QED test of a Bethe-Salpeter solution method

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A method is described of solving Bethe-Salpeter equations for spin-$\frac{1}{2}$ particles using a standard quasipotential approximation with a correction series. A three-dimensional, 16-component bound-state equation is obtained whose form is different from the Breit equation. The equation has two equivalent first-order forms which show each particle obeying a Dirac equation in the presence of the other. The formalism also reconstitutes the four-dimensional bound-state vertex function from the three-dimensional wave function. The method is tested on positronium and the hydrogen atom, using a single-photon-exchange kernel in the Coulomb gauge. In the bound-state equation, in addition to the Coulomb potential and Breit interaction, the formalism gives a subtracted box potential $(1/2E)\alpha^2/r^2$ where E is the mass of the bound state. With this term included, the energy levels (fine structure and hyperfine structure) are correct to order α^4. The reconstituted bound-state vertex function, when substituted in Feynman triangle diagrams, gives the lowest-order atomic dipole transition amplitudes correctly.

I. INTRODUCTION

In this paper we describe and test a formally exact method of solving Bethe-Salpeter (BS) equations[1,2] by mapping the equation from four-space down to three-space, solving the resulting equation, and mapping the solution back to four-space. The method is based on a quasipotential approximation with a correction series.

In the quark model of mesons, it is still undecided what is the best type of equation and interaction potential to use.[3,4] The primary application of the formalism defined in this paper is intended to be the quark model. In the present paper we aim only to derive the method and to test it on two-body QED systems. For positronium and the hydrogen atom, using a single-photon-exchange BS kernel in the Coulomb gauge, we verify that the lowest-order (α^4) fine structure (fs) and hyperfine structure (hfs), as well as the lowest-order bound-state transition amplitudes, come out correctly.

First we review briefly some well-known three-dimensional reductions of the BS equation.

(i) The earliest approach was that of Salpeter,[5] applied by him to hydrogen, and also used by Karplus and Klein[6] on positronium. This method separates the BS kernel into an "instantaneous" Coloumb term, which causes the binding, plus a remainder which is used in perturbation theory.

(ii) A second way is the quasipotential approach, which starts from the fact that the pair of bound particle propagators has a sharp peak in the relative energy variable (for bound states with small binding energy, the usual case). This peak is approximated by a δ function in relative energy, and perturbation theory uses the remainder of this approximation. (a) The oldest work in which the δ function is explicitly expressed seems to be that of Blankenbecler and Sugar;[7] in that paper equal-mass interacting particles were used and the coefficient of the δ function contains a square-root kinematic function in the denominator. (b) A second, simpler type of approximation due to the present author[8] and to Todorov[9] allowed unequal-mass particles and contained no square-root factor in the denominator. (c) A delta function in relative energy is also used by Gross,[10] but with the different objective of putting one bound particle on its mass shell.

We also mention some ways of obtaining bound-state equations using assumptions extraneous to the BS equation.

(iii) Rizov and Todorov,[11] treating unequal-mass particles, start from the Todorov approximation,[9] but use it only to set a constant of proportionality in the identification of a first Born approximation with the first-order interaction terms in an assumed form of the Klein-Gordon equation [Ref. 11, Eq. (185) ff.].

(iv) Sazdjian[12] uses the constraints of quantized relativistic Hamiltonian dynamics to obtain an independent Dirac equation for each particle, with a common interaction between them derived from compatibility conditions. A combination of these equations gives a bound-state eigenvalue equation.

(v) Crater and van Alstine[13] also start from quantized relativistic Hamiltonian dynamics with single-particle constraints and compatibility requirements, but obtain bound-state equations in a different form from Sazdjian's.

(vi) Finally, a common prescription to obtain lowest-order fs and hfs splittings is simply to assume the interaction potential to be the matrix element of the appropriate lowest-order kernel between normalized free-particle spinors (first Born approximation).[14] Using single-photon exchange and the relativistic kinetic-energy operator in the Schrödinger equation, this method gives the QED fs and hfs to order α^4 correctly.

Category (ii)(b) above is most relevant to the present paper. In this category we include the following work. (1) Caswell and Lepage[15] used Todorov's approximation[9]

on positronium and muonium, choosing the numerator of the fermion propagators to be the fermions' positive-energy projection operators. (2) Bodwin, Yennie, and Gregorio[16] use Todorov's approximation on the hydrogen atom, allowing the δ function in relative energy to determine the fermion propagators' numerators. However, they simplify the proton numerator to a constant operator. (3) The present paper, of which Secs. II and III are taken from an unpublished thesis,[8] again uses Todorov's approximation allowing the δ function to determine the numerators, but the masses are kept general and neither numerator is approximated. An overview of this paper follows.

In Sec. II, we give an independent derivation of Todorov's quasipotential approximation using a transformation of Cutkosky.

Section III lays out in a standard fashion, for a pair of spin-$\frac{1}{2}$ particles, the "4→3→4"-dimensional sequence of mappings, which are determined by an operator which can be expanded in a series in the error of the quasipotential approximation. The resulting three-dimensional bound-state equation contains no Casimir projection operators. This second-order bound-state equation is also expressed in two equivalent first-order Dirac-like forms, which show each particle simultaneously obeying a Dirac equation in the presence of the other. Finally the four-dimensional solutions are expressed in terms of the three-dimensional ones using the mapping operator.

Section IV successfully tests the 4→3 part of the formalism by calculating the energy levels of the bound-state equation for positronium and the hydrogen atom to order α^4 using a Coulomb-gauge single-photon-exchange kernel. Relativistic kinematics are automatic and there are no kinetic-energy operator ($\mathbf{p}^4$) corrections. Instead, we find the presence of a potential term $(1/2M_B)\alpha^2/r^2$ (where M_B is the bound-state mass), proportional to the square of the binding potential.

Section V deals with various technical points which arise in Sec. IV: the existence of solutions; small radius; one mass large; iteration of the transverse term in the kernel.

Section VI contains a successful test of the 3→4 mapping, by verifying that the four-dimensional vertex functions reconstructed from the three-dimensional wave functions do give low-energy transition probabilities correctly when substituted in Feynman triangle diagrams.

Section VII contains brief conclusions.

II. ORIGIN OF QUASIPOTENTIAL APPROXIMATION

In this section we define notation and give a derivation from Ref. 8 of the quasipotential approximation to be used in this paper, using the massless-scalar exchange BS equation treated by Wick[17] and Cutkosky[18] as a model.

Consider two scalar particles of masses m,M subject to an attractive interaction due to the exchange of a massless scalar particle. Let the m particle (so named for short) have initial four-momentum $xK+q$ and final four-momentum $xK+p$; let the M particle have initial and final four-momenta $XK-q, XK-p$. K is the total four-momentum of the system, and $x+X=1$. The BS equation for the particles' truncated scattering Green's function $T(p,q;K)$ is

$$T(p,q;K)=\frac{-g^2}{(p-q)^2}+\frac{1}{i}\int\frac{d^4l}{(2\pi)^4}\frac{-g^2}{(p-q)^2}\left[-\frac{2m}{(xK+l)^2+m^2-i\epsilon}\frac{2M}{(XK-l)^2+M^2-i\epsilon}\right]T(l,q;K)\ . \tag{1}$$

(The Feynman diagram is represented by $-iT$.) The factors $2m,2M$ are put in the numerators of the particle propagators instead of in the coupling constant in order to preserve the analogy with the spin-$\frac{1}{2}$ case, where the rationalized propagators at low energies have numerators which approximate $2m$ times an idempotent projection operator. The coupling constant g has the dimensions of e.

We shall use the c.m. system $K=(\mathbf{0},E)$, with E confined to the bound-state region $|M-m|<E<M+m$. With the "bound-state wave number" β defined by $E=(m^2-\beta^2)^{1/2}+(M^2-\beta^2)^{1/2}$, we can further define

$$t=(m^2-\beta^2)^{1/2},\quad T=(M^2-\beta^2)^{1/2}\ ,$$
$$x=t/E,\quad X=T/E,\quad E=t+T\ . \tag{2}$$

Then for example the initial m,M particles carry energies $t+q^0$, $T-q^0$.

The quasipotential approximation to be used in this paper is

$$-\frac{1}{(xK+l)^2+m^2-i\epsilon}\frac{1}{(XK-l)^2+M^2-i\epsilon}\approx-\frac{2\pi i\delta(l^0)}{2E(\mathbf{l}^2+\beta^2)}\ . \tag{3}$$

We now give a derivation of this simple approximation, which represents the well-known peak in relative energy of the pair of particle propagators. This type of approximation was suggested many years ago independently by the present author[8] and by Todorov.[9] In Ref. 8 we substituted $(m+M)$ for E in (3); in the present paper we follow Todorov, who proposed (3) as written.

Following Cutkosky, the relative energy in (1) is Wick-rotated to $l^0=il_4$, with $d^4l=d\mathbf{l}\,dl^0=id\mathbf{l}\,dl_4$, $l^2=\mathbf{l}^2-(l^0)^2=\mathbf{l}^2+l_4^2$. The flat Euclidean four-space $\{\mathbf{p},p_4\}$ is then projected onto the surface of a unit sphere in five-dimensional space $\{u\}$ by means of the transformations $u_i=2\beta p_i/(p^2+\beta^2)$ $(i=1,2,3,4)$, $u_5=(p^2-\beta^2)/(p^2+\beta^2)$. Then this surface is projected back

onto a four-dimensional flat space $\{\xi\}$ by means of the transformations $\xi_\alpha = u_\alpha/(1-u_4)$ ($\alpha=1,2,3,5$). Symbolizing this transformation by $p\to u\to\xi$, we also define $l\to w\to\zeta$, $q\to v\to\eta$.

With the definition

$$T_c(\xi,\eta;E)=\frac{\beta(1-u_4)}{1-u_5}T(p,q;K)\frac{\beta(1-v_4)}{1-v_5}$$

and the easily derived relation $dl\,dl_4=\beta^4[(1-w_4)/(1-w_5)]^4 d^4\zeta$, Eq. (1) becomes[18]

$$T_c(\xi,\eta;E)=\frac{-g^2}{(\xi-\eta)^2}+\int\frac{d^4\zeta}{(2\pi)^4}\frac{-g^2}{(\xi-\zeta)^2}\frac{-4}{\omega\bar{\Omega}(\zeta^2-\bar{\omega}^2)(\zeta^2-\Omega^2)}\times T_c(\zeta,\eta;E)\ , \tag{4}$$

which is manifestly rotationally invariant in four-dimensional ξ space. Here $\omega=(t+i\beta)/m$, $\Omega=(T+i\beta)/M$, and $|\omega|=|\Omega|=1$.

Now assuming that the energy E is near the threshold $m+M$, β is small compared to m,M and the two complex poles in ζ^2 are slightly above and below unity. This makes the function of ζ^2 in the integrand peak sharply in a Lorentz shape near $\zeta^2=1$, behaving rather like a δ function of (ζ^2-1). Estimating the area under this peak by using the residue at either pole, and noting that $\omega\Omega-\bar{\omega}\bar{\Omega}=2i\beta E/mM$, the term in the integrand has the approximate behavior[8]

$$\frac{-4}{\omega\bar{\Omega}(\zeta^2-\bar{\omega}^2)(\zeta^2-\Omega^2)}\approx -4\pi\frac{mM}{\beta E}\delta(\zeta^2-1)\ . \tag{5}$$

The quasipotential approximation (3) follows from (5) by reversing the transformations.

With the substitution of (5) into (4), the integral in (4) becomes an angular integral restricted to the unit sphere $\zeta^2=1$. The variables ξ,η in (4) can be restricted to the same unit sphere and denoted by $\hat{\xi},\hat{\eta}$. Furthermore, mM/E equals the reduced mass μ to a good approximation. Then (4) becomes an equation on the unit sphere:

$$T_c(\hat{\xi},\hat{\eta};E)=\frac{-g^2}{(\hat{\xi}-\hat{\eta})^2}-\frac{4\pi\mu}{\beta}\int\frac{d^3\hat{\zeta}}{(2\pi)^4}\frac{-g^2}{(\hat{\xi}-\hat{\zeta})^2}T_c(\hat{\zeta},\hat{\eta};E)\ . \tag{6}$$

Since

$$1/(\hat{\xi}-\hat{\eta})^2=\sum_{Nlm}[2\pi^2/(N+1)]Y_{Nlm}(\hat{\xi})\bar{Y}_{Nlm}(\hat{\eta})\ ,$$

where Y_{Nlm} are four-dimensional orthonormal surface harmonics and $N\geq l\geq m\geq 0$, the solution to Eq. (6) can be written down by inspection. It is an infinite sum of pole terms. These poles, representing bound states, occur at $\beta=\alpha\mu/(N+1)$, where $\alpha=g^2/4\pi$. Recalling that the binding energy $B=m+M-E$ is $\beta^2/2\mu$ to the lowest order, the energy levels of the bound states are the Bohr levels, as expected.

Equation (6) is Schwinger's transformation[19] of the inhomogeneous Coulomb Schrödinger equation (writing e for g)

$$T(\mathbf{p},\mathbf{q};E)=\frac{-e^2}{(\mathbf{p}-\mathbf{q})^2}+\int\frac{dl}{(2\pi)^3}\frac{-e^2}{(\mathbf{p}-\boldsymbol{l})^2}\frac{-2\mu}{l^2+\beta^2}T(\boldsymbol{l},\mathbf{q};E)\ . \tag{7}$$

For $\xi^2=1$ implies $u_4=0$, giving the coordinate transformation

$$\hat{\xi}_i=\frac{2\beta p_i}{\mathbf{p}^2+\beta^2},\quad i=1,2,3;\quad \hat{\xi}_5=\frac{\mathbf{p}^2-\beta^2}{\mathbf{p}^2+\beta^2}\ ;$$

with which Schwinger obtained (6) from (7).

Cutkosky's unusual O(4)-symmetric space has given a direct derivation of the quasipotential approximation (3), as well as Schwinger's expression for the Coulomb Schrödinger equation and the Bohr levels. We hope this derivation has some historical interest. In the following sections we will use (3) to establish a perturbation formalism and test the formalism on QED.

III. "4→3→4"-DIMENSIONAL FORMALISM

In this section (from the unpublished Ref. 8) we shall lay out a standard type of formalism (see, e.g., Refs. 7 and 15) for the BS equation for spin-$\frac{1}{2}$ particles, based on the quasipotential approximation (3).

The momenta and notation of Sec. II are used, with Dirac indices added to the scattering Green's function T and kernel I. Denote the Dirac matrices of the m,M particles by γ^μ,Γ^μ. Instead of $2m,2M$ the numerators of the rationalized propagators are spinors. The BS equation is

$$T(p,q;K)=I(p,q;K)+\frac{1}{i}\int\frac{d^4l}{(2\pi)^4}I(p,l;K)S(l;K)T(l,q;K)\ , \tag{8}$$

where

$$S(l;K)\equiv-\frac{m-\gamma\cdot(xK+l)}{(xK+l)^2+m^2-i\epsilon}\frac{M-\Gamma\cdot(XK-l)}{(XK-l)^2+M^2-i\epsilon}\ . \tag{9}$$

Now, using the c.m. frame $K=(\mathbf{0},E)$ and (3), we have

$$S(l;K)\approx S_0(l;K)=\hat{S}_0(l;E)2\pi i\delta(l^0)\ , \tag{10}$$

where

$$\hat{S}_0(l;E)\equiv-\frac{(m-\boldsymbol{\gamma}\cdot\boldsymbol{l}+\gamma^0 t)(M+\boldsymbol{\Gamma}\cdot\boldsymbol{l}+\Gamma^0 T)}{2E(l^2+\beta^2)}\ . \tag{11}$$

Recall from (2) that t,T are the particles' individual bound-state energies at zero relative energy.

It is now straightforward to derive a method using the error of the approximation (10). Denote the operation $\int d^4l/i(2\pi)^4$ by $*$, and $i(2\pi)^4\delta^4(p-q)$ by **1**, and (10) by S_0. Then (8) can be written $T=I+I*ST$, and we can successively define

$$R=S-S_0\ , \tag{12}$$

$$U=I+I*RU\ , \tag{13}$$

$$W = S_0 1 + S_0 T S_0 \ . \tag{14}$$

From these it follows that

$$W = S_0 1 + S_0 U * W \ . \tag{15}$$

Evidently R is the error of the approximation (10), and U appears as powers of this error when Eq. (13) (from Blankenbecler and Sugar[7]) is evaluated by iteration. By iteration of (15) it is clear that

$$W(p,q;K) = 2\pi i \delta(p^0) \hat{W}(\mathbf{p},\mathbf{q};E) 2\pi i \delta(q^0) \ . \tag{16}$$

It follows that the equation (15) for W is effectively only three-dimensional; it is equivalent to

$$\hat{W}(\mathbf{p},\mathbf{q};E) = \hat{S}_0(\mathbf{p};E)(2\pi)^3 \delta^3(\mathbf{p}-\mathbf{q}) + \hat{S}_0(\mathbf{p};E) \int \frac{dl}{(2\pi)^3} \hat{U}(\mathbf{p},l;E) \hat{W}(l,\mathbf{q};E) \ , \tag{17}$$

in which $\hat{U}(\mathbf{p},\mathbf{q};E) \equiv U(\hat{p},\hat{q};K)$, $\hat{p} \equiv (\mathbf{p},0)$, $\hat{q} \equiv (\mathbf{q},0)$, $K = (\mathbf{0},E)$ in the c.m. system. Equation (17) is an exact three-dimensional equation for the function $\hat{W}(\mathbf{p},\mathbf{q};E)$ in terms of the function $\hat{U}(\mathbf{p},\mathbf{q};E)$. The inhomogeneous Schrödinger equation (17) is defined on the plane of zero relative energy in the c.m. system, relative energy being offset as in Sec. II.

It is well known that at (nondegenerate) bound-state poles of the scattering Green's function T, the residue factorizes:

$$T(p,q;K) \underset{s \to M_B^2}{\sim} \frac{\Gamma(p;K)\bar{\Gamma}(q;K)}{s - M_B^2} \ , \tag{18}$$

where $s = -K^2$ and M_B is the bound-state mass. From Eq. (14), it follows that the poles of $\hat{W}$ factorize similarly:

$$\hat{W}(\mathbf{p},\mathbf{q};E) \underset{s \to M_B^2}{\sim} \frac{\psi(\mathbf{p};M_B)\bar{\psi}(\mathbf{q};M_B)}{s^2 - M_B^2} \ , \tag{19}$$

where $s = E^2$ since (19) is in the c.m. frame.

From (11), (17), and (19), $\psi(\mathbf{p};M_B)$ evidently obeys the homogeneous equation

$$(\mathbf{p}^2 + \beta^2)\psi(\mathbf{p};M_B) = -\frac{(m - \boldsymbol{\gamma}\cdot\mathbf{p} + \gamma^0 t)(M + \boldsymbol{\Gamma}\cdot\mathbf{p} + \Gamma^0 T)}{2M_B} \times \int \frac{dl}{(2\pi)^3} \hat{U}(\mathbf{p},l,M_B)\psi(l;M_B) \ . \tag{20}$$

Equation (20) is the fundamental bound-state equation which we shall work with in this paper. The function $\hat{U}(\mathbf{p},\mathbf{q};E)$ can be regarded as a potential. Once $\hat{U}$ is calculated from Eq. (13) to the required accuracy, Eq. (20) can be solved to obtain energy levels using the conventional techniques of quantum mechanics.

Each of the first-order operators on the right side of (20) is nonsingular in the bound-state region. Either of them can be exactly divided into the second-order operator on the left side of the equation. Therefore there exists a pair of first-order equations equivalent to (20), and to each other:[8]

$$(M + \boldsymbol{\gamma}\cdot\mathbf{p} - \gamma^0 t)\psi(\mathbf{p};M_B) = -\frac{M + \boldsymbol{\Gamma}\cdot\mathbf{p} + \Gamma^0 T}{2M_B} \int \frac{dl}{(2\pi)^3} \hat{U}(\mathbf{p},l;M_B)\psi(l;M_B) \ , \tag{21}$$

$$(M - \boldsymbol{\Gamma}\cdot\mathbf{p} - \Gamma^0 T)\psi(\mathbf{p};M_B) = -\frac{m - \boldsymbol{\gamma}\cdot\mathbf{p} + \gamma^0 t}{2M_B} \int \frac{dl}{(2\pi)^3} \hat{U}(\mathbf{p},l;M_B)\psi(l;M_B) \ . \tag{22}$$

Equations (21) and (22) could be interpreted as indicating that each particle is obeying a Dirac equation in the field of the other.

Equations (21) and (22) are reminiscent of a pair of independent single-particle Dirac equations with a common interaction found by Sazdjian from quantized relativistic Hamiltonian dynamics [Ref. 12, Eqs. (6.5a) and (6.25b)]. However, the two Dirac-like equations (21) and (22) are not independent. Equations (20), (21), and (22) are the *same* equation, expressed in three different ways.

The full four-dimensional solution of the BS equation (8) can be reconstituted from the three-dimensional solution W using

$$T = U + U * W * U \ , \tag{23}$$

which is easily deduced from Eqs. (12)–(15). From Eqs. (18), (19), and (23), it follows that the exact bound-state vertex function is reconstituted (up to sign) from the three-dimensional wave function by

$$\Gamma(p;K_B) = -\int \frac{dl}{(2\pi)^3} U(p,\hat{l};K_B)\psi(l;M_B) \ , \tag{24}$$

where in the c.m. frame $\hat{l} = (l,0)$ and $K_B = (\mathbf{0},M_B)$. The normalization of $\psi(\mathbf{p};M_B)$ is determined by (19). Since Γ is a four-dimensional vertex function, once it is calculated by (24) it can be used directly in the calculation of bound-state transition amplitudes from Feynman diagrams, as will be done in Sec. VI.

The function U given by Eq. (13) plays two different roles. When specialized to the zero-relative-energy plane, it serves as an interaction potential in the three-dimensional Eqs. (17) and (20). In Eqs. (23) and (24) U serves as the means to reconstitute the full four-dimensional solution of the BS equation from the three-dimensional solution. The value of the method of this paper stands or falls upon the simplicity and speed of the convergence of the series (13) for U in terms of the correction R to the quasipotential approximation (8). In the next section we shall examine the QED problem to fourth order in the Coulomb gauge.

IV. TEST OF 4→3 MAPPING: QED BOUND-STATE ENERGIES

In this section we test the formalism of Sec. III by using the three-dimensional equation (20) to calculate the energy levels of the QED BS equation with a single-

photon-exchange kernel in the Coulomb gauge. This kernel is known[5] to give energy levels correctly to order α^4.

Assuming opposite charges $-e+e$, the kernel will be

$$I(p,q;K)=I_C+I_T\ , \tag{25}$$

where

$$I_C\equiv -e^2\frac{\gamma^0\Gamma^0}{\mathbf{k}^2}\ , \tag{26}$$

$$I_T\equiv +e^2\frac{1}{k^2}(\boldsymbol{\gamma}\cdot\boldsymbol{\Gamma}-\boldsymbol{\gamma}\cdot\hat{\mathbf{k}}\boldsymbol{\Gamma}\cdot\hat{\mathbf{k}})\ , \tag{27}$$

with $k=p-q$, $k^2=\mathbf{k}^2-(k^0)^2$, and $\hat{\mathbf{k}}\equiv\mathbf{k}/|\mathbf{k}|$.

In the method of Salpeter,[5] in which Casimir energy-projection operators naturally emerge, first-order perturbation theory on the terms I_C and I_T alone are sufficient to give results correct to order α^4. In the quasipotential formalism we must allow for the possibility of a next-order correction to the potential $\hat{U}$ in the homogeneous equation (20). From Eq. (13) the correction to U would be

$$I*RI=I_C*RI_C+I_C*RI_T+I_T*RI_C+I_T*RI_T\ . \tag{28}$$

In Sec. V we shall show that the last three terms of (28) do not contribute to fourth order.

The first term

$$U_{\rm CC}=I_C*RI_C=I_C*SI_C-I_C*S_0I_C \tag{29}$$

is easily calculable, since I_C does not contain k^0. One finds

$$\hat{U}_{\rm CC}(\mathbf{p},\mathbf{q};E)=\int\frac{dl}{(2\pi)^3}\frac{e^2\gamma^0\Gamma^0}{(\mathbf{p}-\boldsymbol{l})^2}\hat{R}(\boldsymbol{l};E)\frac{e^2\gamma^0\Gamma^0}{(\boldsymbol{l}-\mathbf{q})^2}\ , \tag{30}$$

where

$$\hat{R}(\boldsymbol{l};E)=\frac{-1}{4EL^2}[(nN+L^2\gamma^0\Gamma^0)(\sin\theta_t+\sin\theta_T) -2nN+L(\gamma^0N-\Gamma^0n)(\cos\theta_t-\cos\theta_T)] \tag{31}$$

in which, referring to Eq. (2),

$$L^2\equiv\boldsymbol{l}^2+\beta^2;\quad \tan\theta_t=\frac{t}{L},\quad \tan\theta_T=\frac{T}{L}\ ;$$
$$n\equiv m-\boldsymbol{\gamma}\cdot\boldsymbol{l}+\gamma^0t,\quad N\equiv M+\boldsymbol{\Gamma}\cdot\boldsymbol{l}+\Gamma^0T\ . \tag{32}$$

To leading order, neglecting $\boldsymbol{\gamma}\cdot\boldsymbol{l},\boldsymbol{\Gamma}\cdot\boldsymbol{l}$, higher orders in $(\boldsymbol{l}^2+\beta^2)/\mu^2$, and the small components, the expression (31) for $\hat{R}(\boldsymbol{l};E)$ is a constant $(1/2E)\frac{1}{2}(1+\gamma^0)\frac{1}{2}(1+\Gamma^0)$, making $\hat{U}_{\rm CC}$ proportional to an iteration of the Coulomb potential.

Then in configuration space, with $U=I_C+I_T+U_{\rm CC}$, the eigenvalue equation (20) reads

$$(-\nabla^2+\beta^2)\psi(r)=-\frac{\left[m-\frac{1}{i}\boldsymbol{\gamma}\cdot\boldsymbol{\nabla}+\gamma^0t\right]\left[M+\frac{1}{i}\boldsymbol{\Gamma}\cdot\boldsymbol{\nabla}+\Gamma^0T\right]}{2M_B} \times\left[-\gamma^0\Gamma^0\frac{\alpha}{r}+\frac{1}{2}(\boldsymbol{\gamma}\cdot\boldsymbol{\Gamma}+\boldsymbol{\gamma}\cdot\hat{\mathbf{r}}\boldsymbol{\Gamma}\cdot\hat{\mathbf{r}})\frac{\alpha}{r}+\frac{1}{2M_B}\frac{1+\gamma^0}{2}\frac{1+\Gamma^0}{2}\frac{\alpha^2}{r^2}\right]\ . \tag{33}$$

The first interaction term of (33) is the usual zero-order Coulomb potential. The second is the usual Breit interaction.[20] The third term, effectively $(1/2M_B)\alpha^2/r^2$, is the potential given by the corrected Coulomb term $\hat{U}_{\rm CC}$ calculated above. In the present formalism, this potential $\hat{U}_{\rm CC}$ contributes to the lowest-order fs and hfs on an equal footing with the Coulomb potential and Breit interaction. [Technically $\hat{U}_{\rm CC}$ is a "subtracted box" potential, which should be accompanied by the corresponding crossed diagram. By a calculation similar to that leading to Eq. (30), the crossed Coulomb diagram is easily shown to give zero contribution to fourth order.]

Equation (33) is an equation for the eigenvalue β^2, which must be inserted into (2) to get the bound-state energy. We shall now obtain an equation for the large component of ψ, using momentum space to allow a clearer comparison of various terms. Let u,d represent Pauli upper and lower components relative to γ^0; similarly for U,D and Γ^0. Then

$$\psi_{dD}(\mathbf{p})=-\frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{2m}\frac{\boldsymbol{\Sigma}\cdot\mathbf{p}}{2M}\psi_{uU}(\mathbf{p})+O(\alpha^4\psi_{uU})\ ,$$
$$\psi_{uD}(\mathbf{p})=\frac{\boldsymbol{\Sigma}\cdot\mathbf{p}}{2M}\psi_{uU}(\mathbf{p})+O(\alpha^3\psi_{uU})\ , \tag{34}$$
$$\psi_{dU}(\mathbf{p})=-\frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{2m}\psi_{uU}(\mathbf{p})+O(\alpha^3\psi_{uU})\ ,$$

in the c.m. frame. The Pauli relations (34) are direct consequences of the pair of equivalent first-order exact "Dirac" equations (21) and (22), with the usual representations

$$\gamma^0=\begin{bmatrix}1&0\\0&-1\end{bmatrix},\quad \boldsymbol{\gamma}=\begin{bmatrix}0&-\boldsymbol{\sigma}\\\boldsymbol{\sigma}&0\end{bmatrix};\quad \Gamma^0,\boldsymbol{\Gamma}\text{ similarly}\ .$$

Using (34), Eq. (33) gives an equation for the large component of the wave function accurate to order α^4:

$$(\mathbf{p}^2+\beta^2)\psi_{uU}(\mathbf{p}) = -2\mu\int\frac{d^3l}{(2\pi)^3}(V_0+V_C+V_T+V_{CC})(\mathbf{p},l)\psi_{uU}(l) \tag{35}$$

in which, with $\mathbf{k}=\mathbf{p}-l$,

$$V_0=z\frac{-e^2}{\mathbf{k}^2},\quad z\equiv\frac{(m+t)(M+T)}{4M_B\mu}\ ; \tag{36}$$

$$V_C=-\frac{e^2}{4}\left[\frac{1}{m^2}+\frac{1}{M^2}\right]\frac{\mathbf{p}\cdot l}{\mathbf{k}^2}-\frac{ie^2}{4}\frac{\mathbf{k}\times l}{\mathbf{k}^2}\cdot\left[\frac{\sigma}{m^2}+\frac{\Sigma}{M^2}\right], \tag{37}$$

$$V_T=-\frac{e^2}{4mM}\frac{1}{\mathbf{k}^2}\{2i\mathbf{k}\times l\cdot(\sigma+\Sigma)+(\mathbf{k}\cdot\sigma\mathbf{k}\cdot\Sigma-\mathbf{k}^2\sigma\cdot\Sigma) +4[l^2-(\hat{\mathbf{k}}\cdot l)^2]\}\ , \tag{38}$$

$$V_{CC}=\text{Fourier transform of }\frac{1}{2M_B}\frac{\alpha^2}{r^2}\ . \tag{39}$$

The terms V_0, V_C, V_T, and V_{CC} come, respectively, from the Coulomb potential on the large component of ψ, the Coulomb potential on the small components of ψ, the Breit interaction, and the Coulomb correction potential $\hat{U}_{CC}$.

Keeping only V_0 in (35), the eigenvalues of the resulting Coulomb Schrödinger equation are $\beta_0=z\alpha\mu/n$, $n=1,2,3,\ldots$, where, from (2) and (36),

$$z=1-\frac{\beta^2}{4\mu^2}\left[1-4\frac{\mu^2}{mM}\right]+O(\alpha^4)\ . \tag{40}$$

From (40), the eigenvalues β_0 are then

$$\beta_0=\frac{\alpha\mu}{n}\left[1-\frac{\alpha^2}{4n^2}\left[1-4\frac{\mu^2}{mM}\right]\right]+O(\alpha^5\mu)\ . \tag{41}$$

The other terms in (35) will give a fourth-order contribution to β^2 and can be calculated from first-order perturbation theory. Then from (2), (35), and (41), the bound-state energy to fourth order is

$$M_B=(m+M)-\frac{\alpha^2\mu}{2n^2}+\frac{\alpha^4\mu}{8n^4}\left[1-5\frac{\mu^2}{mM}\right] +\langle V_C+V_T+V_{CC}\rangle\ . \tag{42}$$

We are now able to compare our results with the standard results of the Breit-Salpeter perturbation prescription. (Salpeter's equation is equivalent to the Breit equation through fourth order.[5])

In V_C, the spin-orbit term agrees with its Breit counterpart, but the scalar terms differ. Their expectation values are

$$\langle V_C^{\text{Scalar}}\rangle=\left\langle-\frac{e^2}{4}\left[\frac{1}{m^2}+\frac{1}{M^2}\right]\frac{\mathbf{p}\cdot l}{\mathbf{k}^2}\right\rangle =\frac{1}{8}\frac{\alpha^4\mu^3}{n^3}\left[\frac{1}{m^2}+\frac{1}{M^2}\right]\left[4\delta_{l0}-\frac{8}{2l+1}+\frac{2}{n}\right], \tag{43}$$

$$\langle V_{\text{Breit}}^{\text{Scalar}}\rangle=\left\langle+\frac{e^2}{4}\left[\frac{1}{m^2}+\frac{1}{M^2}\right]\frac{\mathbf{p}\cdot\mathbf{k}}{\mathbf{k}^2}\right\rangle =\frac{1}{8}\frac{\alpha^4\mu^3}{n^3}\left[\frac{1}{m^2}+\frac{1}{M^2}\right](4\delta_{l0})\ . \tag{44}$$

[Note that the Born approximation between normalized free particle spinors, another standard way of calculating Fermi-Breit terms mentioned in Sec. I, gives a term $+e^2(1/m^2+1/M^2)/8$ instead of (44). However, the two versions have identical expectation values.]

Another difference is that here (as in most quasipotential approaches) the relativistic kinematics is automatic, as the eigenvalue β^2 is simply substituted in the expression $E=(m^2-\beta^2)^{1/2}+(M^2-\beta^2)^{1/2}$ to obtain the bound-state energy. In the Breit-Salpeter approach a kinetic-energy operator $(m^2+\mathbf{p}^2)^{1/2}+(M^2+\mathbf{p}^2)^{1/2}$ is obtained, whose fourth-order expansion must be included. Its expectation value is

$$\left\langle-\frac{1}{8}\mathbf{p}^4\left[\frac{1}{m^3}+\frac{1}{M^3}\right]\right\rangle=\frac{1}{8}\frac{\alpha^4\mu^4}{n^3}\left[\frac{1}{m^3}+\frac{1}{M^3}\right] \times\left[\frac{-8}{2l+1}+\frac{3}{n}\right]. \tag{45}$$

This term does not occur in our formalism.

Finally, the Coulomb correction term occurs here but not in the Breit-Salpeter equation. It is

$$\langle V_{CC}\rangle=\left\langle\frac{1}{2M_B}\frac{\alpha^2}{r^2}\right\rangle=\frac{\mu^2}{mM}\frac{\alpha^4\mu}{n^3}\frac{1}{2l+1}\ . \tag{46}$$

In spite of the differences in details, the atomic energy levels to order α^4 are obtained correctly from each of the following.

(i) The standard Breit-Salpeter prescription, which uses the Bohr levels, V_C with (44) for its scalar part, V_T, and the $\mathbf{p}^4$ kinetic-energy correction (45).

(ii) Our method, using the expansion (42) for the energy, V_C with (43) for its scalar part, V_T, and V_{CC} (46).

For completeness, we also quote those parts of the energy-level expression (42) which agree with the Breit-Salpeter method:

$$\langle V_C^{\text{spin orbit}}+V_T\rangle=\frac{\alpha^4\mu}{n^3}\left[\frac{1-\delta_{l0}}{2(2l+1)}\Xi(l,j) +\delta_{l0}\frac{\mu^2}{mM}\begin{Bmatrix}\frac{2}{3},\ S=1\\ -2,\ S=0\end{Bmatrix} +\frac{\mu^2}{mM}\left[\frac{1}{n}-\frac{3}{2l+1}+\delta_{l0}\right]\right], \tag{47}$$

in which

$$\Xi(l,j)=\begin{cases}\dfrac{1}{l+1}+4\dfrac{\mu^2}{mM}\dfrac{1}{2l+3} & (l=j-1,\ S=1),\\ \dfrac{-\frac{1}{2}\pm\frac{1}{2}[1+4l(l+1)(1-4\mu^2/mM)]^{1/2}}{l(l+1)} & (l=j,\ S \text{ mixed}),\\ -\dfrac{1}{l}-4\dfrac{\mu^2}{mM}\dfrac{1}{2l-1} & (l=j+1,\ S=1).\end{cases}$$

Then the bound-state masses are given to order α^4 by Eq. (42) with $\langle V_C+V_T+V_{CC}\rangle$ replaced by the sum of Eqs. (43), (46), and (47).

These energy levels agree with a standard reference:[20] for positronium without the annihilation term (Ref. 20, Sec. 23); for $M\to\infty$ (Ref. 20, Sec. 17); for $m\ll M$, to order $\alpha^4(m^2/M)$ for the hfs of the hydrogen atom with a Dirac magnetic moment for the proton (Ref. 20, Sec. 22). We see that the bound-state equation (33) does give the energy levels expected from single-photon exchange in the Coulomb gauge at an accuracy of α^4.

V. TECHNICAL NOTES

In this section we discuss (A) the existence of solutions, (B) the small radius r, (C) one mass large, (D) the corrections involving I_T.

A. Existence

Bodwin, Yennie, and Gregorio[16] have objected to the use of the homogeneous wave equation (20) on the grounds that the iteration solution of its originating inhomogeneous equation (17) will diverge if $\hat{U}(\mathbf{p},\mathbf{q};E)$ is the Coulomb potential $-e^2/(\mathbf{p}-\mathbf{q})^2$. In fact at large $\mathbf{p}$, the approximation $\hat{S}_0$ in (17) approaches $-\boldsymbol{\gamma}\cdot\hat{\mathbf{p}}\boldsymbol{\Gamma}\cdot\hat{\mathbf{p}}/2E$, an operator of constant magnitude, and a power-counting argument[16] implies the divergence of (17). We shall illustrate the large-$|\mathbf{p}|$ argument of Ref. 16 in configuration space at small r, to make it easier to discuss. With $\hat{U}$ the Coulomb potential and $\hat{S}_0$ majorized at large $\mathbf{p}$ by $1/2E$, when (17) is Fourier transformed and solved by iteration the function $\hat{W}$ is majorized by

$$\frac{1}{2E}\left[1+\left[\frac{1}{2E}\frac{\alpha}{r}\right]+\left[\frac{1}{2E}\frac{\alpha}{r}\right]^2+\cdots\right], \tag{48}$$

which indeed is divergent for $r<\alpha/2E$, in accordance with Ref. 16.

However, Eq. (48) *converges* for $r>\alpha/2E$, and $\alpha/2E$ is less than α^2 times the Bohr radius. The correct results of Sec. IV indicate that it is not necessary to claim that (20) is valid for such small r, and also that it is possible to use the correction series (13) for U, as long as perturbation theory is used.

B. Small r

Nevertheless, suppose we *do* want to consider Eq. (20) at distances of order α^2 (Bohr radius). (In the quark model, α may be of order $\frac{1}{5}$ instead of $\frac{1}{137}$.) Then the nature of $\hat{U}$ must be considered more carefully.

In such a bound-state problem, assume that the dominant binding interaction I_C is local and instantaneous, as in the Coulomb gauge kernel. Neglect other terms (such as I_T in Sec. IV). Then in Sec. IV we have seen that the largest part of the relative energy integral of the correction R is $1/2E$. For the dominant part of $\hat{U}$, (13) then reads

$$\hat{U}=I_C+I_C\frac{1}{2E}\hat{U}\ . \tag{49}$$

In Sec. IV we used the first two terms of the iteration solution of (49) with $I_C=-\alpha/r$; the exact solution of (49) is

$$\hat{U}=\frac{-\alpha}{r+\alpha/2E}\ . \tag{50}$$

Clearly $|\hat{U}|<2E$ always, and when $\hat{U}$ is substituted into (48) instead of $-\alpha/r$ the series will converge for all r.

It follows that substitution of the dominant binding part of the BS kernel into (13) to get a more exact function $\hat{U}$ at small r, as the formalism does actually require, eliminates the objection of Ref. 16.

From this simple model it appears that the use of Eqs. (17) and (20) with $\hat{U}$ artificially restricted to be I_C (as supposed in Ref. 16) would introduce a *spurious* singularity at small r, which more exact solutions of (49) eliminate. In bound-state problems which are not as well understood as QED, based on a BS equation with a local (instantaneous) binding kernel, consideration would evidently have to be given to using an exact solution of (49) for $\hat{U}$ from the outset.

C. One mass large

A nice and easily understood aspect of Eq. (33) is the hydrogen-atom limit, when one mass is large but not infinite. (The following paragraph is a summary of remarks in Refs. 8 and 16.)

Equation (33) with $M\gg m$, in the first-order form (21), is

$$\left[m+\frac{1}{i}\boldsymbol{\gamma}\cdot\boldsymbol{\nabla}-\gamma^0\sqrt{(m^2-\beta^2)}\right]\psi(\mathbf{r})=\frac{\mu}{m}\frac{\alpha}{r}\gamma^0\frac{1+\Gamma^0}{2}\psi(\mathbf{r})$$

(since the transverse interaction, the Coulomb correction, and the term $\boldsymbol{\Gamma}\cdot\mathbf{p}$ are all suppressed by further powers of $1/M$). This is a Dirac equation for the electron of mass m, with a coupling constant $(\mu/m)\alpha$. Evidently the eigenvalue β is $(\mu/m)\alpha m/n$ to leading order, which is $\alpha\mu/n$ as it should be. So our formalism gives the hydrogen fs (to order $\alpha^4 m$) but with the correct (reduced mass) Bohr levels.

D. Transverse correction terms

We want to show that the last three terms of (28) do not give an α^4 contribution to the energy. The transverse term I_T [Eq. (27)] contains $1/k^2=1/(\mathbf{k}^2-k^{0^2})$, so (31) cannot be used to simplify $R=S-S_0$. To the order

needed, the numerator of S is

$$-\left[2m\frac{1+\gamma^0}{2}+\gamma^0 l^0\right]\left[2M\frac{1+\Gamma^0}{2}-\Gamma^0 l^0\right]; \quad (51)$$

in S_0 the l^0 terms are missing. The terms of (28) have their expectation values taken between large-component spinors, given by the projection operator $\frac{1}{2}(1+\gamma^0)\frac{1}{2}(1+\Gamma^0)$.

In (28) the terms $I_C * RI_T$ and $I_T * RI_C$ contain $\gamma^i\Gamma^j$ once. This mixes the largest and smallest components of the external wave function, reducing the order below α^4.

In the remaining term $I_T * RI_T$, $\gamma^i\Gamma^j$ occurs twice. But a $\gamma^i\Gamma^j$ always occurs between the projection operators in S and S_0 and the external wave function, mixing upper and lower components and reducing the contribution of any projection operator in S and S_0 below α^4. The only part left is $\gamma^0\Gamma^0 l^{0^2}$ in (51), which is not reduced in magnitude by $(\gamma^i\Gamma^j)^2$. The contribution to the energy of this piece is the expectation value of the integral of a product of $e^2/(\hat{p}-l)^2$, $e^2/(l-\hat{q})^2$, 1/(Denominator of S), and l^{0^2}, in which $\hat{p}\equiv(\mathbf{p},0)$, $\hat{q}\equiv(\mathbf{q},0)$. For $m=M$ and small $\mathbf{p},\mathbf{q}$, this expectation value is Eq. (4.13) of the paper of Karplus and Klein,[6] which they show to be one of the contributions of order α^5.

VI. TEST OF 3→4 MAPPING: QED BOUND-STATE TRANSITIONS

In this section we verify the correctness of the reconstruction (24) of the four-dimensional vertex functions from the three-dimensional wave functions by calculating the low-energy transition amplitude for photon emission by an atom such as hydrogen or positronium directly from the Feynman triangle diagram, and showing that the expected result of nonrelativistic quantum mechanics is obtained.

Consider the decay of a bound state of four-momentum K_i into a bound state of four-momentum K_f accompanied by a real or virtual photon of four-momentum $k=K_i-K_f$. It is sufficient to consider a photon emitted from the m particle, of charge q.

The current matrix element (without external kinematic factors) can be written from the Feynman triangle diagram as

$$j^\nu_{\text{Feyn}}=-iq\int\frac{d^4l_i}{(2\pi)^4}\bar{\Gamma}_f(l_f;K_f)g(x_fK_f+l_f)\times\gamma^\nu g(x_iK_i+l_i)G(XK-l)\Gamma_i(l_i;K_i)\ , \quad (52)$$

where g,G are the m,M particle propagators, and $\bar{\Gamma}_f,\Gamma_i$ are the bound-state vertex functions. The momenta are parametrized as in Secs. II and III. Either subscript, i or f, can be used for the variables in the propagator G of the nonradiating M particle, as evidently

$$X_fK_f-l_f=X_iK_i-l_i\ . \quad (53)$$

The loop integral variable could have been l_f.

Consider Eq. (52) in the rest frame of the i bound state. The simplest way to estimate the integral $\int dl_i^0/2\pi$ is to find the residue of the pole of G near the origin, as the near poles of the two propagators g_i,g_f are both on the other side of the l_i^0 contour. That is equivalent to treating G as

$$G(x_iK_i-l_i)=2\pi i\delta(l_i^0-l_{i\,\text{near}}^0)\frac{M+\boldsymbol{\Gamma}\cdot\boldsymbol{l}_i+\Gamma^0\Omega(l_i)}{2\Omega(l_i)}\ , \quad (54)$$

where

$$\Omega(l_i)=\sqrt{(l_i^2+M^2)}\ ,$$
$$l_{i\,\text{near}}^0=\mathrm{T}_i-\Omega(l_i)=-\frac{l_i^2+\beta_i^2}{2M}+O(\alpha^4\mu)\ . \quad (55)$$

Substituting (54) into (52) requires the substitution of (55) into the initial g propagator in (52), giving

$$g(x_iK_i+l_i)=\frac{\mu}{m}\frac{m-\boldsymbol{\gamma}\cdot\boldsymbol{l}_i+\gamma^0 t}{l_i^2+\beta_i^2}[1+0(\alpha^2)]\ . \quad (56)$$

Thus after the operation $\int dl_i^0/2\pi$, the three right-hand terms of (52) become

$$g_iG\Gamma_i=i\frac{\mu}{m}\frac{(m-\boldsymbol{\gamma}\cdot\boldsymbol{l}_i+\gamma^0 t_i)(M+\boldsymbol{\Gamma}\cdot\boldsymbol{l}_i+\Gamma^0 T_i)}{(l_i^2+\beta_i^2)2M}\times\Gamma_i((\boldsymbol{l}_i,l_{i\,\text{near}}^0);K_i)\ , \quad (57)$$

in which terms of relative order α^2 are neglected.

The variables $\mathbf{p},\boldsymbol{l}_i$ are of order $\alpha\mu$, and $l_{i\,\text{near}}^0$ is of order $\alpha^2\mu$. Then $l_{i\,\text{near}}^0$ can be replaced by 0 in the vertex function Γ_i, for in the integral (25) giving Γ_i, we have

$$1/[(\mathbf{p}-\boldsymbol{l})^2-(l_{i\,\text{near}}^0)^2]=[1+0(\alpha^2)]\times 1/(\mathbf{p}-\boldsymbol{l})^2$$

by the usual expansion. The advantage of this is that in situations where $p^0=0$ in the rest frame, our wave function ψ and vertex function Γ have a simple algebraic relationship. Equations (20) and (24) show that in the rest frame, where $K_B=(\mathbf{0},M_B)$, and with $\hat{p}=(\mathbf{p},0)$, it is exactly true that

$$\psi(\mathbf{p};M_B)=\frac{(m-\boldsymbol{\gamma}\cdot\mathbf{p}+\gamma^0 t)(M+\boldsymbol{\Gamma}\cdot\mathbf{p}+\Gamma^0 T)}{2M_B(\mathbf{p}^2+\beta^2)}\Gamma(\hat{p};K_B)\ . \quad (58)$$

Using (58), and the relation $M_{Bi}=m+M+0(\alpha^2\mu)$, the expression (57) with $l_{i\,\text{near}}^0=0$ simplifies to

$$[g_iG\Gamma_i]^{i\text{ c.m.}}=i[\psi_i(\boldsymbol{l}_i;M_{Bi})]^{i\text{ c.m.}}+0(\alpha^2\psi_i)\ , \quad (59)$$

where henceforth the superscripts i c.m., f c.m. denote quantities evaluated in the i,f rest frames.

To treat the first two terms $\bar{\Gamma}_f g_f$ in the expression (52) similarly, the variable $l_f^{f\text{ c.m.}}$ must be used instead of $l_i^{i\text{ c.m.}}$. We will simplify the problem by assuming that in the i rest frame, the f bound state recoils with a speed of order αc at most. That assumption is consistent with the purpose of this section, which is to test whether the eval-

uation of (52) using our formalism agrees with standard nonrelativistic quantum mechanics. From Eq. (53) and a Lorentz transformation, the f rest frame relations $|l_f^{f\ \mathrm{c.m.}}|=0(\alpha\mu)$, $|l_f^{0f\ \mathrm{c.m.}}|=0(\alpha^2\mu)$ are then consistent with the corresponding i frame relations used earlier. This allows us to treat $\bar{\Gamma}_f g_f$ in the f rest frame in the same way that $g_i G\Gamma_i$ was treated in the i rest frame. In the same way (59) was obtained, one finds

$$[\bar{\Gamma}_f g_f]^{f\ \mathrm{c.m.}}=2M\left[\frac{\bar{\psi}_f(l_f;M_f)(M-\boldsymbol{\Gamma}\cdot\mathbf{l}_f-\Gamma^0 T_f)}{l_f^2+\beta_f^2}\right]^{f\ \mathrm{c.m.}}. \tag{60}$$

Then (59) and (60) can be substituted into the Feynman integral (52) to give

$$j^\nu_{\mathrm{Feyn}}=2qM\int\frac{dl_i^{i\ \mathrm{c.m.}}}{(2\pi)^3}\times\left[\frac{\bar{\psi}_f(l_f;M_f)(M-\boldsymbol{\Gamma}\cdot\mathbf{l}_f-\Gamma^0 T_f)}{(l_f^2+\beta_f^2)}\right]^{f\ \mathrm{c.m.}}\times\gamma^\nu[\psi_i(l_i;M_i)]^{i\ \mathrm{c.m.}}. \tag{61}$$

Because the bound states' relative speed is $O(\alpha c)$ at most, to lowest order it is true that $l_f^{f\ \mathrm{c.m.}}=l_i^{i\ \mathrm{c.m.}}-(\mu/m)\mathbf{k}$ and also the spin operators are the same in both frames, as expected in nonrelativistic quantum mechanics. The counterpart to Eq. (34) for $\bar{\psi}$ can be used to express $\bar{\psi}(l_f^{f\ \mathrm{c.m.}})$ in terms of the large component $\bar{\psi}_{uU}(l_f^{f\ \mathrm{c.m.}})$. Also, it follows from Eq. (17) that to lowest order the normalized wave functions ϕ are given by $\phi(\mathbf{p})=\psi_{uU}(\mathbf{p})/(2M_B)^{1/2}$. Finally, inserting kinematic factors using box normalization, the matrix element must be divided by $1/(2E_fV)^{1/2}(2M_{Bi}V)^{1/2}$, which is $1/2(m+M)V$ to lowest order.

Using the preceding paragraph, to leading order the current matrix element (61) becomes, in the i rest frame,

$$\mathbf{j}_{fi}=\frac{1}{V}\frac{q}{m}\int\frac{dl}{(2\pi)^3}\bar{\phi}_f\left[l-\frac{\mu}{m}\mathbf{k}\right]\left[1-\frac{\mathbf{k}}{2}+\frac{i}{2}\mathbf{k}\times\boldsymbol{\sigma}\right]\phi_i(l), \tag{62}$$

$$j^0_{fi}=\frac{1}{V}q\int\frac{dl}{(2\pi)^3}\bar{\phi}_f\left[l-\frac{\mu}{m}\mathbf{k}\right]\phi_i(l). \tag{63}$$

The result (62) agrees with a standard reference[21] for $M\to\infty$; for general m,M we have done a nonrelativistic calculation of the matrix element of the current operator $q\bar{\psi}(\mathbf{r})\gamma^\nu\psi(\mathbf{r})$ and found results identical to (62) and (63). Thus the $3\to4$ mapping, which reconstructed the bound-state vertex functions, has been verified to give correct results in the case of low-energy QED bound states. This is the final step in establishing that the formalism described in this paper can calculate bound-state energies and transition amplitudes correctly.

VII. CONCLUSIONS

For the Bethe-Salpeter equation for two spin-$\frac{1}{2}$ particles, this paper has explored in a straightforward manner the consequences of the Todorov approximation,[9] in which the peak in the pair of propagators of the bound particles in the relative energy variable is approximated by a δ function.

Todorov's derivation of his approximation was on the basis of simplicity and elastic unitarity; in Sec. II we gave an alternative derivation which may have had some historical interest.

The "$4\to3\to4$"-dimensional formalism laid out in Sec. III was of a standard kind once the starting approximation is accepted. Section V gave two arguments for the validity of the method: that it works in perturbation theory, and that a more exact expression for the potential $\hat{U}$ in the bound-state equation eliminates convergence objections expressed in Ref. 16.

The Coulomb-gauge single-photon-exchange kernel used as a test of energy levels to order α^4 in Sec. IV shows the Coulomb correction potential $\hat{U}_{\mathrm{CC}}=(1/2M_B)\alpha^2/r^2$ coexisting an equal basis with the more familiar Coulomb potential and Breit interaction. As explained in Sec. IV, the crossed Coulomb diagram which should accompany such a "subtracted box" term gives a zero contribution to fourth order.

In Sec. VI it was shown that the reconstructed vertex functions do give the low-energy bound-state dipole matrix elements correctly when substituted in Feynman triangle diagrams. This result verifies the "$3\to4$" part of the method.

Two features of the method worth remarking on are the following. (a) Equations (21) and (22) show each particle obeying a Dirac equation in the presence of the other, even though the equations are the same equation in different guises. Of course the Dirac limit (one particle infinitely heavy) is automatically reached in this method. (b) The term $(1/2M_B)\alpha^2/r^2$ in the full (not Pauli reduced) equation (33) shows that the subtracted Coulomb box diagram really contributes at the α^4 level in a formalism without Casimir energy projection operators. As a matter of fact very few BS reductions other than in Ref. 16 seem to have been done without Casimir operators.

In the quark model of mesons no bound-state equation seems to have come to the forefront at present.[3,4] Subjectively, we feel most comfortable with a method that comes directly from a well-used equation of standard quantum field theory such as the BS equation. In the future we hope to investigate whether the present paper's methods provide any new contribution to the quark model.

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