Superspace formulation of Yang-Mills theory

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We give a formulation of gauge theories in a six-dimensional superspace in which the additional two dimensions are anticommuting. We write down an action that is gauge invariant under gauge transformations in six-dimensional space and has invariance under rotations in this space. This action, when reduced to four dimensions, naturally contains the composite operators and their sources that one introduces in order to discuss the renormalization of gauge theories using the Becchi-Rouet-Stora transformation. It is shown how the six-dimensional theory can be reduced to yield the usual results of four-dimensional theory.

I. INTRODUCTION

The renormalization of gauge theories has been studied extensively.[1–9] In the renormalization of gauge theories and especially that of gauge-invariant operators,[5–7] a great simplification results[4] when one takes into account the Becchi-Rouet-Stora (BRS) symmetry,[8] as this enables one to formulate the Ward-Takahashi (WT) identities (which are a result of underlying gauge invariance) in a compact and mathematically convenient form.[4,9] A brief analysis of the formulation of WT identities is given at the beginning of Sec. II.

In the formulation of WT identities using BRS invariance, one has to naturally consider two composite operators (in linear gauges), viz. $D_\mu^{\alpha\beta}c^\beta$ and $\frac{1}{2}f^{\alpha\beta\gamma}c^\beta c^\gamma$ which occur in the BRS variations. As the Green's functions of these composite operators enter the WT identities, the action for the gauge theories has to be modified by introducing sources for these composite operators. The WT identities for gauge theories and for insertion of gauge-invariant operators are then cast in a very simple form:[9]

$$\mathcal{G}'\Gamma=0\ , \tag{1.1}$$

where $\mathcal{G}'$ is a nilpotent differential operator, $\mathcal{G}'^2=0$. (We suppress here the details which are given in Sec. II.) The presence of a nilpotent operator is suggestive of its relation to an anticommuting variable θ through a possible relation of the kind $\mathcal{G}'=\partial/\partial\theta$, for then $\mathcal{G}'^2=0$ would be automatically guaranteed. This suggests that we try to formulate gauge theories in a superspace.

With this aim, in this work we shall attempt to formulate Yang-Mills theories in a six-dimensional superspace in which the two extra dimensions are anticommuting. (The choice of the superspace is justified by the results and by the fact that the *two* composite operators $D_\mu^{\alpha\beta}c_\beta$ and $\frac{1}{2}f^{\alpha\beta\gamma}c^\beta c^\gamma$ must *naturally* arise in this formulation.) The aim is to formulate the gauge theories in this superspace in such a manner that *the whole action including the source terms for the composite operators* is accomodated in a *single superspace action.*

To this end, we first formulate a "larger" theory consisting of superfields $A_\mu(x,\lambda,\theta)$ $(\mu=0,1,\ldots,5)$ and $\zeta(x,\lambda,\theta)$ defined on the six-dimensional superspace (x,λ,θ). This theory has a generalized gauge invariance in this superspace (except for the gauge-fixing term), and the Lagrange density is a scalar under rotations in the six-dimensional superspace (with a metric). In addition, a superfield $K_\mu(x,\lambda,\theta)$ of sources is also introduced. This theory, when broken up in terms of the component fields in four-dimensional space, can be shown to be related to the usual Yang-Mills theory. This formulation thus exhibits an underlying superspace symmetry of the Yang-Mills theory.

We shall briefly present the plan of the paper. In Sec. II we summarize the results on the formulation of WT identities using BRS invariance and introduce superspace notations. In Sec. II we shall introduce the superspace action of a "larger" theory and simplify it in the four-dimensional notation. In Sec. IV we shall establish the correspondence between the generating functionals of the superspace theory and the usual Yang-Mills theory.

The final aim of our work is to establish the WT identities for the Yang-Mills theory in terms of the generating functional of the superspace theory, where we expect Eq. (1.1) to be reexpressed in a simpler form. This has applications in a simplified treatment of the renormalization problem of gauge theories and of gauge-invariant operators. This part of the work will be presented in subsequent work.[10] In this work we shall be content with establishing the contact between the superspace theory and the usual Yang-Mills theory with composite operators introduced.

II. PRELIMINARY

A. Review of gauge theories

The unrenormalized but dimensionally regularized Green's functions of gauge theories are derived from the effective action (in linear gauges) in the usual notation:

$$\mathcal{L}_{\rm eff}[A,c,\zeta]=\mathcal{L}_0[A]-\tfrac{1}{2}\eta_0\int d^4x\,[\partial\cdot A^\alpha(x)]^2 + \int d^4x\,\partial^\mu\zeta^\alpha(x)D_\mu^{\alpha\beta}c_\beta(x)\ , \tag{2.1}$$

where

$$\mathcal{L}_0[A]=-\tfrac{1}{4}\int F^{\alpha}_{\mu\nu}(x)F^{\alpha\mu\nu}(x)d^4x \ , \tag{2.2}$$

which is the gauge-invariant action, and c and ζ are the ghost and antighost fields.

It is well known that the above action $\mathcal{L}_{\rm eff}$ is invariant under the BRS supersymmetry transformations:

$$\delta A^{\alpha}_{\mu}(x)=D^{\alpha\beta}_{\mu}c^{\beta}(x)\delta\Lambda \ ,$$

$$\delta c^{\alpha}(x)=-\tfrac{1}{2}g_0 f^{\alpha\beta\gamma}c^{\beta}(x)c^{\gamma}(x)\delta\Lambda \ , \tag{2.3a}$$

$$\delta\zeta^{\alpha}(x)=-\eta_0\partial\cdot A^{\alpha}(x)\delta\Lambda \ . \tag{2.3b}$$

It is known that the discussion of the renormalization of $\mathcal{L}_{\rm eff}$ (alone or with additional gauge-invariant terms) involves the discussion of the renormalization of the two "external" composite operators $D^{\alpha\beta}_{\mu}c^{\beta}(x)$ and $-\frac{1}{2}g_0 f^{\alpha\beta\gamma}c^{\beta}(x)c^{\gamma}(x)$ and therefore is more convenient to consider instead of $\mathcal{L}_{\rm eff}$:

$$S[A,c,\zeta,\kappa,l]=\mathcal{L}_{\rm eff}+\int d^4x\ \kappa^{\alpha\mu}(x)D^{\alpha\beta}_{\mu}c^{\beta}(x) + \int d^4x\ \tfrac{1}{2}g_0 l^{\alpha}(x)f^{\alpha\beta\gamma}c^{\beta}(x)c^{\gamma}(x) \ . \tag{2.4}$$

Here κ and l are two new sources for the composite operators mentioned above.

This action without its gauge-fixing term, i.e.,

$$\tilde{S}=S+\tfrac{1}{2}\eta_0\int d^4x[\partial\cdot A^{\alpha}(x)]^2 \equiv S+\tfrac{1}{2}\int d^4x\ f^2_{\alpha}[A] \ , \tag{2.5}$$

has the following properties.

(a) On account of the BRS invariance of the two composite operators mentioned above, $\tilde{S}$ is invariant under the BRS transformations of (2.3a), just as $\mathcal{L}_{\rm eff}+\frac{1}{2}\eta_0\int d^4x(\partial\cdot A^{\alpha})^2$ is.

(b) It has a simple property:

$$\int d^4x \left[\frac{\delta\tilde{S}}{\delta A^{\alpha}_{\mu}(x)}+\frac{\delta\tilde{S}}{\delta\kappa^{\alpha}_{\mu}(x)}+\frac{\delta\tilde{S}}{\delta c^{\alpha}(x)}\ \frac{\delta\tilde{S}}{\delta l^{\alpha}(x)}\right]=0 \ . \tag{2.6}$$

Its (renormalized) analog is preserved under renormalization to all orders.

(c) When gauge-invariant operators are included in addition to $\tilde{S}$, the necessary counterterms satisfy

$$\mathcal{G}'O[A,c,\zeta,K,l]=0 \ ,$$

where $\mathcal{G}'$ is expressed simply in terms of $\tilde{S}$:

$$\mathcal{G}'=\int\left\{\frac{\delta\tilde{S}}{\delta A^{\alpha}_{\mu}(x)}\ \frac{\delta}{\delta\kappa^{\alpha}_{\mu}(x)}+\frac{\delta\tilde{S}}{\delta\kappa^{\alpha}_{\mu}(x)}\ \frac{\delta}{\delta A^{\alpha}_{\mu}(x)} + \frac{\delta\tilde{S}}{\delta c^{\alpha}(x)}\ \frac{\delta}{\delta l^{\alpha}(x)}+\frac{\delta\tilde{S}}{\delta l^{\alpha}(x)}\ \frac{\delta}{\delta c^{\alpha}(x)}\right\}d^4x \ . \tag{2.7}$$

For these reasons we shall be interested in the supersymmetry of $\tilde{S}$ rather than that of $\mathcal{L}_{\rm eff}$ only.

We now wish to modify slightly the form of S of Eq. (2.4). In Eq. (2.4) it has not been necessary to introduce a source for the gauge-fixing term $f_{\alpha}[A]$ because in linear gauges $f_{\alpha}[A]$ is not a composite operator. But one can introduce a source for it (which becomes necessary when dealing with bilinear gauges.[4] Then

$$S[A,c,\bar{c},\zeta,K,l,t]=\tilde{S}-\tfrac{1}{2}\int d^4x[f^{\alpha}(A)+t^{\alpha}]^2 \ , \tag{2.5'}$$

where an additional field-independent term $-\frac{1}{2}\int d^4x\ t^{\alpha^2}(x)$ has been added for future convenience.

B. Notation

We shall work in a space of six dimensions, four of which x^{μ} denote the usual space-time, and λ and θ are the other two anticommuting coordinates. Generally, we shall use an overbar to denote things in six-dimensional space. Thus $\bar{x}=(x^{\mu},\lambda,\theta)$. The superfields will be functions of $\bar{x}$.

Unless otherwise stated, we shall always use the left derivaties when they are with respect to anticommuting quantities. [The left derivative of a function $X(\bar{x})$ with respect to, say, λ is defined by

$$\delta X(\bar{x})=\delta\lambda\frac{\partial X(\bar{x})}{\partial\lambda} \ . \tag{2.8}$$

We may for compactness write this left derivative by $X_{,\lambda}$.]

We shall choose the metric in this space to be $\bar{g}_{\mu\nu}=g_{\mu\nu}$ $[0\le\mu,\nu\le 3]$; $-g_{45}=g_{54}=1$. Here $g_{\mu\nu}$ is the standard Lorentz metric diag$(1,-1,-1,-1)$, and the rest of the $\bar{g}_{\mu\nu}$'s not defined above are zero.

Any $X(\bar{x})$ can be expanded in powers of λ and θ $(\lambda^2=\theta^2=0)$ as

$$X(\bar{x})\equiv X(x)+\lambda\tilde{X}_{,\lambda}(x)+\theta\tilde{X}_{,\theta}(x)+\lambda\theta\tilde{X}_{,\lambda\theta} \ . \tag{2.9}$$

All the coefficients (of $1,\lambda,\theta,\lambda\theta$) are functions of x only and, evidently,

$$\tilde{X}_{,\lambda}=X_{,\lambda}|_{\theta=0}=X_{,\lambda}-\theta\tilde{X}_{,\lambda\theta} \ ,$$
$$\tilde{X}_{,\theta}=X_{,\theta}|_{\lambda=0}=X_{,\theta}+\lambda X_{,\lambda\theta}\equiv X_{,\theta}-\lambda X_{,\theta\lambda} \ , \tag{2.10}$$

and also

$$X(\bar{x})=X(x)+\lambda\tilde{X}_{,\lambda}+\theta X_{,\theta}=X(x)+\lambda X_{,\lambda}+\theta\tilde{X}_{,\theta} \ . \tag{2.11}$$

C. Gauge superfields

We shall introduce a gauge superfield on the superspace $\bar{A}^{\alpha}_{\mu}(\bar{x})\equiv(A^{\alpha}_{\mu}(\bar{x}),c^{\alpha}_4(\bar{x}),c^{\alpha}_5(\bar{x}))$ and assume that it transforms like a covariant vector under coordinate transformations in the superspace that preserve $\bar{g}_{\mu\nu}\bar{x}^{\mu}\bar{x}^{\nu}$. These transformations are elements of OSp $(3,1|2)$.] In the coordinate frame in which the first four coordinates in $\bar{x}$ are identified with the space-time, $A^{\alpha}_{\mu}(\bar{x})|_{\lambda=\theta=0}$ can be identified with the gauge field $A^{\alpha}_{\mu}(x)$, while $c^{\alpha}_4(x)$ and $c^{\alpha}_5(x)$ will become related to the ghost field $c^{\alpha}(x)$.

The definitions of covariant derivatives and field strengths can be generalized easily for these superfields. Thus a gauge transformation characterized by infinitesimal parameters $\{\omega_\alpha(\bar{x})\}$ is defined by

$$\bar{A}^{\alpha}_{\mu}(\bar{x}) \to \bar{A}'^{\alpha}_{\mu}(\bar{x}) = \bar{A}^{\alpha}_{\mu}(\bar{x}) + [\delta^{\alpha\beta}\vec{\partial}_\mu - g f^{\alpha\beta\gamma}\bar{A}^{\gamma}_{\mu}(\bar{x})]\omega_\alpha(\bar{x}) . \tag{2.12}$$

The field strengths defined by

$$\bar{F}^{\alpha}_{\mu\nu}(\bar{x}) = \partial_\mu \bar{A}^{\alpha}_{\nu}(\bar{x}) - \bar{A}^{\alpha}_{\mu}(\bar{x})\overleftarrow{\partial}_\nu + g f^{\alpha\beta\gamma}\bar{A}^{\beta}_{\mu}(\bar{x}) A^{\gamma}_{\nu}(\bar{x}) \tag{2.13}$$

are then covariant under the gauge transformation (2.12). We note the slight modification in the second term on the right-hand side (RHS) of Eq. (2.13) done in order to make it applicable for anticommuting dimensions. [$\bar{A}^{\alpha}_{\mu}(x)\overleftarrow{\partial}_\nu$ is the right derivative.] We note that while

$$\bar{F}^{\alpha}_{\mu\nu}(\bar{x}) = -\bar{F}^{\alpha}_{\nu\mu}(\bar{x}) , \quad \mu \le 3 , \quad \text{and/or} \quad \nu \le 3 ,$$

as usual, we have

$$\bar{F}^{\alpha}_{\mu\nu}(\bar{x}) = \bar{F}^{\alpha}_{\nu\mu}(\bar{x}) , \quad 4 \le \mu,\nu \le 5 , \tag{2.14}$$

which is as it should be since μ and ν refer to anticommuting indices.

We shall also introduce the antighost field $\xi^\alpha(\bar{x})$ and assume that it is a scalar under OSp (3,1|2) transformations.

III. SUPERSPACE ACTIONS

In this section we shall make an observation which will have a suggestive significance regarding what will be done in the later sections. Some of the details of the discussion in this section will have no direct use later.

As stated in Sec. II A, what enters naturally in the discussion of renormalization in gauge theories is

$$\tilde{S}[A,c,\xi,\kappa,l] = \mathcal{L}_0[A] + \int d^4x [\kappa^{\alpha\mu}(x) D^{\alpha\beta}_{\mu} c^\beta(x) + \tfrac{1}{2} g_0 l^\alpha(x) f^{\alpha\beta\gamma} c^\beta(x) c^\gamma(x)] . \tag{3.1}$$

We have thus, in addition to the gauge-invariant Lagrangian $\mathcal{L}_0$ and the ghost term $(\partial\xi)_i(Dc)_i$, two additional terms with sources κ_i and l_α for the two composite operators. Our object in this section is to observe that if certain identifications regarding $\bar{A}_{\mu,\lambda}$ and $\bar{A}_{\mu,\theta}$ are made, the above action (with both sources κ and l) can be expressed as a gauge-invariant, coordinate-invariant (in superspace) action expressed in terms of the superfields only.

Consider the expression

$$\bar{\mathcal{L}}_0[\bar{A}] = -\tfrac{1}{4}\bar{g}^{\mu\lambda}\bar{g}^{\nu\sigma}\bar{F}^{\alpha}_{\mu\nu}(\bar{x})\bar{F}^{\alpha}_{\lambda\sigma}(\bar{x}) . \tag{3.2}$$

[$\bar{g}$ is the inverse of g: $\bar{g}^{\mu\nu} = g^{\mu\nu}$, $0 \le \mu,\nu \le 3$; $\bar{g}^{45} = \bar{g}_{54} = 1 = -\bar{g}^{54}$.] Evidently, $\bar{\mathcal{L}}_0[\bar{A}]$ is Lorentz invariant and gauge invariant in the generalized sense, and contains $\mathcal{L}_0[A]$ as a part. We now proceed to evaluate it:

$$\begin{aligned}
F^{\alpha}_{4\sigma} &= \partial_\lambda A^{\alpha}_{\sigma}(\bar{x}) - \partial_\sigma c^{\alpha}_{4}(\bar{x}) + g f^{\alpha\beta\gamma} c^{\beta}_{4}(\bar{x}) A^{\gamma}_{\sigma}(\bar{x}) \\
&= A^{\alpha}_{\sigma,\lambda} + D^{\alpha\beta}_{\sigma} c^{\beta}_{4} = -F^{\alpha}_{\sigma 4} , \\
F^{\alpha}_{5\sigma} &= A^{\alpha}_{\sigma,\theta} + D^{\alpha\beta}_{\sigma} c^{\beta}_{5} = -F^{\alpha}_{\sigma 5} , \\
F^{\alpha}_{44} &= 2c^{\alpha}_{4,\lambda} + g f^{\alpha\beta\gamma} c^{\beta}_{4} c^{\gamma}_{4} , \\
F_{55} &= 2c^{\alpha}_{5,\theta} + g f^{\alpha\beta\gamma} c^{\beta}_{5} c^{\gamma}_{5} , \\
F_{45} &= c^{\alpha}_{5,\lambda} + c^{\alpha}_{4,\theta} + f^{\alpha\beta\gamma} c^{\beta}_{4} c^{\gamma}_{5} = F_{54} .
\end{aligned} \tag{3.3}$$

Then,

$$\bar{\mathcal{L}}_0[\bar{A}] = -\tfrac{1}{4}F_{\mu\nu}(\bar{x})F^{\mu\nu}(\bar{x}) - F^{\alpha}_{4\sigma}(\bar{x})F^{\alpha}_{5\sigma}(\bar{x}) - \tfrac{1}{2}F^{\alpha}_{44}(\bar{x})F^{\alpha}_{55}(\bar{x}) + \tfrac{1}{2}F^{\alpha}_{45}(\bar{x})F^{\alpha}_{54}(\bar{x}) . \tag{3.4}$$

[Note that since $F_{\mu\nu}(\bar{x})$ in the first term does not carry a bar over it, μ,ν range only from 0 to 3. So also $0 \le \sigma \le 3$ in the second term.]

Using Eqs. (3.3), we get

$$\bar{\mathcal{L}}_0[\bar{A}] = -\tfrac{1}{4}F_{\mu\nu}(\bar{x})F^{\mu\nu}(\bar{x}) - (A^{\alpha}_{\sigma,\lambda} + D^{\alpha\beta}_{\sigma}c^{\beta}_{4})(A^{\alpha}_{\sigma,\theta} + D^{\alpha\beta}_{\sigma}c^{\beta}_{5}) - \tfrac{1}{2}(2c^{\alpha}_{4,\lambda} + g f^{\alpha\beta\gamma}c^{\beta}_{4}c^{\gamma}_{4})(2c_{5,\theta} + g f^{\alpha\beta\gamma}c^{\beta}_{5}c^{\gamma}_{5}) + \tfrac{1}{2}(c^{\alpha}_{4,\sigma} + c^{\alpha}_{5,\lambda} + g f^{\alpha\beta\gamma}c^{\beta}_{4}c^{\gamma}_{5})^2 . \tag{3.5}$$

We now wish to make an observation. *Suppose* one makes the following identifications.

(i) λ refers to the same anticommuting parameter that enters the BRS transformations of Eq. (2.3):

$$\text{(ii)} \quad c^{\alpha}_{4}(\bar{x}) = c^{\alpha}_{5}(\bar{x}) = -c^{\alpha}(\bar{x}) ; \tag{3.6}$$

i.e., if we let, in particular,

$$\begin{aligned}
A^{\alpha}_{\sigma,\lambda}(\bar{x}) &= -D^{\alpha\beta}_{\sigma}c^{\beta}(\bar{x}) , \\
c^{\alpha}_{4,\lambda}(\bar{x}) &= c^{\alpha}_{5,\lambda}(\bar{x}) = \tfrac{1}{2} g f^{\alpha\beta\gamma} c^{\beta}(x) c^{\alpha}(x) ,
\end{aligned} \tag{3.7}$$

we obtain

$$\bar{\mathcal{L}}_0[\bar{A}] = -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + 2A^{\alpha}_{\sigma,\theta}D^{\alpha\beta}_{\sigma}c^{\beta} + c^{\alpha}_{,\theta}\tfrac{1}{2} g f^{\alpha\beta\gamma}c^{\beta}c^{\gamma} + \tfrac{1}{2}c^{\alpha 2}_{,\theta} , \tag{3.8}$$

where we have used

$$(f^{\alpha\beta\gamma}c^{\beta}c^{\gamma})^2 = 0 , \tag{3.9}$$

on account of the Jacobi identity.

Then, apart from the fact that all fields are functions of $\bar{x}$, the expressions (3.8) for $\mathcal{L}_0[A]$ and S_0 of Eq. (3.1) are identical if the sources $(\kappa^{\alpha}_{\mu} - \partial_\mu \xi^\alpha)$ and l^α are identified with $2A^{\alpha}_{\sigma,\theta}$ and $c^{\alpha}_{,\theta}$, respectively.

Before discussing the significance of this observation, we must make some technical comments.

(i) Equations (3.7) are consistent with

$$\frac{\partial^2}{\partial\lambda^2} A_\mu^\alpha(\bar{x}) = 0 = \frac{\partial^2}{\partial\lambda^2} c^\alpha(\bar{x}) \ .$$

(ii) In identifying $l_\alpha = c^\alpha_{,\theta}$, for example, we note that $l_{\alpha,\lambda}$ is not independent, but determined by (3.7) via $l_{\alpha,\lambda} = c^\alpha_{,\theta\lambda} = -c_{,\lambda\theta}$.

This serves as a *preliminary indication* that it is possible for $\mathcal{L}_0[\bar{A}]$ of Eq. (3.5) to generate the action of Eq. (3.1); i.e., it has just the right operators in it. In Sec. IV we shall actually derive a relation between the generating functionals of the theory derived from $\mathcal{L}_0[\bar{A}]$ and the usual Yang-Mills theory where this connection will be shown explicitly to appear, but in a *different* way. [It should be remarked that the identifications of Eqs. (3.6) and (3.7) will have no direct bearing on the discussion in Sec. IV.]

IV. RELATION BETWEEN GAUGE THEORY AND SUPERSPACE THEORY

The aim of this section is to establish the link between the generating functional of the usual gauge theory and the generating functional for the theory as formulated in six-dimensional superspace. This is the relation of Eq. (4.7) below.

We first note that the generating functional of the ordinary gauge theory is given by

$$W[j_\mu^\alpha, \bar{\xi}^\alpha, \xi^\alpha, \kappa^{\alpha\mu}, l^\alpha, t^\alpha] = \int DA\, Dc\, D\zeta \times \exp(i\Sigma[A, c, \zeta, \kappa^{\alpha\mu}, l^\alpha, j_\mu^\alpha, \bar{\xi}^\alpha, \xi^\alpha, t^\alpha]) \ ,$$

where [see Eq. (2.5)]

$$\Sigma = S[A, c, \bar{c}, -\kappa, -l, t] + \int d^4x [j^{\alpha\mu} A_\mu^\alpha + \bar{\xi}^\alpha c_\alpha + \zeta^\alpha \bar{\xi}^\alpha] \ . \tag{4.1}$$

We define the generating functional of the superspace theory. Consider the generating functional

$$\bar{W}[\bar{K}(\bar{x}), t(\bar{x})] \equiv \int \{d\bar{A}\}\{d\zeta\} \exp(iS[\bar{A}, \zeta, \bar{K}, l]) \ , \tag{4.2}$$

where (we set $\eta_0 = 1$ for simplicity)

$$\bar{S}[\bar{A}, \zeta, \bar{K}, l] = \mathcal{L}_0[\bar{A}] + \int d^4x \frac{\partial}{\partial\theta} \{\bar{K}^i(\bar{x}) \bar{A}_i^\alpha(\bar{x}) + \zeta^\alpha(\bar{x})[\partial \cdot A^\alpha(\bar{x}) + \tfrac{1}{2}\zeta^\alpha_{,\theta}(\bar{x}) + t^\alpha(\bar{x})]\} \ , \tag{4.3}$$

$$\begin{aligned} \{d\bar{A}\} &\equiv \{dA\}\{dc_5\}\{dc_4\} \ , \\ \{dA\} &= \prod_{\alpha\mu x} dA_\mu^\alpha(\bar{x}) dA_{\mu,\lambda}^\alpha(\bar{x}) dA_{\mu,\theta}^\alpha(\bar{x}) \quad (0 \le \mu \le 3) \ , \\ \{dc_5\} &= \prod_{\alpha x} dc_5^\alpha(\bar{x}) dc_{5,\lambda}^\alpha(\bar{x}) dc_{5,\theta}^\alpha(\bar{x}) \ , \end{aligned} \tag{4.4}$$

and similar definitions for $\{dc_4\}, \{d\zeta\}$.

Here $\bar{K}_\mu^\alpha(\bar{x})$ is a supermultiplet (a contravariant vector) of sources on the superspace. As we shall show, it essentially contains in it j_μ^α, κ_μ^α, ξ^α, and l^α of Eq. (4.1). $t^\alpha(\bar{x})$ is a scalar [under OSp(3,1|2)] source for the scalar superfield $\zeta^\alpha(\bar{x})$.

We note that $\mathcal{L}_0[\bar{A}]$ is (i) generalized gauge invariant and (ii) generalized Lorentz invariant. The second term, which violates both of these invariances, has a relatively simple form and shall be shown to generate (i) a gauge-fixing term, (ii) a ghost term, (iii) source terms for the gague and ghost fields, and (iv) source terms for the composite operators in (4.1). We shall be more precise about these properties soon.

Before proceeding to prove the properties of $\bar{W}[\bar{K}, t]$, we shall make two comments.

(a) A field over the superspace such as $A_\mu^\alpha(\bar{x})$ contains four independent fields over x for each α and μ. [See the decomposition of Eq. (2.9).] For each (α,μ) we are integrating over only three independent fields for each x. Thus $\bar{W}$ could be, in principle, a function of $\tilde{A}_{\mu,\lambda\theta}^\alpha(x)$, say, in addition to that of $\bar{K}$ and t. As we shall see, this is not the case.

(b) For reasons of convenience, we have written the measure as

$$dA_\mu^\alpha(\bar{x}) dA_{\mu,\theta}^\alpha(\bar{x}) dA_{\mu,\lambda}^\alpha(\bar{x}) \cdots \ , \tag{4.5}$$

even though $A_\mu^\alpha(\bar{x})$, $A_{\mu,\lambda}^\alpha(\bar{x})$, and $A_{\mu,\theta}^\alpha(\bar{x})$ contain in them three, two, and two independent quantities in each, respectively. Here it is assumed that the integrations over $A_{\mu,\lambda}^\alpha$ and $A_{\mu,\theta}^\alpha$ are to be performed before integration over $A_\mu^\alpha(\bar{x})$, etc. $A_{\mu,\lambda\theta}^\alpha$ is always held fixed. Thus we could have replaced (4.5) above by

$$dA_\mu^\alpha(x) d\tilde{A}_{\mu,\theta}^\alpha(x) d\tilde{A}_{\mu,\lambda}^\alpha(x) \cdots \ . \tag{4.6}$$

With the form (4.5), it is easy to show invariance properties of the measure, while the form (4.6) will be used while performing the explicit integrations.

We shall now prove the following result, which relates the generating functional $\bar{W}$ of the "larger" theory to that of the ordinary gauge theory W viz.,

$$W[K_{\mu,\theta}^\alpha(\bar{x}), K_{,\theta}^{\alpha 5}(\bar{x}), t^\alpha(\bar{x}), K^{\alpha\mu}(\bar{x}), K^{\alpha 5}(\bar{x})] = \int [dK^4][dK^4_{,\theta}] \bar{W}[\bar{K}, t] \ , \tag{4.7}$$

allowing one to deduce the properties of the gague theory from that of $\bar{W}[\bar{K}, t]$, which contains implicitly all the extra sources, in a manner, so as to exhibit a gauge symmetry in superspace.

We shall first consider the ζ-integration.

$$\int \{d\zeta(\bar{x})\}\exp\left[i\int d^4x \frac{\partial}{\partial\theta}\{\zeta^\alpha(\bar{x})[\partial\cdot A^\alpha(\bar{x})+\tfrac{1}{2}\zeta^\alpha_{,\theta}(\bar{x})+t^\alpha(\bar{x})]\}\right],$$

$$=\int \{d\zeta(\bar{x})\}\exp\left[i\int d^4x[-\zeta^\alpha\partial_\mu A^{\alpha\mu}_{,\theta}+\zeta^\alpha_{,\theta}(\partial\cdot A^\alpha+t^\alpha)+\tfrac{1}{2}(\zeta^\alpha_{,\theta})^2+t^\alpha_{,\theta}\zeta^\alpha]\right],$$

$$=\int [d\zeta(x)][d\tilde{\zeta}_{,\theta}(x)]\exp\left[i\int d^4x\{-[\zeta^\alpha(x)+\theta\zeta^\alpha_{,\theta}(\bar{x})][\partial\cdot A^\alpha_{,\theta}(\bar{x})+t^\alpha_{,\theta}(\bar{x})]\right.$$

$$\left.+\tfrac{1}{2}[\zeta^\alpha_{,\theta}(\bar{x})]^2+\zeta^\alpha_{,\theta}[\partial\cdot A^\alpha(\bar{x})+t^\alpha(\bar{x})]\}\right],$$

$$\times\int d\tilde{\zeta}^\alpha_{,\lambda}(x)\exp\left[-i\lambda\int d^4x\,\tilde{\zeta}^\alpha_{,\lambda}[\partial\cdot A^\alpha_{,\theta}(\bar{x})+t^\alpha_{,\theta}(\bar{x})]\right]. \tag{4.8}$$

Now the $\tilde{\zeta}_{,\lambda}$ integration is

$$\int [d\tilde{\zeta}^\alpha_{,\lambda}(x)]\left[1-i\lambda\int d^4x\,\tilde{\zeta}^\alpha_{,\lambda}(\bar{x})(t^\alpha_{,\theta}+\partial\cdot A^\alpha_{,\theta})\right]=\int [d\zeta^\alpha_{,\lambda}(x)]=(\text{an infinite})\text{ const.}$$

Here we have performed a symmetric integration over $\tilde{\zeta}^\alpha_{,\lambda}$.

Thus the expression on the right-hand side of (4.8) becomes ($\simeq$ means up to a constant),

$$\simeq\int [d\zeta]\exp\left[-i\int d^4x\zeta^\alpha(x)[\partial\cdot A^\alpha_{,\theta}(\bar{x})+t^\alpha_{,\theta}(\bar{x})]\right]$$

$$\times\int [d\tilde{\zeta}_{,\theta}]\exp\left[i\int d^4x\{\tfrac{1}{2}(\zeta^\alpha_{,\theta})^2+\zeta^\alpha_{,\theta}[\partial\cdot A^\alpha(\bar{x})+t^\alpha(\bar{x})-\theta\partial\cdot A^\alpha_{,\theta}(\bar{x})-\theta t^\alpha_{,\theta}(\bar{x})]\}\right]. \tag{4.9}$$

We perform the $\tilde{\zeta}_{,\theta}$ integration by completing the square. The result is

$$\simeq\int [d\zeta]\exp\left[i\int d^4x\{-\zeta^\alpha(x)[\partial\mu A^{\alpha\mu}_{,\theta}(\bar{x})+t^\alpha_{,\theta}(\bar{x})]-\tfrac{1}{2}[\partial\cdot A^\alpha(\bar{x})+t^\alpha(\bar{x})]^2+\theta(\partial\cdot A^\alpha+t^\alpha)(\partial\cdot A^\alpha_{,\theta}+t^\alpha_{,\theta})\}\right]$$

$$=\int [d\zeta]\exp\left[i\int d^4x\{\zeta^\alpha[-t^\alpha_{,\theta}-\partial\cdot A^\alpha_{,\theta}(\bar{x})]-\tfrac{1}{2}[\partial\cdot A^\alpha(\bar{x})+t^\alpha(\bar{x})]^2+\theta(\partial\cdot A^\alpha+t^\alpha)[\partial\cdot A^\alpha_{,\theta}(\bar{x})+t^\alpha_{,\theta}(\bar{x})]\}\right]. \tag{4.10}$$

The third term in the curly brackets vanishes because of the equation of motion for $\zeta(x)$.

Thus we get the expression for $\overline{W}$:

$$\overline{W}[\overline{K},t]\simeq\int \{d\overline{A}(\bar{x})\}[d\zeta(x)]\exp\left[i\int d^4x\{\mathcal{L}_0[\overline{A}]-\tfrac{1}{2}[\partial\cdot A^\alpha(\bar{x})+t^\alpha(\bar{x})]^2+\overline{K}^{\alpha i}_{,\theta}(\bar{x})\overline{A}^\alpha_i(\bar{x})\right.$$

$$\left.+\overline{K}^{\alpha i}(\bar{x})\overline{A}^\alpha_{i,\theta}(\bar{x})-\zeta^\alpha[\partial\cdot A^\alpha_{,\theta}(\bar{x})+t^\alpha_{,\theta}(\bar{x})]\}\right]$$

$$=\int \{d\overline{A}(\bar{x})\}[d\zeta(x)]\exp\left[i\int d^4x\{\mathcal{L}_0[A(\bar{x})]-(A^\alpha_{\sigma,\lambda}+D^{\alpha\beta}_\gamma c^\beta_4)(A^\alpha_{\sigma,\theta}+D^{\alpha\beta}_\sigma c^\beta_5)\right.$$

$$-\tfrac{1}{2}[\partial\cdot A^\alpha(\bar{x})+t^\alpha(\bar{x})]^2+\overline{K}^{\alpha i}_{,\theta}(\bar{x})\overline{A}^\alpha_i(\bar{x})+\overline{K}^{\alpha i}(\bar{x})\overline{A}^\alpha_{i,\theta}(\bar{x})$$

$$\left.-\zeta^\alpha[\partial\cdot A^\alpha_{,\theta}(\bar{x})+t^\alpha_{,\theta}(\bar{x})]+\text{pure ghost terms}\}\right]. \tag{4.11}$$

We shall now perform the $\tilde{A}^\alpha_{\sigma,\lambda}$ integration. We can safely omit terms proportional to $\tilde{A}^\alpha_{\sigma,\lambda}$ that arise out of $\tilde{A}^\alpha_{\sigma,\lambda}$ contained in $A^\alpha_\sigma(\bar{x})$ [i.e., expanding by the use of $A^\alpha_\sigma(\bar{x})\equiv A^\alpha_\sigma(x)+\lambda\tilde{A}^\alpha_{\sigma,\lambda}+\theta A^\alpha_{\sigma,\theta}$]. This is on account of the equation of motion for $A^\alpha_\sigma(x)$. To see this, we write the exponent as

$$S[A(\bar{x});A^\alpha_{\sigma,\lambda};A^\alpha_{\sigma,\theta};\ldots]=\left[S-\lambda\int\frac{\delta S}{\delta A^\alpha_\mu(x)}\tilde{A}^\alpha_{\mu,\lambda}(x)d^4x\right]+\lambda\int\frac{\delta S}{\delta A^\alpha_\mu(x)}\tilde{A}^\alpha_{\mu,\lambda}(x)d^4x\,. \tag{4.12}$$

Now,

$$\int [dA(x)]e^{iS}=\int [dA(x)]\exp\left[i\left[S-\lambda\int\frac{\delta S}{\delta A^\alpha_\mu(x)}\tilde{A}^\alpha_{\mu,\lambda}(x)d^4x\right]\right]\left[1+i\lambda\int\frac{\delta S}{\delta A^\alpha_\mu(x)}\tilde{A}^\alpha_{\mu,\lambda}(x)d^4x\right], \tag{4.13}$$

while an equation of motion for $A(x)$ is

$$0=\int [dA(x)]\exp\left[i\left[S-\lambda\int \frac{\delta S}{\delta A_{\mu}^{\alpha}(x)}\tilde{A}^{\alpha}_{\mu,\lambda}(x)\right]\right]\left[\frac{\delta S}{\delta A_{\nu}^{\beta}(y)}-\lambda\int\frac{\delta^2 S}{\delta A_{\nu}^{\beta}(y)\delta A_{\mu}^{\alpha}(x)}\tilde{A}^{\alpha}_{\mu,\lambda}(x)d^4x\right]. \tag{4.14}$$

Thus multiplying the above equation by λ, we learn that the second term on the RHS of Eq. (4.13) vanishes, proving the earlier statement. This sort of result will be used again several times later. In short only the terms explicit in $\tilde{A}^{\alpha}_{\sigma,\lambda}$ matter. The result of the $\tilde{A}^{\alpha}_{\sigma,\lambda}$ integration is

$$\overline{W}[\overline{K},t]\simeq\int\{dc_4\}\{dc_5\}[d\zeta][dA]\int[d\tilde{A}_{,\theta}]\prod_{\alpha,\sigma,x}(A^{\alpha}_{\sigma,\theta}(\bar{x})+D^{\alpha\beta}_{\sigma}c^{\beta}_{5})$$
$$\times\exp\left[i\int d^4x\{\mathcal{L}_0[A]-D^{\alpha\beta}_{\sigma}c^{\beta}_{4}(A^{\alpha}_{\sigma,\theta}+D^{\alpha\beta}_{\sigma}c^{\beta}_{5})-\tfrac{1}{2}(\partial\cdot A^{\alpha}+t^{\alpha})^2+K^{-\alpha i}\overline{A}_{i,\theta}-\zeta^{\alpha}\partial\cdot A^{\alpha}_{,\theta}\right.$$
$$\left.+\overline{K}^{\alpha i}_{,\theta}\overline{A}^{\alpha}_{i}-\zeta^{\alpha}t^{\alpha}_{,\theta}+\text{pure ghost terms}\}_{A^{\alpha}_{\sigma,\lambda}=0}\right]. \tag{4.15}$$

Now we perform integration with respect to $\tilde{A}^{\alpha}_{\sigma,\theta}$, yielding

$$\overline{W}[\overline{K},t]\simeq\int\{dc_4\}\{dc_5\}[d\zeta][dA]$$
$$\times\exp\left[i\int d^4x\left\{\mathcal{L}_0[A]-\tfrac{1}{2}(\partial\cdot A^{\alpha}+t^{\alpha})^2-(\overline{K}^{\alpha\mu}+\partial^{\mu}\zeta^{\alpha})D^{\alpha\beta}_{\mu}c^{\beta}_{5}-\tfrac{1}{2}(2c^{\alpha}_{4,\lambda}+gf^{\alpha\beta\gamma}c^{\beta}_{4}c^{\gamma}_{4})\right.\right.$$
$$\times(2c^{\alpha}_{5,\theta}+gf^{\alpha\beta\gamma}c^{\beta}_{5}c^{\gamma}_{5})+\tfrac{1}{2}(c^{\alpha}_{4,\theta}+c^{\alpha}_{5,\lambda}+gf^{\alpha\beta\gamma}c^{\beta}_{4}c^{\gamma}_{5})^2+\overline{K}^{\alpha\mu}_{,\theta}A^{\alpha}_{\mu}$$
$$\left.\left.-\zeta^{\alpha}t^{\alpha}_{,\theta}+\frac{\partial}{\partial\theta}(K^{\alpha4}c^{\alpha}_{4}-K^{\alpha5}c^{\alpha}_{5})\right\}_{A_{,\lambda}=0=A_{,\theta}}\right]. \tag{4.16}$$

Next, we integrate with respect to $\tilde{c}^{\alpha}_{4,\lambda}$. [Again, we recall that the terms proportional $\lambda\tilde{c}^{\alpha}_{4,\lambda}$ arising out of expanding $c^{\alpha}_{4}(\bar{x})$ are irrelevant on account of the equation of motion for $c^{\alpha}_{4}(x)$.] Performing the $\tilde{c}^{\alpha}_{4,\lambda}$ integration and then the $\tilde{c}^{\alpha}_{5,\theta}$ integration successively (and, of course, using the equation of motion for c^{α}_{5}), we arrive at

$$\overline{W}[\overline{K},t]\simeq\int[dA\,d\zeta\,dc_4dc_5][dc_{4,\theta}][dc_{5,\lambda}]$$
$$\times\exp\left[i\int d^4x\{\mathcal{L}_0[A]-\tfrac{1}{2}(\partial\cdot A^{\alpha}+t^{\alpha})^2-(K^{\alpha\mu}+\partial^{\mu}\zeta^{\alpha})(D^{\alpha\beta}_{\mu}c^{\beta}_{5})+\tfrac{1}{2}(c^{\alpha}_{4,\theta}+c^{\alpha}_{5,\lambda}+gf^{\alpha\beta\gamma}c^{\beta}_{4}c^{\gamma}_{5})^2\right.$$
$$\left.+K^{\alpha\mu}_{,\theta}A^{\alpha}_{\mu}-\zeta^{\alpha}t^{\alpha}_{,\theta}+\frac{\partial}{\partial\theta}(K^{\alpha4}c^{\alpha}_{4})+K^{\alpha5}_{,\theta}c^{\alpha}_{5}-\tfrac{1}{2}gK^{\alpha5}f^{\alpha\beta\gamma}c^{\beta}_{5}c^{\gamma}_{5}\}_{\substack{A_{,\lambda}=0=A_{,\theta}\\ \tilde{c}_{4,\lambda}=0=c_{5,\theta}}}\right]. \tag{4.16b}$$

We change the variables to $\xi^{\alpha}=c^{\alpha}_{4,\theta}-c^{\alpha}_{5,\lambda}$ and $2\eta=c^{\alpha}_{4,\theta}+c^{\alpha}_{5,\lambda}$ and perform the straightforward Gaussian integration over η. One thus obtains

$$\overline{W}[\overline{K},t]=\int[dc_4dc_5dA\,d\zeta\,d\xi]$$
$$\times\exp\left[i\int d^4x\{\mathcal{L}_0[A(x)]-\tfrac{1}{2}(\partial\cdot A^{\alpha}(x)+t^{\alpha})^2-[K^{\alpha\mu}(\bar{x})+\partial^{\mu}\zeta^{\alpha}(x)]D^{\alpha\beta}_{\mu}(x)c^{\beta}_{5}(\bar{x})+K^{\alpha\mu}_{,\theta}(\bar{x})A^{\alpha}_{\mu}(x)\right.$$
$$-\zeta^{\alpha}(x)t^{\alpha}_{,\theta}(\bar{x})+K^{\alpha4}_{,\theta}(\bar{x})c^{\alpha}_{4}(\bar{x})+K^{\alpha5}_{,\theta}(\bar{x})c^{\alpha}_{5}(\bar{x})-\tfrac{1}{2}gK^{\alpha5}\overline{(x})f^{\alpha\beta\gamma}c^{\beta}_{5}(\bar{x})c^{\gamma}_{5}(\bar{x})$$
$$-\tfrac{1}{2}gf^{\alpha\beta\gamma}c^{\beta}_{4}(\bar{x})c^{\gamma}_{5}(\bar{x})(K^{\alpha4}(\bar{x})-\theta K^{\alpha4}_{,\theta}+\lambda K^{5}_{,\theta})$$
$$\left.+\tfrac{1}{2}\xi^{\alpha}(K^{\alpha4}-\theta K^{\alpha4}_{,\theta}-\lambda K^{\alpha5}_{,\theta})\}_{\eta=0=\tilde{c}_{4,\lambda}=c_{5,\theta}}\right]. \tag{4.17}$$

The terms containing ξ explicitly (only which matter in the integration over ξ on account of the equations of motion for c_4 and c_5—the proof is a little more tedius, which is needed also in the η (integration) are

$$\int[d\xi]\exp\left[i\int d^4x\,\xi^{\alpha}(x)[K^{\alpha4}(x)+\lambda(K^{\alpha4}_{,\lambda}(x)+K^{\alpha5}_{,\theta}(x))]\right]\simeq\prod_{\alpha,x}\{\delta(K^{\alpha4}(x))+\lambda[\tilde{K}^{\alpha4}_{,\lambda}(x)+K^{\alpha5}_{,\theta}(x)]\delta'(K^{\alpha4}(x))\}$$

Thus,

$$\overline{W}[\overline{K},t]=\prod_{\alpha,y}\delta(K^{\alpha 4}(y))\int[dc_4dc_5dA\,d\zeta]\{1+\tfrac{1}{2}gi\lambda[\widetilde{K}^{\alpha 4}_{,\lambda}(y)+K^{\alpha 5}_{,\theta}(y)]f^{\alpha\beta\gamma}c^{\beta}_{4}(y)c^{\gamma}_{5}(y)\}$$
$$\times\exp(i\{\Sigma[A,c_5,\zeta,K^{\mu},K^{5},K^{\alpha\mu}_{,\theta};K^{\alpha 5}_{,\theta};-t^{\alpha}_{,\theta}]+\int K^{\alpha 4}_{,\theta}(\overline{x})c^{\alpha}_{4}x\})\ , \qquad (4.18)$$

where Σ was defined in (4.1). Note that here the field arguments of Σ are functions of x_{μ} only while the sources in Σ are functions of $\overline{x}$ generally.

In particular, performing an integration over $K^{4}_{,\theta}$ and then over c_4,

$$\int[dK^{4}_{,\theta}]\overline{W}[\overline{K},t]=\left[\prod_{\alpha,y}\delta(K^{\alpha 4}(y))\right]W[K^{\alpha\mu}_{,\theta};K^{\alpha 5}_{,\theta};t^{\alpha},K^{\alpha\mu},K^{\alpha 5}]\ , \qquad (4.19)$$

i.e.,

$$\int[dK^{4}][dK^{4}_{,\theta}]\overline{W}[\overline{K},t]=W[K^{\alpha\mu}_{,\theta}(\overline{x}),K^{\alpha 5}_{,\theta}(\overline{x}),-t^{\alpha}_{,\theta}(\overline{x}),K^{\alpha\mu}(\overline{x}),K^{\alpha 5}(\overline{x}),t^{\alpha}]\ . \qquad (4.20)$$

Thus the result of Eq. (4.7) is proven.

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