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Self-duality and nonrelativistic Maxwell-Chern-Simons solitons

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We consider a model of nonrelativistic charged matter interacting with a massive Abelian gauge field and a neutral scalar field in (2+1)-dimensional space-time. As in the model considered by Jackiw and Pi with a pure Chern-Simons interaction, the model admits classical static self-dual solutions. These solutions are nontopological solitons which may be threaded by a number of vortices. Furthermore, these self-dual classical solutions are zero-energy configurations.

I. INTRODUCTION

There has recently been much revived interest in (2+1)-dimensional Abelian Higgs models in which a Chern-Simons term is contained in the gauge action.[1–10] Such models may be considered as field-theoretic models for anyons[11] or as effective theories (of the Landau-Ginzburg type) for anyon superconductivity.[12] While there have been many variations on this theme, a common thread has been the analysis of classical vortexlike solutions and the associated role of self-duality. These Chern-Simons solitons will presumably play an important role in the complete understanding of realistic models of fractional statistics and anyonic superconductivity.

Originally,[1] the relativistic Abelian Higgs model had a Chern-Simons term added to the usual Maxwell kinetic term for the gauge field. More recently,[3,4,7–10] relativistic models in which the Chern-Simons term accounts for the entire gauge field action have been considered. For a specific form of the Higgs potential such models admit static self-dual vortexlike solutions.[3,4] The nonrelativistic limit of such theories has also been discussed, revealing a rich structure related once again to self-duality.[5] A relativistic theory with both a Maxwell and a Chern-Simons term in the gauge field action has recently been analyzed by Lee, Lee, and Min.[6] In this paper we shall investigate the role of self-dual solitons in a theory in which a charged *nonrelativistic* matter field interacts with a massive gauge field, having both a Maxwell and a Chern-Simons term in the action.

Before turning to this model we first briefly summarize the approaches and results in other related theories. One important issue in these (2+1)-dimensional matter-gauge field systems is whether to take the gauge field action to be given by a Maxwell term, a Chern-Simons term, or both.[13] As mentioned above, Paul and Khare initially considered both.[1] However, since the Chern-Simons term is of lower order in space-time derivatives, it dominates the large-distance properties of the theory and substantially modifies the characteristics of the classical solutions. This led Hong, Kim, and Pac[3] and Jackiw and Weinberg[4] to consider a truncation in which the Chern-Simons term accounts for the whole gauge field action. They found that for a specific form of the Higgs potential the model admits static self-dual vortexlike solutions. These self-dual Chern-Simons vortices differ in several respects from the familiar Nielsen-Olesen vortices.[14] First of all, they are necessarily charged, due to the Chern-Simons Gauss law relating the magnetic field and the matter charge density. Second, the radially symmetric solutions exhibit an unusual magnetic field profile which is concentrated on a ring at a finite distance from the origin. Third, being charged, the Chern-Simons vortices always have a nonvanishing (fractional) angular momentum. The specific form of the Higgs potential which admits static self-dual solutions has two degenerate minima: a symmetry-breaking minimum and a symmetry-preserving minimum. Correspondingly, the theory has two types of classical solutions: (i) *topological* vortices with quantized magnetic flux, and (ii) *nontopological* solitons with nonvanishing but not-necessarily-quantized magnetic flux.[7]

The nonrelativistic limit of these nontopological solitons has been studied by Jackiw and Pi.[5] In the nonrelativistic limit this theory provides a second-quantized description of point particles moving (nonrelativistically) in δ-function potentials and interacting via a Chern-Simons term. The matter equation of motion becomes a gauged nonlinear Schrödinger equation which can be integrated with a self-dual ansatz. In fact, the nonrelativistic charge density for static self-dual solutions satisfies the Liouville equation, the solutions of which are known analytically.[15] An interesting feature of these nonrelativistic self-dual solitons is that they saturate the lower bound $\mathcal{E}=0$ of the static energy functional.

We now recall that self-dual equations also arise in the Abelian Higgs model (with only the standard Maxwell gauge action) when parameters are chosen to make the vector and scalar masses equal. This is the Bogomol'nyi model[16] and provides a relativistic version of the phenomenological Landau-Ginzburg model for a system on the boundary between type-I and type-II superconductivity.[17]

Given this ubiquitous role of self-duality, it is natural to ask if self-dual solitons exist when the gauge field action contains both a Maxwell and a Chern-Simons term. (Note that non-self-dual solitons also exist in such models, and have been extensively studied in Refs. 2, 8, and 9.) At first sight this does not look promising as self-duality plays no obvious role when one simply adds a Maxwell term to the actions considered in Refs. 3 and 4 (relativistic case) or in Ref. 5 (nonrelativistic case). However, an important clue towards the answer was provided by Lee, Lee, and Weinberg,[10] who observed that the relativistic Chern-Simons Higgs model[3,4] admitting self-dual solitons is the bosonic portion of a model with $N=2$ supersymmetry.

This has led Lee, Lee, and Min[6] to consider the bosonic portion of an $N=2$ supersymmetric model with gauge field action consisting of *both* a Maxwell *and* a Chern-Simons term. The requirement of $N=2$ supersymmetry prescribes the matter content of the theory, the couplings, and the potentials. An important novelty is the appearance of a *neutral scalar* field (with mass equal to the gauge field mass) in addition to the usual charged scalar field. In this paper we shall consider this model of Lee, Lee, and Min[6] in the *nonrelativistic limit for the charged matter.*

This paper is organized as follows. In Sec. II we define our model and consider various important physical limits of parameters in the Lagrangian. In Sec. III we discuss the equations of motion for our model and show that with a static self-dual ansatz these reduce to two coupled nonlinear equations for the charge density and the neutral scalar field. In the pure Chern-Simons limit in which the Maxwell term is removed from the action, the neutral scalar decouples and the equation for the charge density reduces to the Liouville equation as found directly in the Jackiw-Pi model.[5] In Sec. IV we show that the static energy may be written as a sum of manifestly positive terms. The minimum ($\mathcal{E}=0$) of the energy is attained when these terms vanish. This gives (first-order) Bogomol'nyi equations[16] which are in fact equivalent to the (second-order) equations of motion supplemented with the self-dual ansatz. In Sec. V we discuss the asymptotic behavior of the coupled equations derived in Sec. III, and use this to deduce important global properties of the solitons such as charge, flux, and angular momentum. Finally, we conclude with some discussion and some comments regarding future directions of investigation.

II. THE MODEL

Our starting point is the $(2+1)$-dimensional relativistic Lagrange density

$$\mathcal{L}^R=-\frac{1}{4e^2}F_{\mu\nu}F^{\mu\nu}+\frac{\kappa}{4}\epsilon^{\mu\nu\alpha}F_{\mu\nu}A_\alpha+(D_\mu\phi)^*(D^\mu\phi)+\frac{1}{2e^2}\partial_\mu N\partial^\mu N-\frac{1}{c^2}|\phi|^2\left[N+\frac{1}{\kappa c}v^2\right]^2-\frac{e^2}{2c^2}(|\phi|^2+\kappa c N)^2 \; . \tag{2.1}$$

We use the convention that $\hbar=1$ and keep explicit the velocity of light c since we are interested in the nonrelativistic limit $c\rightarrow\infty$. Our convention for the metric is $g_{\mu\nu}=\mathrm{diag}(1,-1,-1)$; $\epsilon^{\mu\nu\alpha}$ is the totally antisymmetric tensor, with sign convention fixed by $\epsilon^{012}=1$. (We shall use Greek letters for space-time indices and Latin letters for spatial indices.) N is a real, neutral scalar field, while ϕ is a complex, charged scalar field. The coupling to the Abelian gauge field A_μ is given by the usual covariant derivative

$$D_\mu=\partial_\mu+\frac{i}{c}A_\mu \; . \tag{2.2}$$

The gauge action involves the field-strength tensor

$$F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu \tag{2.3}$$

and contains both the usual Maxwell term [first term in (2.1)] and the Chern-Simons term [second term in (2.1)]. The strength of the Maxwell term is governed by the gauge coupling e^2 [with dimension mass$\times$(velocity)^3] while the strength of the Chern-Simons term is governed by the Chern-Simons coupling κ [with dimension (velocity)$^{-2}$]. Note that since we shall restrict our attention to the *classical* theory, gauge invariance is guaranteed without any quantization condition on the dimensionless parameter κc^2. [Even in the *quantum* theory such a condition would not be necessary when the gauge group is U(1).]

The model (2.1) is the bosonic portion of a model with $N=2$ supersymmetry and admits static self-dual solitons.[6] The role of $N=2$ supersymmetry is to prescribe the matter content and the couplings so as to guarantee the existence of self-dual solitons. Evidence of a deep relationship between $N=2$ supersymmetry and self-duality has been known for some time in other related contexts.[18] The model (2.1) interpolates between the model of Hong, Kim, and Pac[3] and Jackiw and Weinberg[4] with pure Chern-Simons interaction and the Abelian Higgs model at the self-dual point,[16] with pure Maxwell interaction, as we now show.

In the *pure Chern-Simons limit* $e^2\rightarrow\infty$ with κ fixed, both the Maxwell term and the kinetic term for the N field disappear from the Lagrange density. Moreover, this limit requires the evaluation of N at $N=-(1/\kappa c)|\phi|^2$, as can be seen from the last term in (2.1). The resulting Lagrange density is given by

$$\mathcal{L}^R_{\mathrm{CS}}=\frac{\kappa}{4}\epsilon^{\mu\nu\alpha}F_{\mu\nu}A_\alpha+(D_\mu\phi)^*(D^\mu\phi)-\frac{1}{\kappa^2c^4}|\phi|^2(|\phi|^2-v^2)^2 \tag{2.4}$$

and coincides with the model of Hong, Kim, and Pac[3] and Jackiw and Weinberg.[4] In the *pure Maxwell limit* $\kappa\rightarrow 0$ with e^2 fixed the Chern-Simons term disappears from the Lagrange density and the N field must be shifted by $-(1/\kappa c)v^2$, $N=n-(1/\kappa c)v^2$. Further, upon taking $n=0$ we arrive at the Lagrange density of the Abelian Higgs model with parameters such that the scalar and vector masses are equal, $m_{\mathrm{Higgs}}=m_{\mathrm{photon}}=ev/c^2$:

$$\mathcal{L}_M^R = -\frac{1}{4e^2}F_{\mu\nu}F^{\mu\nu} + (D_\mu\phi)^*(D^\mu\phi) - \frac{e^2}{2c^2}(|\phi|^2 - v^2)^2 . \tag{2.5}$$

This is the Bogomol'nyi model[16] which describes a system on the boundary between type-I and type-II superconductivity and admits self-dual solitons.

We now turn to the nonrelativistic limit of the model (2.1). To this end we note that the theory (2.1) involves *two mass scales*. Actually all three fields A_μ, ϕ, and N are massive but the (topological) vector mass m_A generated by the Chern-Simons term[19] is equal to the mass of the neutral scalar N:

$$m_\phi = \frac{v^2}{|\kappa|c^3} \equiv m \ , \quad m_A = m_N = \frac{|\kappa|e^2}{c} \ . \tag{2.6}$$

This shows that in the limit $c \to \infty$ (with κ and e^2 fixed) both the photon and the neutral scalar become massless. The mass of the charged scalar instead remains constant if we accompany the limit $c \to \infty$ with the limit $v^2 \to \infty$ such that v^2/c^3 is constant. Accordingly, a nonrelativistic expansion can be made only for the charged scalar ϕ.

First of all we rewrite the matter Lagrange density in terms of the masses $m \equiv m_\phi$ and m_A:

$$\begin{aligned}\mathcal{L}^R_{\text{matter}} = & \frac{1}{c^2}|(\partial_t + iA_0)\phi|^2 - (D_i\phi)^*(D_i\phi) + \frac{1}{2e^2}\partial_\mu N\partial^\mu N \\ & - m^2c^2|\phi|^2 - \frac{1}{2e^2}m_A^2c^2N^2 \\ & - \frac{e^2}{2c^2}(|\phi|^2)^2 - \tau(2m + m_A)|\phi|^2 N - \frac{1}{c^2}|\phi|^2N^2 \ ,\end{aligned} \tag{2.7}$$

where we have introduced $\tau = \text{sgn}(\kappa)$. Note that we have defined m and m_A to be positive. To consider the nonrelativistic limit[5] we first substitute in (2.7)

$$\phi = \frac{1}{\sqrt{2m}}(e^{-imc^2t}\psi + e^{imc^2t}\tilde{\psi}^*) \tag{2.8}$$

and drop all terms which oscillate as $c \to \infty$. Keeping only dominant inverse powers of c gives

$$\begin{aligned}\mathcal{L}^R_{\text{matter}} \to & \psi^* i(\partial_t + iA_0)\psi - \frac{1}{2m}(D_i\psi)^*(D_i\psi) \\ & + \tilde{\psi}^* i(\partial_t - iA_0)\tilde{\psi} - \frac{1}{2m}(D_i\tilde{\psi})^*(D_i\tilde{\psi}) \\ & + \frac{1}{2e^2}\partial_\mu N\partial^\mu N - \frac{1}{2e^2}m_A^2c^2N^2 \\ & - \frac{e^2}{8m^2c^2}(\rho + \tilde{\rho})^2 - \tau\left[1 + \frac{m_A}{2m}\right](\rho + \tilde{\rho})N \ .\end{aligned} \tag{2.9}$$

Here ρ and $\tilde{\rho}$ are the particle and antiparticle densities, respectively, $\rho = \psi^*\psi$, $\tilde{\rho} = \tilde{\psi}^*\tilde{\psi}$. Particles and antiparticles are separately conserved by the interactions in (2.9). Therefore we may work in the zero-antiparticle sector by setting $\tilde{\psi} = 0$. This gives us a Lagrange density

$$\begin{aligned}\mathcal{L} = & -\frac{1}{4e^2}F_{\mu\nu}F^{\mu\nu} + \frac{\kappa}{4}\epsilon^{\mu\nu\alpha}F_{\mu\nu}A_\alpha + \frac{1}{2e^2}\partial_\mu N\partial^\mu N \\ & - \frac{1}{2e^2}m_A^2c^2N^2 + \psi^* i(\partial_t + iA_0)\psi - \frac{1}{2m}(D_i\psi)^*(D_i\psi) \\ & - \frac{e^2}{8m^2c^2}\rho^2 - \tau\left[1 + \frac{m_A}{2m}\right]\rho N \ ,\end{aligned} \tag{2.10}$$

which provides a second-quantized description of a fixed number of nonrelativistic particles moving in δ-function potentials with strength $(e^2/4m^2c^2)$ and interacting with massive relativistic photons and neutral scalars. In the next sections we will perform a classical analysis of the equations of motion following from (2.10) and we will show that (2.10) admits static self-dual solitons.

In the pure Chern-Simons limit $e^2 \to \infty$ (and therefore $m_A \to \infty$) with κ (and therefore m) fixed, the neutral scalar N has to be evaluated at $N = -(1/2mc\kappa)\rho$ and the Maxwell term and the kinetic term for N disappear from the Lagrange density. This leaves us with

$$\begin{aligned}\mathcal{L}_{\text{CS}} = & \frac{\kappa}{4}\epsilon^{\mu\nu\alpha}F_{\mu\nu}A_\alpha + \psi^* i(\partial_t + iA_0)\psi \\ & - \frac{1}{2m}(D_i\psi)^*(D_i\psi) + \frac{1}{2mc|\kappa|}\rho^2 \ ,\end{aligned} \tag{2.11}$$

which is the Lagrange density of the nonrelativistic Jackiw-Pi model,[5] with self-dual solitons governed by the Liouville equation.

III. EQUATIONS OF MOTION WITH STATIC, SELF-DUAL ANSATZ

In this section we analyze the equations of motion for the "nonrelativistic" Maxwell-Chern-Simons model described by the Lagrange density (2.10). Varying with respect to the fields ψ^* and N we obtain the equations of motion

$$\begin{aligned}i(\partial_t + iA_0)\psi + \frac{1}{2m}D_iD_i\psi - \tau\left[1 + \frac{m_A}{2m}\right]N\psi & \\ - \frac{m_A}{4c|\kappa|m^2}\rho\psi = 0 \ , &\end{aligned} \tag{3.1a}$$

$$\left[\frac{1}{c^2}\partial_t^2 - \nabla^2 + m_A^2c^2\right]N^2 + \frac{m_Ac}{|\kappa|}\tau\left[1 + \frac{m_A}{2m}\right]\rho = 0 \ , \tag{3.1b}$$

where ρ is the nonrelativistic number density $\rho = \psi^*\psi$. Varying with respect to the gauge fields A_μ leads to the equations of motion

$$\frac{1}{e^2}\partial_\mu F^{\mu\nu} + \frac{\kappa}{2}\epsilon^{\nu\alpha\beta}F_{\alpha\beta} = \frac{1}{c}J^\nu \ , \tag{3.1c}$$

where J^ν is the (conserved) matter current

$$\begin{aligned}J^\nu & = (c\rho, J^i) \\ & = \left[c\psi^*\psi, -\frac{i}{2m}[\psi^* D_i\psi - \psi(D_i\psi)^*]\right] \ .\end{aligned} \tag{3.2}$$

Note that the time component ($\nu=0$) of (3.1c) is the Chern-Simons modified Gauss law

$$\partial_i E^i - e^2\kappa B = e^2\rho \tag{3.3}$$

where B is the magnetic field $B=-\epsilon^{ij}\partial_i A_j$ and E^i is the electric field $E^i=-\partial_i A^0-(1/c)\partial_t A^i$. [The e^2 (rather than the familiar e) on the right-hand side of (3.3) is simply due to the fact that we have absorbed the coupling constant e in the gauge field, according to (2.2).]

We shall seek *static* solutions to the equations of motion, in which case (3.1) simplify to

$$-A_0\psi+\frac{1}{2m}D_iD_i\psi-\tau\left[1+\frac{m_A}{2m}\right]N\psi-\frac{m_A}{4c|\kappa|m^2}\rho\psi=0, \tag{3.4a}$$

$$(-\nabla^2+m_A^2c^2)N^2+\frac{m_Ac}{|\kappa|}\tau\left[1+\frac{m_A}{2m}\right]\rho=0\ , \tag{3.4b}$$

$$\nabla^2A_0+cm_A\tau B+\frac{cm_A}{|\kappa|}\rho=0\ , \tag{3.4c}$$

$$(\partial_2+i\partial_1)(B+cm_A\tau A_0)=\frac{im_A}{2|\kappa|m}\{\psi[(D_1+iD_2)\psi]^*-\psi^*(D_1-iD_2)\psi\}\ , \tag{3.4d}$$

$$(\partial_2-i\partial_1)(B+cm_A\tau A_0)=\frac{im_A}{2|\kappa|m}\{\psi[(D_1-iD_2)\psi]^*-\psi^*(D_1+iD_2)\psi\}\ . \tag{3.4e}$$

Note that the combination of (3.4c) with (3.4d) and (3.4e) exhibits the massive nature of the photon. We can further simplify this system of equations by making the self-dual (anti-self-dual) ansatz

$$(D_1\pm iD_2)\psi=0 \tag{3.5}$$

for the gauge fields A_i and the nonrelativistic charged matter field ψ. Such an ansatz is motivated by previous work[3-7] in other (2+1)-dimensional matter-gauge field systems involving Chern-Simons terms where self-duality has been found to play an important role. It is of course clear from Eqs. (3.4d) and (3.4e) that the ansatz (3.5) will simplify the equations of motion considerably. Furthermore, we shall see in the next section that the self-dual Bogomol'nyi equations (3.5) *must* be satisfied for a static solution of *minimum* energy.

Expressing the charged nonrelativistic matter field ψ as

$$\psi=e^{i\omega/c}\rho^{1/2}\ , \tag{3.6}$$

the self-dual (anti-self-dual) ansatz (3.5) implies that

$$A_i=-\partial_i\omega\mp\frac{c}{2}\epsilon^{ij}\partial_j\ln\rho\ . \tag{3.7}$$

Thus the magnetic field is related to the nonrelativistic number density ρ as

$$B=\mp\frac{c}{2}\nabla^2\ln\rho\ . \tag{3.8}$$

Given the ansatz (3.5), the equations of motion (3.4a)–(3.4e) reduce to

$$-A_0\mp\frac{1}{2mc}B-\tau\left[1+\frac{m_A}{2m}\right]N-\frac{m_A}{4c|\kappa|m^2}\rho=0\ , \tag{3.9a}$$

$$(-\nabla^2+m_A^2c^2)N+\frac{m_Ac}{|\kappa|}\tau\left[1+\frac{m_A}{2m}\right]\rho=0\ , \tag{3.9b}$$

$$\nabla^2A_0+cm_A\tau B+\frac{cm_A}{|\kappa|}\rho=0\ , \tag{3.9c}$$

$$B+cm_A\tau A_0\pm\frac{m_A}{2|\kappa|m}\rho=0\ . \tag{3.9d}$$

Note that Eq. (3.4a) becomes (3.9a) since $D_iD_i\psi=\mp(1/c)B\psi$ when the self-dual ansatz (3.5) is satisfied. Also, given (3.5), Eqs. (3.4d) and (3.4e) may be combined and integrated to give the single equation (3.9d). These equations (3.9a)–(3.9d) must of course be supplemented with the self-duality relation between B and ρ given by (3.8).

This system of equations (3.8) and (3.9a)–(3.9d) may be further simplified by simple algebraic manipulations. In fact, as we have five equations relating four fields we expect further reductions. For example, combining (3.9a) and (3.9d) we find a linear relation between N and A_0:

$$N=-\tau\frac{1\mp\tau\dfrac{m_A}{2m}}{1+\dfrac{m_A}{2m}}A_0\ . \tag{3.10}$$

Inserting this into (3.9b) we find that consistency with (3.9c) and (3.9d) demands that the + (−) sign corresponding to self-duality (anti-self-duality) must be correlated with the sign τ of the Chern-Simons coupling constant κ as

$$\tau=\mp \tag{3.11}$$

[i.e., self-duality (anti-self-duality) requires κ to be negative (positive), just as in Ref. 5]. This correlation then implies that $N=-\tau A_0$, and that Eqs. (3.9b) and (3.9c) become identical [using (3.9d)].

With this sign correlation the ansatz (3.5) may be conveniently written as

$$(D_1-i\tau D_2)\psi=0\ , \tag{3.12}$$

and the A_0 and B fields may be eliminated from the equations of motion (3.9) leaving the two coupled equations

$$(-\nabla^2+m_A^2c^2)N=-\tau\frac{m_Ac}{|\kappa|}\left[1+\frac{m_A}{2m}\right]\rho\ , \tag{3.13a}$$

$$\nabla^2\ln\rho=\frac{m_A}{m|\kappa|c}\rho+2\tau m_AN\ . \tag{3.13b}$$

These equations may of course be further combined to yield a single highly nonlinear equation for the nonrelativistic charged matter density ρ:

$$\nabla^2 \ln\rho = -\frac{2}{c|\kappa|}\left[\frac{m_A^2c^2+\frac{m_A}{2m}\nabla^2}{m_A^2c^2-\nabla^2}\right]\rho$$

$$= -\frac{2}{c|\kappa|}\rho - \frac{2}{c|\kappa|}\left[1+\frac{m_A}{2m}\right]\sum_{j=1}^{\infty}\frac{1}{(m_A^2c^2)^j}(\nabla^2)^j\rho \ . \tag{3.14}$$

It is interesting to note that in the pure Chern-Simons limit $m_A \to \infty$ (with $|\kappa|$ and m fixed) this equation reduces directly to the Liouville equation[15]

$$\nabla^2\ln\rho + \frac{2}{c|\kappa|}\rho = 0 \ . \tag{3.15}$$

This is not altogether surprising since in this limit the Maxwell term and the N field disappear from the Lagrange density (as discussed in Sec. II) leaving the nonrelativistic Lagrange density of the Jackiw-Pi model, in which the static self-dual ansatz leads directly to the Liouville equation (3.15) for the matter density ρ.[5]

We shall postpone a detailed discussion of the equations (3.13a) and (3.13b) until Sec. V where we will show that the asymptotic behavior of solutions may be deduced and is sufficient to determine such global quantities as charge and flux.

IV. SELF-DUALITY AND MINIMUM ENERGY CONDITIONS

In this section we show that in the static case the energy of our system with Lagrangian (2.10) may be written as a sum of manifestly positive terms. Thus the energy is bounded from below by zero. Clearly this lower bound is saturated when each of these terms vanishes separately. We will see that these conditions for minimizing the static energy functional lead to (first-order) Bogomol'nyi[16] equations which are the same as the (second-order) static equations of motion with self-dual ansatz (3.5). Thus, just as in the Jackiw-Pi model[5] (nonrelativistic, with no Maxwell term), the static self-dual solutions correspond to zero-energy configurations.

Given the Lagrangian (2.10), the total energy is the spatial integral of the energy density cT^{00},

$$\mathcal{E} = \int d^2x\, cT^{00} \ , \tag{4.1}$$

where the time component of the nonrelativistic energy-momentum tensor is[5]

$$T^{00} = \frac{1}{2e^2}(E_i^2+B^2) + \frac{1}{2e^2c^2}(\partial_t N)^2 + \frac{1}{2e^2}(\partial_i N)^2 + \frac{1}{2e^2}m_A^2c^2N^2 + \frac{1}{2m}|D_i\psi|^2 + \tau\left[1+\frac{m_A}{2m}\right]N\rho + \frac{e^2}{8m^2c^2}\rho^2 \ . \tag{4.2}$$

With static fields, $\partial_t N=0$ and the electric field is just $E^i = -\partial_i A^0$, where A^0 is expressed in terms of the magnetic field B and the number density ρ via Gauss' law [see (3.4c)]

$$\nabla^2 A_0 + \frac{cm_A}{|\kappa|}(\kappa B+\rho) = 0 \ . \tag{4.3}$$

Thus the static energy may be written as

$$\mathcal{E} = c\int d^2x\left[\frac{1}{2e^2}B^2 - \frac{cm_A}{2|\kappa|}(\kappa B+\rho)\frac{1}{\nabla^2}(\kappa B+\rho) + \frac{1}{2e^2}N(m_A^2c^2-\nabla^2)N + \frac{1}{2m}|D_i\psi|^2 + \tau\left[1+\frac{m_A}{2m}\right]N\rho + \frac{e^2}{8m^2c^2}\rho^2\right] \ . \tag{4.4}$$

We now note the *identity*

$$|D_i\psi|^2 = |(D_1 \pm iD_2)\psi|^2 \mp m\epsilon^{ij}\partial_i J_j \pm \frac{1}{c}B\rho \ , \tag{4.5}$$

where J^i is the spatial part of the conserved current (3.2). The conservation law (or continuity equation) for this current

$$\partial_t\rho + \partial_i J^i = 0 \tag{4.6}$$

implies that in the static case J^i is transverse: $J^i = \epsilon^{ij}\partial_j j$. Thus for sufficiently well-behaved currents we can neglect the second term in (4.5) when it is integrated over all space in the energy.

Before using the identity (4.5) in the expression (4.4) for the static energy $\mathcal{E}$ we note that the expression for $\mathcal{E}$ involves $\tau \equiv \text{sgn}(\kappa)$. Thus in inserting (4.5) into (4.4) we can choose to correlate the $\pm$ sign with τ in one of two ways. Motivated by the preceding section we choose $\tau = \mp$. In fact, this choice will lead to an expression for $\mathcal{E}$ which is a sum of manifestly positive terms, each of which vanishes when the equations of motion together with the self-dual ansatz are satisfied—had we chosen the other correlation ($\tau = \pm$) this would not be the case. Thus, with this choice we write

$$\int d^2x\,|D_i\psi|^2 = \int d^2x\,|(D_1 - i\tau D_2)\psi|^2 - \frac{1}{c}\tau\int d^2x\, B\rho \ , \tag{4.7}$$

so that the energy is given by

$$\mathcal{E} = c\int d^2x\left[\frac{1}{2e^2}B\left[1-\frac{m_A^2c^2}{\nabla^2}\right]B - \tau m_A cB\frac{1}{\nabla^2}\rho - \frac{cm_A}{2|\kappa|}\rho\frac{1}{\nabla^2}\rho + \frac{1}{2e^2}N(m_A^2c^2-\nabla^2)N + \frac{1}{2m}|(D_1-i\tau D_2)\psi|^2 - \frac{\tau}{2mc}B\rho + \tau\left[1+\frac{m_A}{2m}\right]N\rho + \frac{e^2}{8m^2c^2}\rho^2\right] \ . \tag{4.8}$$

We can now complete the square in the N field to obtain

$$\mathcal{E}=\frac{c}{2m}\int d^2x\,|(D_1-i\tau D_2)\psi|^2+\frac{c}{2e^2}\int d^2x\left[N+\tau\frac{cm_A}{|\kappa|}\left[1+\frac{m_A}{2m}\right]\frac{1}{(m_A^2c^2-\nabla^2)}\rho\right]$$
$$\times(m_A^2c^2-\nabla^2)\left[N+\tau\frac{cm_A}{|\kappa|}\left[1+\frac{m_A}{2m}\right]\frac{1}{(m_A^2c^2-\nabla^2)}\rho\right]$$
$$+c\int d^2x\left[\frac{1}{2e^2}B\left[1-\frac{m_A^2c^2}{\nabla^2}\right]B-\tau m_A cB\frac{1}{\nabla^2}\rho-\frac{cm_A}{2|\kappa|}\rho\frac{1}{\nabla^2}\rho-\frac{\tau}{2mc}B\rho\right.$$
$$\left.+\frac{e^2}{8m^2c^2}\rho^2-\frac{cm_A}{2|\kappa|}\left[1+\frac{m_A}{2m}\right]^2\rho\left[\frac{1}{m_A^2c^2-\nabla^2}\right]\rho\right]. \tag{4.9}$$

The first two terms in (4.9) are manifestly positive, the first being a square and the second a quadratic form of the *positive* operator $(m_A^2c^2-\nabla^2)$. Remarkably, the rest of (4.9) is also manifestly positive, as is clear when it is expressed as a quadratic form of the positive operator $[(m_A^2c^2-\nabla^2)/-\nabla^2]$. In fact,

$$\mathcal{E}=\frac{c}{2m}\int d^2x\,|(D_1-i\tau D_2)\psi|^2+\frac{c}{2e^2}\int d^2x\left[N+\tau\frac{cm_A}{|\kappa|}\left[1+\frac{m_A}{2m}\right]\frac{1}{(m_A^2c^2-\nabla^2)}\rho\right]$$
$$\times(m_A^2c^2-\nabla^2)\left[N+\tau\frac{cm_A}{|\kappa|}\left[1+\frac{m_A}{2m}\right]\frac{1}{(m_A^2c^2-\nabla^2)}\rho\right]$$
$$+\frac{c}{2e^2}\int d^2x\left[B+\tau\frac{m_A}{2m|\kappa|}\left[\frac{2mm_Ac^2+\nabla^2}{m_A^2c^2-\nabla^2}\right]\rho\right]\left[\frac{m_A^2c^2-\nabla^2}{-\nabla^2}\right]\left[B+\tau\frac{m_A}{2m|\kappa|}\left[\frac{2mm_Ac^2+\nabla^2}{m_A^2c^2-\nabla^2}\right]\rho\right]. \tag{4.10}$$

This expression (4.10) for the static energy functional proves our claim that it may be written as a sum of manifestly positive terms. The Bogomol'nyi[16] minimum-energy conditions are now seen to be

$$(D_1-i\tau D_2)\psi=0\ , \tag{4.11a}$$

$$(m_A^2c^2-\nabla^2)N+\tau\frac{cm_A}{|\kappa|}\left[1+\frac{m_A}{2m}\right]\rho=0\ , \tag{4.11b}$$

$$B+\tau\frac{m_A}{2m|\kappa|}\left[\frac{2mm_Ac^2+\nabla^2}{m_A^2c^2-\nabla^2}\right]\rho=0\ . \tag{4.11c}$$

Note that (4.11a) is the self-duality condition (3.12) with the consistent choice (3.11) for the sign of the Chern-Simons coupling parameter κ. Equations (4.11b) and (4.11c) are identical to (3.13a) and (3.13b), respectively, with the relation (3.8) between the magnetic field B and the charge density ρ implied by (4.11a). This shows that, while in Sec. III self-duality was a convenient ansatz for simplifying the static equations of motion, here the condition of *minimum* static energy *forces* the self-dual (anti-self-dual) condition (4.11a). We may then use the results of Sec. III to see immediately that the minimum (static) energy configuration, given by the conditions (4.11), satisfies the equations of motion. We shall call nontrivial solutions of (4.11) static Maxwell-Chern-Simons solitons.

V. ASYMPTOTICS AND GLOBAL PROPERTIES OF THE SOLITONS

In this section we discuss the asymptotic behavior of the static self-dual solutions. This will enable us to derive certain global quantities characterizing these solitons, such as charge, magnetic flux, and angular momentum. At the end we will specialize to radially symmetric solutions.

The starting point for the asymptotic analysis is the system of coupled equations (3.13) (we recall that $N=-\tau A_0$). The key observation is that the Chern-Simons term is lower order in space-time derivatives with respect to the Maxwell term and therefore dominates the long-distance physics. For large $r\equiv|\mathbf{x}|$ we should therefore recover the equations of the Jackiw-Pi model.[5] In fact, for $r\gg 1/m_Ac$ the Laplacian can be neglected with respect to the mass term in (3.13a) and therefore N is proportional to ρ:

$$r\gg\frac{1}{m_Ac}:\ N=-\tau A_0=-\frac{\tau}{m_Ac|\kappa|}\left[1+\frac{m_A}{2m}\right]\rho\ . \tag{5.1}$$

Inserting this in (3.13b), we obtain the Liouville equation[15]

$$\nabla^2\ln\rho+\frac{2}{c|\kappa|}\rho=0\ , \tag{5.2}$$

which, as expected, coincides with the number density equation in the Jackiw-Pi model.

The most general radially symmetric and positive solution of (5.2) with ρ vanishing at spatial infinity involves two parameters r_0 and n (which can be chosen to be positive):

$$\rho(r)=\frac{4n^2c|\kappa|}{r^2}\left[\left[\frac{r_0}{r}\right]^n+\left[\frac{r}{r_0}\right]^n\right]^{-2}. \tag{5.3}$$

Using (3.8) and (5.1) this implies the following asymptotic behavior

$$\begin{aligned} r\to\infty:\ \rho &\sim r^{-\epsilon}\,, \\ A_0 &= -\frac{1}{\tau}N \sim r^{-\epsilon}\,, \\ B &\sim r^{-\epsilon}\,, \\ E^i &\sim x^i r^{-\epsilon-2}\,, \\ \epsilon &\geq 2\,. \end{aligned} \tag{5.4}$$

An immediate consequence of these asymptotic behaviors is that the electric field does not contribute to the integrated Gauss law,

$$\int d^2x\, \partial_i E^i = 0\,, \tag{5.5}$$

and therefore we obtain the usual relationship between flux and charge dictated by the Chern-Simons part of the Gauss law (3.3) [note that the $1/e$ in the definition of the magnetic flux Φ is due to our absorbing the coupling constant e into the gauge field, according to (2.2)]

$$\begin{aligned} \Phi &= -\frac{1}{e^2\kappa}Q\,, \\ \Phi &\equiv \frac{1}{e}\int d^2x\, B\,, \\ Q &\equiv e\int d^2x\, \rho\,. \end{aligned} \tag{5.6}$$

This is a consequence of the electric field being short ranged due to the damping caused by the topological mass $m_A=|\kappa|e^2/c$. The magnetic field is also short ranged for the same reason; however, the gauge-variant potential A_i is long ranged, giving a nonvanishing contribution to the magnetic flux.[13]

We must distinguish two types of solutions[7] to the coupled system of equations (3.13). The first type is when $|\psi|=\rho^{1/2}$ has no zeros in the finite plane. In this case B is given everywhere in the finite plane by $(\tau c/2)\nabla^2\ln\rho$ and the only contribution to the flux and charge comes from spatial infinity:

$$\Phi = -\tau\frac{2\pi c}{e}\frac{\epsilon}{2}\,,\quad Q = 2\pi c|\kappa|e\frac{\epsilon}{2}\,. \tag{5.7}$$

These are the "*bare soliton*" solutions. A second type of solution arises when $|\psi|$ has zeros in the finite plane. At the locations of the zeros $\nabla^2\ln\rho$ becomes singular. However, singularities in the magnetic field can be avoided by choosing the phase ω in (3.7) to be discontinuous, so that singularities in $-\partial_i\omega$ cancel those in $(\tau c/2)\epsilon^{ij}\partial_j\ln\rho$, and the total gauge field A_i and the magnetic field B are regular. Such cancellations require a discontinuity in ω of $(-\tau 2\pi c n)$ when an nth-order zero of $|\psi|$ is encircled. To ensure single valuedness of $\psi=\exp(i\omega/c)|\psi|$ when a zero of $|\psi|$ is encircled, one must then require the quantization of n in integer units. This shows that zeros of $|\psi|$ in the finite plane lead to a *nonvanishing vorticity*. We will give a detailed example of this mechanism below, where we discuss radially symmetric solutions.

The flux and charge of this second type of solution can be computed by adding to the contribution from spatial infinity the contributions of all the zeros of $|\psi|$ in the finite plane. These are obtained by encircling the locations of the zeros with infinitesimal lines, integrating $[-(1/e)A_i]$ along these lines, and adding the contribution from the rest of the finite plane [where $B=(\tau c/2)\nabla^2\ln\rho$] of which the infinitesimal lines form now the boundary. Thus the contributions to flux and charge from zeros of $|\psi|$ in the finite plane arise purely from the vorticity, giving a total flux and charge of

$$\Phi = -\tau\frac{2\pi c}{e}\left[n+\frac{\epsilon}{2}\right]\,,\quad Q = 2\pi c|\kappa|e\left[n+\frac{\epsilon}{2}\right]\,, \tag{5.8}$$

where n is the number of zeros of $|\psi|$ in the finite plane (counted with multiplicity). This second type of solution can be viewed as a composite of the "bare" soliton with n vortices located at the zeros of $|\psi|$.

The total angular momentum of the solitons is the sum of the contributions from the nonrelativistic matter ψ and the gauge field A_i. In fact, the momentum density of the neutral scalar N vanishes for static solutions, so

$$\begin{aligned} \mathcal{J} &= \mathcal{J}_\psi + \mathcal{J}_A\,, \\ \mathcal{J}_\psi &= \int d^2x\, x^i\epsilon^{ij}\Pi^j_\psi\,,\quad \mathcal{J}_A = \int d^2x\, x^i\epsilon^{ij}\Pi^j_A\,. \end{aligned} \tag{5.9}$$

The momentum densities Π^i_ψ and Π^i_A are given by

$$\Pi^i_\psi = -\frac{ic}{2}[\psi^*(D_i\psi)-(D_i\psi)^*\psi] = c\,\mathrm{Im}\psi^* D_i\psi\,, \tag{5.10a}$$

$$\Pi^i_A = -\frac{1}{e^2}F^{0j}F^i{}_j\,. \tag{5.10b}$$

Using (3.6) and (3.7) we can rewrite the matter momentum density of the static self-dual solutions as

$$\Pi^i_\psi = \frac{\tau c}{2}\epsilon^{ij}\partial_j\rho\,. \tag{5.11}$$

An integration by parts (with no boundary term since ρ vanishes more rapidly than r^{-2} at spatial infinity) then leads to

$$\mathcal{J}_\psi = \frac{\tau c}{e}Q\,, \tag{5.12}$$

where Q is the total charge (5.8). The gauge field contribution to the angular momentum can be written in compact notation by using $F_{ij}=-\epsilon^{ij}B$:

$$\mathcal{J}_A = -\frac{1}{e^2}\int d^2x\, B(x^iE^i)\,. \tag{5.13}$$

This gives the total angular momentum of the soliton as

$$\mathcal{J} = \frac{\tau c}{e}Q - \frac{1}{e^2}\int d^2x\, B(x^iE^i)\,. \tag{5.14}$$

In the pure Chern-Simons limit $e^2\to\infty$ (κ fixed) the gauge contribution $\mathcal{J}_A$ vanishes, as is clear from (5.14). This is because there is no energy-momentum tensor associated with the Chern-Simons term, it being independent of the metric.

We now discuss radially symmetric solutions. In this case the field ψ takes the form

$$\psi(\mathbf{x})=e^{i(\omega/c)\theta}\rho^{1/2}(r) , \tag{5.15}$$

and the solution for $r \gg 1/m_A c$ takes the form (5.3). In the pure Chern-Simons limit $e^2=m_A c/|\kappa| \to \infty$ (κ fixed) this solution extends all the way to $r=0$ and absence of singularities and single valuedness of ψ require $\lim_{e^2\to\infty} n=S$, with $S=\text{integer}\geq 1$ and $\omega=-\tau c(S-1)\theta$.[5] However, since n is independent of e^2 [clearly $e^2=m_A c/|\kappa|$ does not enter (5.2)] and *assuming smoothness of the limit* $e^2\to\infty$, the solution for $r\gg 1/m_A c$ is given by

$$r \gg \frac{1}{m_A c}:\ \rho(r)=\frac{4S^2 c|\kappa|}{r^2}\left[\left(\frac{r_0}{r}\right)^S+\left(\frac{r}{r_0}\right)^S\right]^{-2} ,$$
$$S=\text{integer}\geq 1 . \tag{5.16}$$

In the region $r \ll 1/m_A c$ instead, the mass term is negligible with respect to the Laplacian in (3.13a). Therefore $N(=-\tau A_0)$ vanishes with two powers more rapidly than ρ for $r\to 0$; for $r\ll 1/m_A c, r \ll 1/mc$, the inhomogeneous term in (3.13b) can be neglected, leaving the Liouville equation

$$\nabla^2 \ln\rho-\frac{m_A}{2m}\frac{2}{c|\kappa|}\rho=0 \tag{5.17}$$

with positive, radially symmetric solution

$$\rho=\frac{4M^2 c|\kappa|}{r^2}\frac{2m}{m_A}\left[\left(\frac{r_1}{r}\right)^M-\left(\frac{r}{r_1}\right)^M\right]^{-2} , \tag{5.18}$$

where M can be chosen to be positive. In order to avoid singularities the scale r_1 has to be chosen $r_1=O(1/m_A c, 1/mc)$ or bigger, leaving

$$\begin{aligned} r \ll \frac{1}{m_A c},\frac{1}{mc}:\ & \rho\sim r^{2M-2} ,\\ & B\sim r^{2M-2} ,\\ & A_0\sim r^{2M} ,\\ & E^i\sim x^i r^{2M-2} ,\\ & A_i=-\partial_i\omega+\tau c\epsilon^{ij}\frac{x^j}{r^2}(M-1) . \end{aligned} \tag{5.19}$$

To avoid a singularity at the origin we require $M\geq 1$. A singularity in A_i can then be avoided by choosing

$$\omega=-\tau c(M-1)\theta , \tag{5.20}$$

and single valuedness of ψ given by (5.15) requires M to be an integer. Moreover, since ω is independent of r we have to identify M with the integer S characterizing the solution for $r\gg 1/m_A c$. Note that, as we have argued in general before, the solution is characterized by the number ϵ determining the behavior at spatial infinity, in this case $\epsilon=2+2S$, and by the number of zeros in $|\psi|$, in this case $S-1$ zeros all located at the origin (additional zeros of ρ between $r=0$ and $r=\infty$ would cause a singularity in A_i). Therefore the flux and charge of the radially symmetric solitons are given by

$$\Phi=-\tau\frac{2\pi c}{e}2S ,\quad Q=2\pi ce|\kappa|2S . \tag{5.21}$$

Note that the matter contribution to the angular momentum of the "bare" soliton ($S=1$), $\mathcal{J}_\psi^{\text{bare}}=4\pi c^2\kappa$, is "dual" to the angular momentum $\mathcal{J}_P$ of an elementary matter particle[13] in the sense

$$\mathcal{J}_\psi^{\text{bare}}=\frac{1}{\mathcal{J}_P} . \tag{5.22}$$

VI. CONCLUSION

To conclude we first briefly summarize our results. We have considered a nonrelativistic limit of the relativistic model of Lee, Lee, and Min[6] which itself is the bosonic portion of an $N=2$ supersymmetric theory. This relativistic theory describes a charged scalar field interacting with a massive Abelian gauge field (having both a Maxwell and a Chern-Simons term in the Lagrangian) and a neutral scalar field of the same mass. In this paper we have analyzed the nonrelativistic limit in the charged matter sector. A self-dual ansatz plays a natural role in the analysis of this nonrelativistic theory and reduces the equations of motion to two coupled nonlinear equations for the number density and the neutral scalar field. In the limit in which the Maxwell term is removed the neutral scalar decouples and the remaining equation for the number density is just the Liouville equation, as found directly with the pure Chern-Simons coupling by Jackiw and Pi.[5] In the full Maxwell-Chern-Simons theory the static energy functional is shown to be a sum of manifestly positive terms. Minimizing the energy leads to a self-dual configuration which also solves the second-order equations of motion. Finally we have discussed the asymptotic behavior of the important fields in the static, self-dual case and thereby determined the charge, magnetic flux, and angular momentum of these Maxwell-Chern-Simons solitons.

This model raises a number of interesting questions which deserve further investigation. First of all, it would be important to have a more clear physical understanding of the relationship between the $N=2$ supersymmetry condition (in the origin model of Lee, Lee, and Min[6]), the self-duality condition in the static theory, and the vanishing of the static energy in the *nonrelativistic* theory (both with the pure Chern-Simons coupling[5] and with the Maxwell-Chern-Simons coupling discussed in this paper). On a more fundamental level, the *quantum* significance of these classical soliton solutions in the nonrelativistic theories remains to be understood. This may also have important implications for the theoretical analysis of quantum phenomena in planar condensed-matter systems where the charged particles are nonrelativistic.

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