground state yields a supercoherent state $|Z\rangle$:

$$|Z\rangle=T(g)|0,0\rangle=\exp(\tfrac{1}{2}\theta\bar{\theta})\exp\left[-\frac{|A|^2}{2}\right]\times\exp(Aa^{\dagger})(|0,0\rangle+\theta|0,1\rangle)\ , \qquad (3.8)$$

where we have used $e^{\theta b^{\dagger}}|0,0\rangle=(1+\theta b^{\dagger})|0,0\rangle=|0,0\rangle+\theta|0,1\rangle$. The factor e^B has been dropped because with B pure imaginary $e^B|n,\alpha\rangle$ represents the same state as $|n,\alpha\rangle$.

For simplicity, define

$$|A,\nu\rangle=\exp\left[-\frac{|A|^2}{2}\right]\exp(Aa^{\dagger})|0,\nu\rangle\ . \qquad (3.9)$$

Then, the supercoherent state is

$$|Z\rangle=(1+\tfrac{1}{2}\theta\bar{\theta})|A,0\rangle+\theta|A,1\rangle\ . \qquad (3.10)$$

With $\theta=0$, the supercoherent state $|Z\rangle$ has the form of a canonical harmonic-oscillator coherent state $|A\rangle$.

The supercoherent states are unity normalized, $\langle Z|Z\rangle=1$. Distinct states are not orthogonal:

$$\langle Z_1|Z_2\rangle=(1+\tfrac{1}{2}\theta_1\bar{\theta}_1+\tfrac{1}{2}\theta_2\bar{\theta}_2+\tfrac{1}{4}\theta_1\bar{\theta}_1\theta_2\bar{\theta}_2)\times\langle A_1,0|A_2,0\rangle+\bar{\theta}_1\theta_2\langle A_1,1|A_2,1\rangle\ . \qquad (3.11)$$

To verify (over)completeness, calculate the resolution of unity. We find

$$\int|Z\rangle\langle Z|d\bar{\theta}\,d\theta\,dA=\pi\mathbb{1}\ . \qquad (3.12)$$

Note that the identity operator on the right-hand side acts in the space of even states only.

The supercoherent states above are defined via a generalization of the group-theoretic approach to ordinary coherent states. This in turn was an extension of the displacement-operator approach, the second definition of the harmonic-oscillator coherent states described in the Introduction. It is interesting to determine whether these states also satisfy generalizations of the first and third definitions.

The key to generalizing the first definition lies in the observation that for the harmonic oscillator, the first definition follows uniquely from the second. From the BCH relation, one has

$$e^{-\bar{\alpha}a}D(\alpha)e^{\bar{\alpha}a}=e^{-|\alpha|^2}D(\alpha)\ . \qquad (3.13)$$

Coupled with the definition $|\alpha\rangle=D(\alpha)|0\rangle$, this implies $e^{-\bar{\alpha}a}|\alpha\rangle=e^{-|\alpha|^2}|\alpha\rangle$. The first definition $a|\alpha=\alpha|\alpha\rangle$ follows directly.

For the supercoherent states, the supersymmetric displacement operator $D(A,\theta)$ is $T(g)$ in Eq. (3.6) with $B=0$. The analogue of Eq. (3.13) may be found using theorem 3 of Ref. 17:

$$e^{-\bar{A}a}D(A,\theta)e^{\bar{A}a}=e^{-\bar{A}A}D(A,\theta)\ ,\qquad e^{\bar{\theta}b}D(A,\theta)e^{-\bar{\theta}b}=e^{-\bar{\theta}\theta}D(A,\theta)\ . \qquad (3.14)$$

These relations yield two independent annihilation-operator conditions to be satisfied:

$$\bar{A}a|Z\rangle=\bar{A}A|Z\rangle,\quad \bar{\theta}b|Z\rangle=-\bar{\theta}\theta|Z\rangle\ . \qquad (3.15)$$

These provide the generalization of the first definition to supercoherent states. Further, direct calculation shows that the supercoherent states are eigenstates of a and b: $a|Z\rangle=A|Z\rangle$, $b|Z\rangle=-\theta|Z\rangle$.

For the third definition, it may be verified that the uncertainty $\Delta q\Delta p=\frac{1}{2}$ and that this relation is preserved in time, as before. The key point is that the expectation value of an even operator $O_E(A,a^{\dagger},t)$ in the supercoherent state $|Z\rangle$ reduces to the expectation value in the state $|Z\rangle$:

$$\langle Z|O_E|Z\rangle=\langle A|O_E|A\rangle\ . \qquad (3.16)$$

Thus, the supercoherent states are minimum-uncertainty coherent states, just as for the ordinary harmonic oscillator.

Another operator whose expectation value is of interest is the Hamiltonian H. We have $\langle Z|H|Z\rangle=(A\bar{A}-\theta\bar{\theta})$. There is a relation between the expectation values of the Hamiltonian, the supersymmetry generators Q and $Q^{\dagger}$, and the sum $a+b$ of the annihilation operators. We find

$$\langle Z|H|Z\rangle+\langle Z|Q+Q^{\dagger}|Z\rangle=\overline{\langle Z|a+b|Z\rangle}\langle Z|a+b|Z\rangle\ , \qquad (3.17)$$

which is a supersymmetric generalization of the relation $\langle\alpha|H|\alpha\rangle=\overline{\langle\alpha|a|\alpha\rangle}\langle\alpha|a|\alpha\rangle$ valid for the coherent states $|\alpha\rangle$ of the ordinary harmonic oscillator.

IV. ELECTRON IN A CONSTANT MAGNETIC FIELD

A. Basics

In this section, we consider a nonrelativistic spin-$\frac{1}{2}$ particle of mass M and charge e moving in a constant uniform magnetic field $\mathbf{B}=B\hat{\mathbf{z}}$. This system is relevant to the quantum Hall effect. It is known to provide a physical realization of supersymmetric quantum mechanics.[28,29]

At the quantum level, the relevant equation is the two-component Pauli equation, whose solutions of the form $e^{-iEt}\psi(\mathbf{r})$ satisfy the equation

$$H\psi\equiv\frac{1}{2M}[\boldsymbol{\sigma}\cdot(\mathbf{p}-e\,\mathbf{A})]^2\psi=E\psi\ . \qquad (4.1)$$

The vector potential $\mathbf{A}$ is defined in terms of $\mathbf{B}$ only up to a gauge transformation. A common choice is the planar gauge, $A_y=A_z=0$, $A_z=-By$. However, this breaks the natural cylindrical symmetry; also, the solutions have plane-wave behavior not only in the z coordinate but also in the y coordinate. We are interested in supercoherent states that are closest to classical. Since solutions of the corresponding classical problem have cylindrical symmetry, it is desirable to work with eigenstates that also possess this symmetry. Hence, we choose the cylindrical gauge

$$A_x = -\tfrac{1}{2}By, \quad A_y = \tfrac{1}{2}Bx \ . \tag{4.2}$$

To simplify further the discussion we consider the two-dimensional problem, effectively setting $p_z = 0$.

Equation (4.1) in the planar gauge describes a supersymmetric quantum-mechanical system.[28,29] To verify this property in the cylindrical gauge (4.2), we write the right-hand side of Eq. (4.1) in matrix form. Define the operators

$$\begin{aligned} D_- &= \partial_x + i\partial_y + \tfrac{1}{2}eB(x+iy) \ , \\ D_+ &\equiv D_-^\dagger = -\partial_x + i\partial_y + \tfrac{1}{2}eB(x-iy) \ , \\ [D_-, D_+] &= 2eB \ . \end{aligned} \tag{4.3}$$

Then, Eq. (4.1) becomes

$$2ME\psi = \begin{bmatrix} D_+D_- & \\ & D_-D_+ \end{bmatrix} \psi \ . \tag{4.4}$$

Renormalize D_+ and D_-,

$$a = \frac{1}{\sqrt{2eB}} D_- , \quad a^\dagger = \frac{1}{\sqrt{2eB}} D_+ \ , \tag{4.5}$$

and introduce a two-by-two matrix representation for the fermionic creation and annihilation operators:

$$b = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad b^\dagger = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} . \tag{4.6}$$

Then, Eq. (4.4) has the form of a supersymmetric quantum-mechanical system [cf. Eq. (3.4)]:

$$\hat{H}\psi \equiv (a^\dagger a + b^\dagger b)\psi = \hat{E}\psi \ , \tag{4.7}$$

where $\hat{H} = MH/eB$ and $\hat{E} = ME/eB$. Note that here the fermionic creation and annihilation operators $b^\dagger$ and b act as operators reversing the particle spin.

We can now carry over the whole apparatus of Sec. III, including the supercoherent states, to this physical system. However, to study supercoherent states in more detail, we need explicit eigenstates. The eigenvalue problem (4.4) in cylindrical coordinates can be reduced to a confluent hypergeometric equation, with two-component solutions labeled by two quantum numbers k and l. The non-negative-integer quantum number k is directly related to the eigenvalue $\hat{E}$ via $k = \hat{E} - \frac{1}{2}(|l| - l) - \delta_{\nu 1}$, where the upper and lower components of ψ are labeled by $\nu = 0$ and $\nu = 1$, respectively. The integer quantum number l labels degenerate eigenstates.

The explicit form of a unity-normalized solution $\psi = |k,l;\nu\rangle$ in polar coordinates is

$$|k,l;\nu\rangle = (-1)^k \left[\frac{eB}{2}\right]^{(|l|+1)/2} \left[\frac{k!}{\pi(k+|l|)!}\right]^{1/2} \times e^{il\varphi} r^{|l|} e^{-eBr^2/4} L_k^{(|l|)}(eBr^2/2) \begin{bmatrix} \delta_{\nu 0} \\ \delta_{\nu 1} \end{bmatrix} , \tag{4.8}$$

where $L_k^{(|l|)}(eBr^2/2)$ denotes a Sonine-Laguerre polynomial. The solutions are orthonormal:

$$\langle k_1, l_1; \nu_1 | k_2, l_2; \nu_2 \rangle = \delta_{k_1 k_2} \delta_{l_1 l_2} \delta_{\nu_1 \nu_2} \ . \tag{4.9}$$

The factor $(-1)^k$ is included in the expression for the wave function so that the operators

$$\begin{aligned} a^\dagger &= -\frac{1}{\sqrt{2eB}} e^{-i\varphi} \left[\partial_r - \frac{i}{r}\partial_\varphi - \frac{1}{2}eBr\right] , \\ a &= \frac{1}{\sqrt{2eB}} e^{i\varphi} \left[\partial_r + \frac{i}{r}\partial_\varphi + \frac{1}{2}eBr\right] \end{aligned} \tag{4.10}$$

can be identified with bosonic raising and lowering operators. We introduce for convenience an alternative set of quantum numbers $n = k + (|l| - l)/2$, $m = k + (|l| + l)/2$ and label states as $|n,m\rangle$. Then, the action of a and $a^\dagger$ changes the quantum number n of the eigenstates:

$$\begin{aligned} a^\dagger |n,m;\nu\rangle &= \sqrt{n+1}\, |n+1,m;\nu\rangle \ , \\ a|n,m;\nu\rangle &= \sqrt{n}\, |n-1,m;\nu\rangle \ . \end{aligned} \tag{4.11}$$

The operators a, $a^\dagger$, b, $b^\dagger$, and $\mathbb{1}$ do not form a complete set. Additional operators are required to distinguish between the degenerate eigenstates. These operators are

$$\begin{aligned} c^\dagger &= -\frac{1}{\sqrt{2eB}} e^{i\varphi} \left[\partial_r + \frac{i}{r}\partial_\varphi - \frac{1}{2}eBr\right] , \\ c &= \frac{1}{\sqrt{2eB}} e^{-i\varphi} \left[\partial_r - \frac{i}{r}\partial_\varphi + \frac{1}{2}eBr\right] . \end{aligned} \tag{4.12}$$

Direct calculation shows that they act as raising and lowering operators for the quantum number l:

$$\begin{aligned} c^\dagger |n,m;\nu\rangle &= \sqrt{m+1}\, |n,m+1;\nu\rangle \ , \\ c|n,m;\nu\rangle &= \sqrt{m}\, |n,m-1;\nu\rangle \ . \end{aligned} \tag{4.13}$$

Their commutator is $[c, c^\dagger] = 1$. They both commute with a, $a^\dagger$, b, $b^\dagger$. Thus, the eigenstates are labeled not only by the quantum numbers associated with the super Heisenberg-Weyl algebra but also with an additional degree of freedom.

B. Supercoherent states and physical interpretation

To construct the supercoherent states, we follow the procedure discussed in Sec. III. However, a modification is needed because the full symmetry superalgebra for the system (4.4) is larger than the super Heisenberg-Weyl algebra. Two procedures are possible. In one, supercoherent states are based on the super Heisenberg-Weyl algebra with generators a, $a^\dagger$, b, $b^\dagger$, and $\mathbb{1}$, i.e., with fixed quantum number m corresponding to the additional degree of freedom. In the other, the supercoherent states are obtained for the full superalgebra, including also the operators c and $c^\dagger$. We use the latter approach.

Since c and $c^\dagger$ have properties of harmonic-oscillator lowering and raising operators, coherent states with respect to them are canonical harmonic-oscillator states. Moreover, since c and $c^\dagger$ commute with the remaining generators, BCH relations involving these two groups of operators are immediate. Therefore, using Eq. (3.1) we can immediately write an explicit expression for a supercoherent state:

$$|Z\rangle = \exp(\tfrac{1}{2}\theta\bar{\theta})\exp\left[-\frac{|A|^2}{2}\right]\exp\left[-\frac{|C|^2}{2}\right] \times \sum_{n,m}\frac{A^n C^m}{\sqrt{n!}\sqrt{m!}}(|n,m;0\rangle + \theta|n,m;1\rangle) \ . \tag{4.14}$$

These states are parametrized by the three Grassmann-valued variables A, C, and θ. It can be verified that they have all the desirable properties of the harmonic-oscillator supercoherent states discussed in Sec. III.

Insight into the physical content of the supercoherent states can be gained by construction the expectation value of the Hamiltonian:

$$\langle Z|H|Z\rangle = \frac{eB}{M}(A\bar{A} - \theta\bar{\theta}) \ . \tag{4.15}$$

To understand the role of the Grassmann-valued variables in Eq. (4.15), consider a *classical* particle with mass M, charge e, and magnetic moment $\boldsymbol{\mu}$ moving in a field $\mathbf{B} = B\hat{\mathbf{z}}$. Part of the total energy is the interaction energy $-\boldsymbol{\mu}\cdot\mathbf{B}$. The eigensolutions of the two-component equation for the quantum problem may be viewed as corresponding to two groups of classical particles with opposite magnetic moments $\boldsymbol{\mu} = \pm(e/2M)\hat{\mathbf{z}}$. Therefore, the magnetic-moment interaction energy is represented by a matrix

$$U = -\frac{eB}{2M}\begin{bmatrix} 1 & \\ & -1 \end{bmatrix} \ . \tag{4.16}$$

The expectation value of this operator for the supercoherent state Z is

$$\langle Z|U|Z\rangle = -\frac{eB}{2M}(1 + 2\theta\bar{\theta}) \ . \tag{4.17}$$

Subtracting this value from the expectation value (4.15) of the total energy yields the expectation value of the energy for a charged particle without the magnetic moment:

$$\langle Z|H - U|Z\rangle = \frac{eB}{M}(A\bar{A} + \tfrac{1}{2}) \ . \tag{4.18}$$

Comparison with (4.15) shows the $\theta\bar{\theta}$ term is absent but an additional term $eB/2M$ appears.

The physical picture corresponding to the above discussion is as follows. Consider two groups of charged classical particles, each with mean energy $(eB/M)(A\bar{A} + \frac{1}{2})$. Equip the first with the magnetic moment $(e/2M)\hat{\mathbf{z}}$ and the second with the opposite moment. The energy of the first group decreases to $(eB/M)A\bar{A}$ while that of the second set increases to $(eB/M)(A\bar{A} + 1)$. This information is contained in the Grassmann-valued expectation value (4.15). Naively, one might expect the $A\bar{A}$ term to describe the mean energy of the first group and the $\theta\bar{\theta}$ term to describe that of the second. This is incorrect; the nilpotency of θ leads to a more complicated relationship. In fact, the $A\bar{A}$ term describes the mean energy of the first group but the $\theta\bar{\theta}$ term describes the energy *splitting* between the two groups.

It is natural to ask in what sense the states (4.14) are closest to classical. By construction, they are supercoherent states of the supersymmetric harmonic oscillator with respect both to a, $a^\dagger$ and c, $c^\dagger$. Therefore, they satisfy the minimum-uncertainty relation $\Delta q\,\Delta p = \frac{1}{2}$ with two sets of generalized coordinates and momenta, one defined via a and $a^\dagger$ and the other via c and $c^\dagger$. However, a, $a^\dagger$ and c, $c^\dagger$ are rather complicated operators.

Instead, let us consider the motion of a classical nonrelativistic charged particle in a constant uniform magnetic field. Assume the particle has no velocity component parallel to the field, corresponding to the earlier assumption $p_z = 0$. Then, the trajectory is a circle whose radius q is related to the energy $E_{\rm cl}$ by $2ME_{\rm cl} = (eB)^2 q^2$.

We now find analogs of $E_{\rm cl}$ and q^2 for the supercoherent states. Since the energy $E_{\rm cl}$ does not include the magnetic-moment interaction, its quantum-mechanical equivalent is the expectation value (4.18) of $H - U$. The equivalent of q^2 is the expectation value of r^2, which is

$$\langle Z|r^2|Z\rangle = \frac{2}{eB}(|A + \bar{C}|^2 + 1) \ . \tag{4.19}$$

Using these results, two subsets of supercoherent states can be identified for which a classical-type relation is valid. The first is obtained by taking the large-A limit:

$$\begin{aligned} &2M\langle Z|H|Z\rangle \to 2eB|A|^2 \ , \\ &(eB)^2\langle Z|r^2|Z\rangle \to 2eB|A|^2 \ . \end{aligned} \tag{4.20}$$

For this subset, the supercoherent states attain the classical behavior. The second subset corresponds to $C = 0$, i.e., supercoherent states constructed from the eigenstates with $m = 0$. In this case,

$$2M\langle Z|H - U|Z\rangle = (eB)^2\langle Z|r^2|z\rangle - eB \ ; \tag{4.21}$$

i.e., the energy is related to the radius in almost the same way as for the classical states. The only difference is the constant term $-eB$.

Equations (4.20) and (4.21) show that the evolution of supercoherent states belonging to these two subsets can be analyzed classically. Note that the subsets specified are not complete and therefore in general are insufficient to represent arbitrary operators or states. The first subset may be useful in the high-energy limit, to which it corresponds physically. The second subset is complete on the subspace with $m = 0$ and can be used as a basis there.

V. THE ELECTRON-MONOPOLE SYSTEM

The physical superalgebra

In this section, we consider a spin-$\frac{1}{2}$ particle of charge e and mass M in the presence of the field of a magnetic monopole. This system possesses a dynamical OSP(1/2) supersymmetry;[30] i.e., the superalgebra generators do not commute with the Hamiltonian but their total time derivatives $d/dt = \partial/\partial t + i[H,\ ]$ vanish.

The Hamiltonian is

$$H = \frac{1}{2M}(p_i - eA_i)^2 - \frac{e}{2M}B_i\sigma_i \ , \tag{5.1}$$

where $i=1,2,3$ and $B_i = gr_i/r^3$ is the magnetic field of the monopole. The particular form of the vector potential A_i is unimportant because only gauge-independent quantities appear.[38] The charges e and g satisfy the Dirac quantization condition.

The Cartan basis of generators R, $B_\pm$, $F_\pm$ for the dynamical superalgebra osp(1/2) is

$$R = \frac{1}{2k^2}K + \frac{k^2}{2}H, \quad B_\pm = \frac{1}{2k^2}K - \frac{k^2}{2}H \pm iD \ ,$$
$$F_\pm = \frac{1}{2k}S \mp \frac{ik}{2}Q \ . \tag{5.2}$$

Here, k is an arbitrary scale-fixing parameter and the explicitly time-dependent operators D, K, Q, S are given by

$$D = tH - \frac{M}{2}r_i\dot{r}_i, \quad K = -t^2H + 2tD + \frac{M}{2}r_i^2 \ ,$$
$$Q = \left[\frac{M}{2}\right]^{1/2}\dot{r}_i\sigma_i, \quad S = -tQ + \left[\frac{M}{2}\right]^{1/2}r_i\sigma_i \ , \tag{5.3}$$

where $\partial_i B_i = 0$. The operators R, $B_\pm$, and $F_\pm$ satisfy the nonvanishing graded commutation relations of osp(1/2):

$$\tfrac{1}{2}[B_-, B_+] = \{F_+, F_-\} = R \ ,$$
$$\pm[R, B_\pm] = \{F_\pm, F_\pm\} = B_\pm \ , \tag{5.4}$$
$$[F_\pm, B_\mp] = 2[R, F_\pm] = \pm F_\pm \ .$$

As occurs for the system considered in Sec. IV, the full invariance supergroup of H is actually larger than OSP(1/2). The Hamiltonian is also invariant under spatial rotations. The angular-momentum operators

$$J_i = M\epsilon_{ijk}r_j\dot{r}_k - eg\hat{r}_i + \frac{\sigma_i}{2} \tag{5.5}$$

commute with the generators of osp(1/2). Therefore, the full dynamical supersymmetry of H is $G_H = \mathrm{SO}(3) \times \mathrm{OSP}(1/2)$. There is one complication:[30] the osp(1/2) generators cannot be defined for states of lowest angular momentum because then the wave functions are nonzero at the origin where the condition $\partial_i B_i = 0$ is violated. The maximal symmetry for these states is so(2,1). We therefore consider here OSP(1/2) supercoherent states with fixed angular-momentum quantum numbers. This results in a family of supercoherent-state systems, distinguished by the eigenvalues $j(j+1)$ and m of J^2 and J_z, respectively. The eigenvalue ranges are $j = j_0, j_0+1, j_0+2, \ldots$ and $m = -j, -j+1, \ldots, j-1, j$, where $j_0 = |eg| - \frac{1}{2}$.

The generator R is a compact operator and has a discrete spectrum.[38] It is therefore more convenient to treat eigenstates of R rather than eigenstates of H, which is noncompact and has a continuous spectrum. The states will thus be labeled by the angular-momentum quantum numbers m and j, by the eigenvalue $\eta = \pm 1$ of the "fermion number" operator $\Xi_0 = i[Q,S] - \frac{1}{2}$, and by the eigenvalue $n = 0, 1, 2, \ldots$ of R.

The action of the superalgebra generators on these states is[30]

$$R|j,m;\eta,n\rangle = (\delta_{j,\eta} + n)|j,m;\eta,n\rangle \ ,$$
$$B_\pm|j,m,\eta,n\rangle = [(\delta_{j,\eta}+n)(\delta_{j,\eta}+n\pm 1) - \delta_{j,\eta}(\delta_{j,\eta}-1)]^{1/2}|j,m;\eta,n\pm 1\rangle \ , \tag{5.6}$$
$$F_\pm|j.m;\eta,n\rangle = [\tfrac{1}{2}(\delta_{j,\eta}+n) \pm \tfrac{1}{8} \pm \tfrac{1}{4}\eta d_j]^{1/2}|j,m;-\eta,n-\tfrac{1}{2}\eta \pm \tfrac{1}{2}\rangle \ .$$

Here, $\delta_{j,\eta} = \frac{1}{2}d_j - (\eta/4) + \frac{1}{2}$ and $d_j = \sqrt{j(j+1) - j_0(j_0+1)}$. The action of the fermion number operator is

$$\Xi_0|j,m;\eta,n\rangle = \eta d_j|j,m;\eta,n\rangle \ . \tag{5.7}$$

The connection between the eigenstate $|j,m;\eta,n\rangle$ of R and the eigenstates $|j,m;\eta,E\rangle$ of H is

$${}_H\langle j,m;\eta,E|j,m;\eta,n\rangle_R = \eta\left[\frac{\eta n!}{E\Gamma(2\delta_{j,\eta}+n)}\right]^{1/2}(2a^2E)^{\delta_{j,\eta}}e^{-a^2E}L_n^{2\delta_{j,\eta}-1}(2a^2E) \ . \tag{5.8}$$

We now fix the quantum numbers $j > j_0, m$ and for simplicity write the eigenstates of R as $|\eta,n\rangle$. In this subspace, the ground state is $|1,0\rangle$. It is annihilated by both B_- and F_-.

It can be verified that R is a Hermitian operator, that $B_+^\dagger = B_-$, and that $F_+^\dagger = F_-$. Therefore, in accordance with the discussion in Sec. II C, we use the following unitary operator to construct supercoherent states:

$$T(g) = \exp(aR + bB_+ - \bar{b}B_- + dF_+ + \bar{d}F_-) \ . \tag{5.9}$$

Using a BCH relation derived in Ref. 19, $T(g)$ can be rewritten as

$$T(g) = \exp(\beta' B_+)\exp(\delta' F_+)\exp(\gamma' B_-)$$
$$\times \exp(\epsilon' F_-)\exp(\alpha' R) \ , \tag{5.10}$$

where α', β', and δ' are given in terms of a, b, and d, by

$$\alpha' = -2\ln S + S^2 A_1 d\bar{d} \ , \tag{5.11}$$

$$\beta'=\frac{b}{K}S^{-1}\sinh K+\left[\frac{b}{2K^3}S^{-2}(\sinh K-K)\right]d\bar{d} , \quad (5.12)$$

$$\delta'=\frac{1}{S}\left[\frac{1}{K^2}\left[S+\frac{a}{2}\right]d+\frac{b}{K^2}(\cosh K-1)\bar{d}\right] . \quad (5.13)$$

The quantities K, S, $\dot{S}$, and A_1 are given by

$$K=(\tfrac{1}{4}a^2+b\bar{b}) , \quad (5.14)$$

$$S=\cosh K-\frac{a}{2K}\sinh K , \quad (5.15)$$

$$\dot{S}=K\sinh K-\frac{a}{2}\cosh K , \quad (5.16)$$

and

$$A_1=\frac{1}{K^2}S^{-2}\left[S^{-1}\left[1+\frac{2b\bar{b}}{a}\cosh K\right]-\frac{2K^2}{a}-1\right] , \quad (5.17)$$

The quantities ϵ' and γ' are also determined by the results of Ref. 19 but are not needed in the sequel.

B. The supercoherent states

To construct supercoherent states, we act with $T(g)$ given by Eq. (5.17) on the ground state $|1,0\rangle$. Since $|1,0\rangle$ is an eigenstate of R with eigenvalue $\delta_{j,1}$ and is annihilated by both F_- and B_-, a supercoherent state $|Z\rangle$ is

$$|Z\rangle=\exp(\alpha'\delta_{j,1})\left[\sum_{n=0}^{\infty}\frac{(\beta' B_+)^n}{n!}|1,0\rangle+\delta'\sum_{n=0}^{\infty}\frac{(\beta' B_+)^n}{n!}|-1,0\rangle\right] . \quad (5.18)$$

Using Eq. (5.6), we obtain the following expression for $|Z\rangle$:

$$|Z\rangle=\exp(\alpha'\delta_{j,1})\left[\sum_{n=0}^{\infty}\frac{\beta'^n}{n!}\left[\frac{n!\Gamma(2\delta_{j,1}+n)}{\Gamma(2\delta_{j,1})}\right]^{1/2}|1,n\rangle+\delta'\sum_{n=0}^{\infty}\frac{\beta'^n}{n!}\left[\frac{n!\Gamma(2\delta_{j,-1}+n)}{\Gamma(2\delta_{j,-1})}\right]^{1/2}|-1,n\rangle\right] . \quad (5.19)$$

Direct calculation shows that $\langle Z|Z\rangle=1$, as expected. One can also show that the osp(1/2) generators have the following expectation values in the supercoherent state $|Z\rangle$:

$$\begin{aligned}
&\langle Z|R|Z\rangle=(1+|\beta'|^2)W ,\\
&\langle Z|B_+|Z\rangle=2\bar{\beta}'W ,\\
&\langle Z|B_-|Z\rangle=2\beta'W ,\\
&\langle Z|F_+|Z\rangle=-(\bar{\delta}'+\bar{\beta}'\delta')W ,\\
&\langle Z|F_-|Z\rangle=-(\delta'+\bar{\delta}'\beta')W ,
\end{aligned} \quad (5.20)$$

where

$$W=|S^{-2}+A_1 d\bar{d}|^{2\delta_{j,1}}(1-|\beta'|^2)^{-(2\delta_{j,1}+1)} \times[1+\tfrac{1}{2}(2\delta_{j,1}+1)\bar{\delta}'\delta'(1-|\beta'|^2)^{-1}]\delta_{j,1} . \quad (5.21)$$

It can be shown that the supercoherent states are closest to classical in the sense of minimizing the dispersion (2.1). The point is that the quadratic Casimir operator C_2 for osp(1/2) is

$$C_2\propto R^2-\tfrac{1}{2}B_+B_--\tfrac{1}{2}B_-B_+-\tfrac{1}{2}F_+F_--\tfrac{1}{2}F_-F_+ \propto\tfrac{1}{4}(J^2-E^2g^2+\tfrac{1}{4}) . \quad (5.22)$$

This implies that all the states $|\eta,n\rangle$ are eigenstates of C_2 and hence that the supercoherent states are also eigenstates of C_2. However, it is known[39] that if an eigenstate of the Casimir operator is also an eigenstate of the operator $P=g^{jk}\langle X_j\rangle X_k$, then it has minimum dispersion in the sense of Eq. (2.1). For osp(1/2), we find that the supercoherent states are indeed eigenstates of P. Thus, they minimize the dispersion (2.1), as desired.

VI. DISCUSSION

In summary, this work develops a supersymmetric generalization of the idea of a coherent state. A general framework is presented for obtaining supercoherent states, based on an extension to supergroups of the group-theoretic method. Knowledge of the supermanifold structure of supergroups permits the construction of unitary supergroup representations. The use of Grassmann-valued quantities is central to the approach. The framework allows connections to physical systems.

The first explicit example considered here was the supersymmetric harmonic oscillator, where the relevant physical symmetry group is the supersymmetric extension of the Heisenberg-Weyl group. The supercoherent states were found and shown to have all the desirable features associated with the standard harmonic-oscillator coherent states.

To explore the physical meaning of these states, we studied a nonrelativistic electron in a constant uniform magnetic field. Appropriate combinations of the supercoherent states for the supersymmetric harmonic oscillator provide the supercoherent states for this system. These correspond physically to two groups of classical charged particles with an added energy due to oppositely oriented magnetic moments. The supercoherent states have closest-to-classical behavior in the sense that they minimize the product of uncertainties of the generalized momentum and position operators. In addition, certain subsets exhibit physical classical behavior in the sense that they satisfy a classical relation between the energy and the radius of the orbit.

Another system analyzed here was that of an electron in the field of a magnetic monopole. This system exhibits a dynamical OSP(1/2) supersymmetry. We used a BCH relation to determine the associated supercoherent states. These have the desirable properties of coherent states: they are unity normalized and are closest to classical in the sense that they minimize the dispersion defined in terms of the quadratic Casimir operator.

The role played by supersymmetry in the physical quantum-mechanical systems considered here is to provide a natural and group-theoretical means of incorporating fermionic spins and the Pauli principle. This role is comparable to those arising in other contexts. For example, the implementation of the Pauli principle through symmetry is one underlying feature of atomic supersymmetry.[25]

In the remainder of this section, we comment briefly on some other approaches to coherent states involving fermionic degrees of freedom.

Anticommuting quantities have been used to describe coherent states for the case of *purely* fermionic systems. For a single anticommuting fermionic degree of freedom ψ, fermionic coherent states are defined[40] as $|\psi\rangle \equiv e^{-\bar{\psi}\psi/2}(|0\rangle + |1\rangle\psi)$. These states are included as a subset of the supercoherent states for the supersymmetric oscillator, Eq. (3.10), by taking $A=0$ and $\psi=-\theta$. Thus, fermionic coherent states can also be treated within the framework of supergroup theory. This feature and the structure of these special supercoherent states are consequences of the simplicity of the super Heisenberg-Weyl algebra, Eq. (3.2).

A different construction of coherent states for the supersymmetric harmonic oscillator *without* Grassmann variables was proposed in Ref. 41. In this approach, supercoherent states are defined as eigenstates of the supersymmetric annihilation operator. They are linear combinations of the eigenstates of H but with complex-number coefficients for both towers of states. These states contain a canonical subset of purely bosonic coherent states. However, the fermionic contribution of our supercoherent states is quite different. In particular, our states also satisfy both the minimum-uncertainty and the group-theoretic definitions.

Coherent states for OSP(1/2) of a similar form to Eq. (5.18) have been considered in Ref. 42. These are a subset of the supercoherent states considered here. This can be seen as follows. Since the BCH relation (5.10) was unavailable, no connection could be made in Ref. 42 between the coherent states and the unitary supergroup operator $T(g)$. Therefore, the parameter α' was taken to be a complex number rather than Grassmann valued. When the parameters a, b, and d in Eq. (5.9) are constrained such that α' in Eq. (5.18) is pure body, the coherent states of Ref. 42 result. Another consequence of the absence of unitarity is the need to introduce explicitly a normalization factor to obtain unity-normalized states. Also, while the supercoherent states (5.18) have the simple integration measure $da\, db\, d\bar{b}\, dd\, d\bar{d}$, the integration measure for the states defined in Ref. 42 is not automatically given and requires calculation.

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[1]See, for example, J. R. Klauder and B. -S. Skagerstam, *Coherent States* (World Scientific, Singapore, 1985); W. -M. Zhang, D. H. Feng, and R. Gilmore, Rev. Mod. Phys. **62**, 867 (1990).
[2]E. Schrödinger, Naturwiss. **14**, 664 (1926).
[3]J. R. Klauder, Ann. Phys. (N.Y.) **11**, 123 (1960); J. Math. Phys. **4**, 1055 (1963); **4**, 1058 (1963).
[4]R. J. Glauber, Phys. Rev. **130**, 2529 (1963); **131**, 2766 (1964).
[5]E. C. G. Sudarshan, Phys. Rev. Lett. **10**, 227 (1963).
[6]A. M. Perelomov, Commun. Math. Phys. **26**, 222 (1972); *Generalized Coherent States and Their Applications* (Springer-Verlag, Berlin, 1986).
[7]M. M. Nieto, *Coherent States* (Ref. 1), p. 429; M. M. Nieto and L. M. Simmons, Jr., Phys. Rev. D **20**, 1321 (1979).
[8]J. E. Campbell, Proc. London Math. Soc. **28**, 381 (1987).
[9]H. F. Baker, Proc. London Math. Soc. **34**, 347 (1902); **35**, 33 (1903); **2**, 293 (1905).
[10]F. Hausdorff, Ber. Verh. Saechs. Akad. Wiss. Leipzig Math. Phys. Kl. **58**, 19 (1906).
[11]R. M. Wilcox, J. Math. Phys. **8**, 962 (1967).
[12]B. Mielnik, Ann. Inst. Henri Poincaré A **12**, 215 (1970).
[13]R. Gilmore, *Lie Groups, Lie Algebras, and Some of Their Applications* (Wiley, New York, 1974).
[14]A. Rogers, J. Math. Phys. **21**, 1352 (1980).
[15]A. Rogers, J. Math. Phys. **22**, 443 (1981).
[16]A. Rogers, J. Math. Phys. **22**, 939 (1981).
[17]V. A. Kostelecký, M. M. Nieto, and D. R. Truax, J. Math. Phys. **27**, 1419 (1986).
[18]V. A. Kostelecký and D. R. Truax, J. Math. Phys. **28**, 2480 (1987).
[19]B. W. Fatyga, V. A. Kostelecký, and D. R. Truax, J. Math. Phys. **30**, 291 (1989).
[20]H. Nicolai, J. Phys. A **9**, 1497 (1976).
[21]E. Witten, Nucl. Phys. **B185**, 513 (1981).
[22]P. Salomonson and J. W. Van Holten, Nucl. Phys. **B196**, 509 (1982).
[23]M. De Crombrugghe and V. Rittenberg, Ann. Phys. (N.Y.) **151**, 99 (1983).
[24]L. F. Urrutia and E. Hernández, Phys. Rev. Lett. **51**, 755 (1983).
[25]V. A. Kostelecký and M. M. Nieto, Phys. Rev. Lett. **53**, 2295 (1984); Phys. Rev. A **32**, 1293 (1985); **32**, 3243 (1985); V. A. Kostelecký, M. M. Nieto, and D. R. Truax, *ibid*, **38**, 4413 (1988).
[26]B. Freedman and F. Cooper, in *Supersymmetry in Physics*, edited by V. A. Kostelecký and D. K. Campbell (North-Holland, Amsterdam, 1985).
[27]V. A. Kostelecký, M. M. Nieto, and D. R. Truax, Phys. Rev. D **32**, 2627 (1985).
[28]R. Jackiw, Phys. Rev. D **29**, 2375 (1984).

[29]R. J. Hughes, V. A. Kostelecký, and M. M. Nieto, Phys. Lett. B **171**, 226 (1986); Phys. Rev. D **34**, 1100 (1986).
[30]E. D'Hoker and L. Vinet, Commun. Math. Phys. **97**, 391 (1985).
[31]F. A. Berezin, *The Method of Second Quantization* (Academic, New York, 1966).
[32]B. DeWitt, *Supermanifolds* (Cambridge University Press, Cambridge, England, 1984).
[33]F. A. Berezin, *Introduction to Superanalysis* (Reidel, Dordrecht, 1987).
[34]Alternatively, one can take the generators as real: $\beta_a = \overline{\beta_a}$. See V. A. Kostelecký and J. M. Rabin, J. Math. Phys. **25**, 2744 (1984).
[35]Several definitions of supermanifolds and supergroups exist. See, for example, the discussion and references in Ref. 19.
[36]D. R. Truax, V. A. Kostelecký, and M. M. Nieto, J. Math. Phys. **27**, 354 (1986).
[37]I. Bars and M. Günaydin, Commun. Math. Phys. **91**, 31 (1983).
[38]R. Jackiw, Ann. Phys. (N.Y.) **129**, 183 (1980).
[39]R. Delbourgo and J. R. Fox, J. Phys. A **10**, L233 (1977).
[40]Y. Onuki and T. Kashiwa, Prog. Theor. Phys. **60**, 449 (1978).
[41]C. Aragone and F. Zypman, J. Phys. A **19**, 2267 (1986).
[42]A. B. Balantekin, H. A. Schmitt, and B. R. Barrett, J. Math. Phys. **29**, 1634 (1988).