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General covariance of the path integral for quantum gravity

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We construct the functional integration measure over four-geometries in the path integral for quantum gravity by means of a geometric, manifestly covariant approach, similar to that used by Polyakov for string theory. This generalizes the previous one-loop method of Mazur and Mottola to all orders of perturbation theory. We compare this measure to that obtained by the gauge-fixed method of Becchi-Rouet-Stora-Tyutin invariance exploited by Fujikawa and co-workers. The path integral defined by these two different procedures is one and the same.

I. INTRODUCTION

The path integral is a powerful tool in quantum field theory. In addition to its conceptually simple appeal as a "sum over histories," it provides an equally good basis to derive the perturbative Feynman rules or study nonperturbative aspects of a quantum theory. Moreover, since it begins with the classical action, invariances of that action should naturally reflect invariances at the quantum level, as expressed by the corresponding Ward identities.

Unfortunately, the path integral has not always lived up to this promise, particularly in the case of gravity. The reason is that, in addition to the classical action, one must specify the measure over the (functional) space of metrics in the path integral. Because of the local symmetry group of general relativity, one encounters the problem of overcounting of gauge-equivalent configurations, a problem familiar in non-Abelian gauge theories. The dynamical variables of the gravitational field are the geometries of spacetime itself, and the local symmetry group is the group of diffeomorphisms; so this overcounting problem becomes more difficult, both technically and conceptually, than in Yang-Mills theories. We should emphasize at the outset that the problem of defining the path-integral measure with which we are concerned is a problem distinct from the ultraviolet nonrenormalizability of the Einstein theory. If the Langrangian were modified or a cutoff inserted by hand to control the divergences, we would still be faced with the problem of the correct measure over the space of metrics. The divergences are strictly an ultraviolet problem, whereas the question of the measure arises at any scale, including the semiclassical (long-distance) limit of the theory.

The usual method of dealing with the overcounting of gauge-equivalent configurations in gauge theories is to fix the gauge and introduce Faddeev-Popov ghost fields.[1] This procedure is well suited to perturbative calculations, but has the disadvantage of obscuring the geometrical significance and general covariance of the full theory. The status of general covariance at the quantum level was investigated by Fradkin and Vilkovisky,[2] who made an extensive analysis of the theory and concluded that the measure must contain apparently noncovariant factors of $\prod_x g^{00}(x)$ and $\prod_x \sqrt{-g(x)}$. Otherwise, they claimed that the Ward identities are violated. However, the ill-defined nature of infinite products of such factors at each spacetime point makes the interpretation of this claim problematic. The result is that the status of general covariance of the path integral for quantum gravity has remained unclear. This is a prime example of technical difficulties underlying an important conceptual issue in quantum gravity, which underscores the need for a different approach.

A major technical advance came with the application to quantum gravity of the supersymmetry of Becchi, Rouet, Stora, and Tyutin[3] (BRST), discovered originally in gauge theories. Fujikawa realized that the gauge-fixed path-integral measure is determined by the requirement that it be invariant under BRST transformations.[4] This is a necessary ingredient in the proof that the Ward identities of the quantum theory are anomaly-free, which in turn is necessary for the theory to be unitary. A method of regularization that respects this supersymmetry was subsequently presented in Ref. 5. Although this solves the problem of the measure in principle, it involves the full algebraic machinery of BRST, while the geometric principle of general covariance is still far from being transparent. Because of these features, it has not been as widely understood or applied to questions of current interest as it might be. As a concrete example, Fujikawa's determination of the measure in the path integral should be equivalent to determining the correct operator ordering of factors in the Wheeler-DeWitt equation, which has been much discussed recently in the context of quantum cosmology and topology change. Yet no use of this connection has been made. Again, pure technicalities have served to obstruct the understanding and limit the usefulness of the path-integral approach.

Recently, a different approach to the path-integral measure for quantum gravity was proposed.[6] A covariant measure for the Feynman integral over four-geometries was constructed at one-loop order, by employing the same techniques used by Polyakov for string theory,[7] and suggested by the pioneering work of De Witt more than a quarter century ago.[8] As in string theory, the Polyakov approach, though quite elegant and manifestly covariant, comes, so to speak, "out of the blue." It is not derived from any strictly canonical approach in which only physical excitations propagate. Hence it is not manifest that it is equivalent to the Hamiltonian or unitary gauge form of the theory[9] (such as the light-cone gauge of string theory). Nor is its connection with the BRST-invariant construction of Fujikawa obvious, since no gauge fixing or ghosts in the usual sense are used in the geometric method of dividing out the gauge-orbit volume.

Our purpose in this article is to extend the formal construction of the generally covariant measure of Ref. 6 to *all* loop orders and to connect it with the BRST-invariant measure in the gauge-fixed path integral of Fujikawa. We shall see that the two approaches yield the same answer, thus indicating the correctness of geometric methods while exposing the intuitive meaning of the more formal BRST approach. Our construction is formal in the sense that we do not consider explicit regulator techniques for defining (for example) functional determinants. Rather, we assume that all such formal quantities are defined by covariant regulators such as those provided by the ζ-function method. The geometric approach to path integrals has been discussed in Ref. 10 as well. The connection with canonical Hamiltonian methods and applications to problems of current interest will be left to future publications.

The paper is organized as follows. In Sec. II we summarize and compare the standard Faddeev-Popov quantization to the geometric method for the case of non-Abelian gauge theory. This is intended partly as a pedagogical review, in order to introduce notation and make the paper reasonably self-contained. In Sec. III we construct the measure for quantum gravity by defining an inner product and volume element on the (co)tangent space of spacetime metrics. The tangent-space measure so constructed induces a functional measure on the full space of metrics. This extends the approach of Ref. 6 and defines a path-integral measure for quantum gravity that is fully covariant to all orders of perturbation theory. No ill-defined infinite factors of $\prod_x g^{00}(x)$ and $\prod_x \sqrt{-g(x)}$ ever arise in this approach. In Sec. IV the geometric approach is compared to the usual gauge-fixing procedure and the method of BRST invariance followed by Fujikawa *et al.*[4,5] The equivalence between the two approaches is then evident. We work in $D=4$ Lorentzian spacetime dimensions, except when otherwise indicated.

II. YANG-MILLS THEORY

In this section we review the standard Faddeev-Popov method for path-integral quantization and outline the geometric method for the case of Yang-Mills theory.

A. Gauge-fixed approach

The Yang-Mills action

$$S_{\rm inv}[A]=-\frac{1}{4g^2}\int d^4x\, G^i_{\mu\nu}G^{i\mu\nu} \tag{2.1}$$

is invariant under any non-Abelian gauge transformation, the infinitesimal form of which is

$$\begin{aligned}A^i_\mu \to A^{i(\theta)}_\mu &\equiv A^i_\mu+(\nabla_\mu\theta)^i\\ &\equiv A^i_\mu+\partial_\mu\theta^i+f^{ijk}A^j_\mu\theta^k\ .\end{aligned} \tag{2.2}$$

The f^{ijk} are the structure constants of some non-Abelian gauge group.

The gauge invariance of the classical action implies that the path integral contains an infinite factor of the gauge-orbit volume. Hence one should define the vacuum amplitude for the theory with this infinite factor divided out.

The usual method of dealing with this formally infinite factor is to "fix the gauge." To do so, first decompose the gauge field into a background piece (denoted by an overbar) and a fluctuating piece:

$$A^i_\mu=\bar{A}^i_\mu+a^i_\mu\ . \tag{2.3}$$

The gauge transformation (2.2) is defined to act only on a^i_μ while the background field remains fixed. Then the path integration over the gauge-field configurations may be thought of as an integration over the fluctuating quantum piece alone:

$$[\mathcal{D}A^i_\mu]=[\mathcal{D}a^i_\mu]\ . \tag{2.4}$$

"Fixing the gauge" means imposing a condition of the form

$$(F\cdot a)^i\equiv F^{ij\mu}a^j_\mu=0\ , \tag{2.5}$$

on the field configurations to be integrated over. The resulting path integral is

$$\begin{aligned}Z=\int[\mathcal{D}a^i_\mu][\mathcal{D}b^i][\mathcal{D}\bar{c}^i][\mathcal{D}c^i]&\\ \times\exp\{i[S_{\rm inv}(\bar{A}+a)+S_{\rm gf}+S_{\rm gh}]\}&\ ,\end{aligned} \tag{2.6}$$

where b^i is an auxiliary Lagrange multiplier field and c^i and $\bar{c}^i$ are the Faddeev-Popov ghost fields. The gauge-fixing and ghost terms in the action, respectively, are

$$\begin{aligned}S_{\rm gf}&=\int d^4x\, b^i(F\cdot a)^i\ ,\\ S_{\rm gh}&=\int d^4x\, \bar{c}^i(F\circ\nabla)^{ij}c^j\ ,\end{aligned} \tag{2.7}$$

where $(F\circ\nabla)^{ij}\equiv F^{\mu ik}\nabla^{kj}_\mu$. Integrating over b^i yields the gauge-fixing condition (2.5) as a δ-function constraint. Integrating over the ghost fields yields the Faddeev-Popov determinant

$$\Delta_{\rm FP}=\det(F\circ\nabla)\ . \tag{2.8}$$

A remarkable property of the path integral (2.6) is its invariance under the BRST transformations, defined in the Yang-Mills case by

$$\begin{aligned}Qa^i_\mu&=(\nabla_\mu c)^i=\partial_\mu c^i+f^{ijk}A^j_\mu c^k\ ,\\ Qc^i&=-\tfrac{1}{2}f^{ijk}c^jc^k\ ,\\ Q\bar{c}^i&=b^i\ ,\\ Qb^i&=0\ .\end{aligned} \tag{2.9}$$

It is easily verified with this definition that Q is nilpotent,

$$Q^2=0\ , \tag{2.10}$$

by the antisymmetry of the structure constants $f^{ijk}=-f^{ikj}$. Since the ghost plus gauge-fixing terms in the action may be written as an exact form,

$$S_{\rm gh}+S_{\rm gf}=Q\int d^4x\,\bar{c}^{\,i}(F\cdot a)^i\ , \tag{2.11}$$

we immediately discover that it is BRST invariant by (2.10). The original classical invariant action $S_{\rm inv}[A]$ is obviously BRST invariant since the BRST transformation on the gauge potential has exactly the form of a gauge transformation with the gauge parameter θ^i replaced by c^i. Thus the entire path integral in Eq. (2.6) is BRST invariant, provided that the measure is.

The BRST invariance of the measure may be verified by a direct computation of the Jacobian of the transformation (2.9) on the set of fields $\Psi\equiv(a^i_\mu,b^i,c^i,\bar{c}^{\,i})$:

$$J_Q={\rm sdet}\left[1+\frac{\partial(Q\Psi)}{\partial\Psi}\right]=1+{\rm str}\left[\frac{\partial(Q\Psi)}{\partial\Psi}\right]\ , \tag{2.12}$$

since we may regard the BRST transformation (2.9) as an infinitesimal one. The transformation on the antighost and auxiliary fields do not contribute to the Jacobian, and so the product of the last two measure factors $[\mathcal{D}\bar{c}^{\,i}][\mathcal{D}b^i]$ is separately BRST invariant. The remaining terms in the gauge and ghost sectors give

$$J_Q=1+{\rm Tr}_a(f^{ijk}c^k)-{\rm Tr}_c(f^{ijk}c^k)\ . \tag{2.13}$$

The two traces are over different spaces, viz., vector gauge fields and anticommuting ghost fields, respectively, so that they do not cancel. However, each vanishes separately by the antisymmetry of f^{ijk}. Hence the Jacobian is unity, and the functional measure is BRST invariant, which is what we wished to prove. This formal BRST invariance is the key ingredient in the establishing of Ward identities which are the reflection of the gauge invariance (2.2) at the quantum level. These are essential also to the proof of the renormalizability and unitarity of the theory.

B. Geometric approach

The standard gauge-fixing procedure reviewed above has implicit in it the geometric structure of the space of gauge fields. The choice of F really amounts to a choice of a slice on the field configuration space. Given this slice, one could introduce, for example, Gaussian normal coordinates to establish coordinates on field configuration space in a local neighborhood of the point on the slice. This suggests that if one were to introduce a metric on the space of gauge fields, then the path-integral measure would naturally be the invariant volume form corresponding to this metric, and the Faddeev-Popov determinant would arise as a Jacobian of transformation to these new coordinates on the field configuration space. We shall see that this is indeed the case.

In the geometric approach we begin by regarding each gauge-field configuration $A^i_\mu(x)$ as a "point" in a continuous function space. In ordinary Riemannian geometry one introduces the notion of a tangent space at each point of the manifold with a corresponding cotangent space of coordinate one-forms dx^μ. Then a metric on the manifold may be introduced by defining an invariant line element ds^2 as a certain (position-dependent) quadratic form in the dx^μ. The analogs of the coordinate one-forms here are the small deformations δA^i_μ of the gauge field. The only Poincaré-invariant bilinear inner product is

$$\langle\delta A,\delta A\rangle=\int d^4x\,\delta A^i_\mu(x)\eta^{\mu\nu}\delta A^i_\nu(x)\ . \tag{2.14}$$

This natural, invariant quadratic form is completely analogous to the invariant line element ds^2 of finite-dimensional Riemmanian manifolds and therefore induces a natural, invariant volume form on the space of gauge fields. In the present case the metric is flat and independent of the point $A^i_\mu(x)$. Hence the invariant volume form is just the product of coordinate differentials. We may fix the irrelevant normalization of this functional measure by a Gaussian normalization condition:

$$\int[\mathcal{D}\delta A^i_\mu]\exp\left[-\frac{i}{2}\langle\delta A,\delta A\rangle\right]=1\ . \tag{2.15}$$

The measure defined in this way is manifestly Lorentz invariant. Moreover, it is gauge invariant under (2.2). This is because it is translationally invariant under $\delta A^i_\mu(x)\rightarrow\delta A^i_\mu(x)+v^i_\mu(x)$, for any δA-independent shift v, which eliminates the translation term in the gauge transformation (2.2), leaving only the rotation, represented by the final term of (2.2). Since the inner product in Eq. (2.15) is invariant under global gauge rotations, this guarantees that the measure is as well and completes the proof of its full gauge invariance.

A key point now is the observation that the metric and measure defined on the *tangent* space at A induces a measure on the *full* space of vector gauge fields. The extension runs as follows: Let X^i be the coordinates of any manifold (for example, the space of gauge fields). Then any tangent vector at a given point $\bar{X}^i$ may be written $V=V^i\partial_i|_{(\bar{X}^i)}$ and the V^i are coordinates for the tangent space to $\bar{X}^i$. If for all $\bar{X}^i$ there is a measure on the tangent space at $\bar{X}^i$ of the form $f(\bar{X}^i)dV^1\wedge dV^2\wedge\cdots$, then there is induced a measure on the original manifold defined by $f(X^i)dX^1\wedge dX^2\wedge\cdots$. Moreover, the Jacobians of any change of coordinates on the tangent space and the original manifold at $\bar{X}^i$ are the same. In the present case, the inner product (2.14) defines a *flat* metric on the function space, so that $f(X^i)=$const, and most of the above remarks seem trivial. This will not be the case in gravity, where their significance will be much more apparent.

We may now define

$$Z[\bar{A}]=[{\rm Vol}(\mathcal{G})]^{-1}\int[\mathcal{D}a^i_\mu]\exp(iS_{\rm inv}[\bar{A}+a])\ , \tag{2.16}$$

with respect to this induced measure. Here ${\rm Vol}(\mathcal{G})$ denotes the (infinite) gauge-orbit volume that must be divided out. However, instead of Faddeev-Popov gauge fixing, we proceed by introducing the coordinatization of the gauge field:

$$\begin{aligned} A^i_\mu &= \bar{A}^i_\mu + a^i_\mu \\ &= [\mathcal{U}^{-1}(\theta)(\bar{A}^i_\mu + a^{\perp i}_\mu)\mathcal{U}(\theta)]^i - i[\mathcal{U}^{-1}(\theta)\partial_\mu \mathcal{U}(\theta)]^i \\ &= \bar{A}^i_\mu + a^{\perp i}_\mu + (\nabla_\mu \theta)^i + \mathcal{O}(\theta^2) \,, \end{aligned} \quad (2.17)$$

where $\mathcal{U}$ is an arbitrary element of the gauge group at each spacetime point. By construction, the quantity $a^{\perp i}_\mu$ is gauge invariant and Lorentz covariant. In order to specify the field coordinates completely, one still needs to fix a condition on the $a^{\perp i}_\mu$ of the form (2.5). However, in the present context this is not a gauge "fixing" that breaks the gauge invariance of the theory, but rather a choice of coordinates in a coordinate-invariant formulation. This coordinate choice may depend on the background $\bar{A}$, such as a covariant, background field gauge:

$$(F\cdot a^\perp)^i = (\bar{\nabla}^\mu a^\perp_\mu)^i = 0 \,. \quad (2.18)$$

Note, however, the important point, that $a^{\perp i}_\mu$ is gauge invariant for *any* choice of condition on it. Hence gauge invariance is assured by this construction from the start.

The task now is to express the gauge-invariant quantity (2.16) in the coordinates (gauge) specified by (2.17) and (2.18). We first compute the Jacobian of the transformation to the new coordinates:

$$[\mathcal{D}a^i_\mu] = J[\mathcal{D}^{\perp i}_\mu][\mathcal{D}\theta^i] \,, \quad (2.19)$$

in the tangent space

$$\begin{aligned} 1 &= \int [\mathcal{D}a^i_\mu] \exp\left[-\frac{i}{2}\langle a,a\rangle\right] \\ &= \int J[\mathcal{D}a^{\perp i}_\mu][\mathcal{D}\theta^i] \exp\left[-\frac{i}{2}(\langle a^\perp,a^\perp\rangle + 2\langle \nabla\theta, a^\perp\rangle + \langle\nabla\theta,\nabla\theta\rangle)\right] \,. \end{aligned} \quad (2.20)$$

This integral may be computed by completing the square and using condition (2.15). We find that

$$J = \{\det_S(-\nabla^2)\det_{\perp V}[\delta^\nu_\mu - \nabla_\mu(\nabla^2)^{-1}\nabla^\nu]\}^{1/2} \,, \quad (2.21)$$

where the vector determinant is to be evaluated over the space of fields $a^{\perp i}_\nu$ obeying the constraint (2.18).

The vector determinant may be converted into a scalar determinant by the following manipulations. First of all, introduce a complete orthonormal set of transverse vector modes $\{v^{\perp(n)}\}$ with respect to the inner product (2.14):

$$\begin{aligned} &\langle v^{\perp(n)}, v^{\perp(n')}\rangle = \delta^{nn'} \,, \\ &\sum_n v^{\perp(n)i}_\mu(x) v^{\perp(n)\nu j}(x') = [\delta^\nu_\mu \delta^{ij} - (\bar{M}^{\ \nu}_\mu)^{ij}]\delta^4(x,x') \equiv \bar{P}^{\perp\nu ij}_\mu(x,x') \,, \end{aligned} \quad (2.22)$$

where

$$(\bar{M}^{\ \nu}_\mu)^{ij} \equiv [\bar{\nabla}_\mu(\bar{\nabla}^2)^{-1}\bar{\nabla}^\nu]^{ij} \,, \quad (2.23)$$

and P is the projector onto the space of transverse vectors. Then we may write the vector determinant as

$$\det_{\perp V}\{\delta^\nu_\mu - M^\nu_\mu\} = \exp\left[-\sum_{k=1}^\infty \frac{\mathrm{Tr}_{\perp V}(M)^k}{k}\right] \,, \quad (2.24)$$

where M is the operator (2.23) without the overbars. Now use

$$\begin{aligned} \mathrm{Tr}_{\perp V}(M)^k &= \sum_{n_1,n_2,\ldots,n_k} \langle v^{\perp(n_1)}, Mv^{\perp(n_2)}\rangle\langle v^{\perp(n_2)}, Mv^{\perp(n_3)}\rangle \cdots \langle v^{\perp(n_k)}, Mv^{\perp(n_1)}\rangle \\ &= \mathrm{Tr}_V(M\bar{P}^\perp)^k \,, \end{aligned} \quad (2.25)$$

by (2.22). It is easily verified by a direct computation that

$$[(M\bar{P}^\perp)^k]^\nu_\mu = \nabla_\mu(\nabla^2)^{-1}(1-W)^{k-1}[\nabla^\nu - (\nabla\cdot\bar{\nabla})(\bar{\nabla}^2)^{-1}\bar{\nabla}^\nu] \,, \quad (2.26)$$

where W is the scalar operator defined by

$$W \equiv (\nabla\cdot\bar{\nabla})(\bar{\nabla}^2)^{-1}(\bar{\nabla}\cdot\nabla)(\nabla^2)^{-1} \,. \quad (2.27)$$

The cyclic property of the trace may now be employed to derive the formal identity

$$\mathrm{Tr}_V(M\bar{P}^\perp)^k = \mathrm{Tr}_S(1-W)^k \,, \quad (2.28)$$

so that

$$\det_{\perp V}(\delta^\nu_\mu - M^\nu_\mu) = \exp\left[-\mathrm{Tr}_S \sum_{k=1}^\infty \frac{(1-W)^k}{k}\right] = \exp\{\mathrm{Tr}_S \ln[1-(1-W)]\} = \det_S W \,, \quad (2.29)$$

and the full Jacobian (2.21) is given finally by

$$J=\{\det_S(-\nabla\cdot\overline{\nabla})\det_S(-\overline{\nabla}\cdot\nabla)\det_S[(-\overline{\nabla}^2)^{-1}]\}^{1/2}$$
$$=\det_S^{-1/2}(-\overline{\nabla}^2)\det_S(-\overline{\nabla}\cdot\nabla)\ , \qquad (2.30)$$

because $\det_S(-\nabla\cdot\overline{\nabla})=\det_S(-\overline{\nabla}\cdot\nabla)$. The second factor in (2.30) is precisely the Faddeev-Popov Jacobian $\Delta_{\rm FP}$ of the gauge-fixing method for the specific choice of background gauge $(F^\mu)^{ij}=(\overline{\nabla}^\mu)^{ij}$, while the first factor is a constant, independent of the field point A_μ, and therefore may be taken out of the path integral without affecting the result. It is important to retain this factor if one calculates the effective action as a function of the background field, however.

Having determined the correct Jacobian of the transformation to field coordinates $(a^\perp_\mu,\theta)$, we may now express Z in the form

$$Z=[\mathrm{Vol}(\mathcal{G})]^{-1}\int[\mathcal{D}\theta^i]\int J[\mathcal{D}a^{\perp i}_\mu]\exp(iS_{\rm inv}[\overline{A}+a])=\det_S^{-1/2}(-\overline{\nabla}^2)\int[\mathcal{D}a^{\perp i}_\mu]\det_S(-\overline{\nabla}\cdot\nabla)|_{A=\overline{A}+a^\perp}\exp(iS_{\rm inv}[\overline{A}+a^\perp])\ , \qquad (2.31)$$

where the integral is over the gauge-invariant field coordinate $a^{\perp i}_\mu$, and the gauge volume factor has been canceled explicitly. It is clear that this form generates the same Feynman rules as the gauge-fixed path integral (2.6), and that a similar equivalence holds for other choices for F. In the present approach, however, the geometric significance of gauge fixing as simply a choice of coordinates in a coordinate-invariant expression is manifest and proofs of gauge invariance are unnecessary. Moreover, the nonperturbative aspects of the path integral and the correct normalization factor $\det_S^{-1/2}(-\overline{\nabla}^2)$ are also apparent.

To one-loop order around the background field, we may replace ∇ by $\overline{\nabla}$ in the Jacobian factor and arrive at

$$Z^{(1)}[\overline{A}]=\det_S^{1/2}(-\overline{\nabla}^2)\det_{V\perp}^{-1/2}(-\overline{\nabla}^2)e^{iS_{\rm inv}[\overline{A}]}\ , \qquad (2.32)$$

provided the background field satisfies the classical equations of motion $\overline{\nabla}^\mu\overline{F}_{\mu\nu}=0$. The factors of the determinants in (2.30) are just those needed for unitarity at one-loop order, since the transverse vector determinant corresponds to the propagation of $4-1=3$ modes, whereas the scalar determinant enters with the opposite power and hence subtracts one additional mode. This leaves precisely the two physical propagating modes per spacetime point, which we expect on the basis of the canonical quantization of the pure Yang-Mills gauge theory. Since the construction of the measure used in the covariant approach $[\mathcal{D}A^i_\mu]$ is identical to the BRST-invariant measure of the first method, we are guaranteed that the Ward identities of the covariant approach are anomaly-free, and that the entire formal apparatus necessary to prove the renormalizability and unitarity of the theory are in place. With this warm-up in Yang-Mills theory, we now apply exactly the same principles to the construction of the covariant measure for gravity.

III. COVARIANT PATH INTEGRAL FOR QUANTUM GRAVITY: GEOMETRIC APPROACH

The action of classical general relativity is invariant under general coordinate transformations $x^\mu\to X^\mu=X^\mu(x)$, the infinitesimal form of which is

$$x^\mu\to x^\mu+\xi^\mu(x)\ . \qquad (3.1)$$

We shall construct the functional measure in the path integral for quantum gravity by requiring that it also be invariant under (3.1). This is accomplished by transcribing the methods of Riemannian geometry on manifolds to the function space of spacetime metrics.

The first step is to consider an arbitrary spacetime metric $g_{\mu\nu}(x)$ to be the coordinate of a point in the function space of all metrics, denoted by $\mathcal{M}$. The infinitesimal one-form $\delta g_{\mu\nu}(x)\equiv h_{\mu\nu}(x)$ lies in the (co)tangent space to $\mathcal{M}$ at the point $g_{\mu\nu}(x)$. In ordinary Riemannian geometry, the metric on a manifold is specified by defining a scalar, bilinear in such coordinate one-forms, and identifying it with the geodesic distance ds^2. In a completely analogous manner, we may define a scalar inner product which is quadratic in the tangent-space one-forms of the manifold of functions $\mathcal{M}$:

$$\langle h,h\rangle_T\equiv\int d^4x\sqrt{-g}\,h_{\mu\nu}(x)G^{\mu\nu\rho\sigma}h_{\rho\sigma}(x)\ . \qquad (3.2)$$

The subscript T reminds us that this is an inner product for tensors. Now the scalar $ds^2=g_{\mu\nu}(x)dx^\mu dx^\nu$ is invariant under the (passive) relabeling of the coordinates of spacetime [Eq. (3.1)] that leaves the geometric spacetime point unchanged. The corresponding (active) transformation of the metric on the spacetime manifold is

$$h_{\mu\nu}(x)\to h_{\mu\nu}(x)+\nabla_\mu\xi_\nu+\nabla_\nu\xi_\mu\ . \qquad (3.3)$$

This may now be regarded as a relabeling of coordinates on $\mathcal{M}$ which leaves the "point," i.e., the geometry corresponding to $g_{\mu\nu}(x)$, unchanged. Hence we must require that the measure be invariant under the transformations (3.1) and (3.3) as well. Since $h_{\mu\nu}(x)$ transforms covariantly as a symmetric tensor under (3.1), $G^{\mu\nu\rho\sigma}$ must transform like a contravariant four-tensor. Like $g_{\mu\nu}(x)$, $G^{\mu\nu\rho\sigma}(g)$ has evident symmetry properties: It is symmetric under interchange of its first or last two indices, as well as interchange of the first two with the last two. Finally, again like $g_{\mu\nu}(x)$, $G^{\mu\nu\rho\sigma}(g)$ must be a purely local function of the coordinates of $\mathcal{M}$. That is, it should contain no derivatives of $g_{\mu\nu}(x)$. The unique ultralocal tensors with these properties are[8]

$$\tfrac{1}{2}(g^{\mu\rho}g^{\nu\sigma}+g^{\mu\sigma}g^{\nu\rho}),\quad g^{\mu\nu}g^{\rho\sigma}\ . \tag{3.4}$$

If we did not demand ultralocality, an infinite number of tensors involving higher derivatives would appear in this list. Using such tensors in the definition of the inner product and functional measure ultimately would have the effect of defining a different set of dynamical coordinates for the theory. Since we assume that the metric is the fundamental field coordinate, and derivatives of it in the action introduce dynamics, we do not wish to introduce derivatives and spurious dynamics into the essentially *kinematic* definition of the inner product or functional measure. In fact, this is the only principle which justifies an otherwise quite arbitrary distinguishing of the functional measure in the path integral from the action functional. The real proof that these statements are correct can come only *a posteriori,* when equivalence to the canonical approach is demonstrated.[9]

Restricting the metric on $\mathcal{M}$ to be covariant and ultralocal determines it (up to an overall irrelevant normalization) to be

$$G^{\mu\nu\rho\sigma}=\tfrac{1}{2}(g^{\mu\rho}g^{\nu\sigma}+g^{\mu\sigma}g^{\nu\rho}+Cg^{\mu\nu}g^{\rho\sigma})\ , \tag{3.5}$$

where C is an undetermined constant.[8]

Having endowed the function space of metrics $\mathcal{M}$ itself with a metric, we are now in a position to define an invariant volume form on the space. In order to avoid confusion, it is useful to introduce the vielbein field e_μ^m, which converts spacetime vector indices to Lorentz indices. If we then define the density

$$\tilde{h}_{mn}\equiv\sqrt{e}\,e_m^\mu e_n^\nu h_{\mu\nu}\ , \tag{3.6}$$

together with the relations

$$g_{\mu\nu}=e_\mu^m e_\nu^n\eta_{mn} \tag{3.7}$$

and

$$e\equiv\det(e_\mu^m)=\sqrt{-g}\ , \tag{3.8}$$

then the inner product (3.2) may be expressed in terms of the *flat* metric $\eta_{mn}=\mathrm{diag}(-1,1,1,1)$ as

$$\langle h,h\rangle_T=\int d^4x\,\tilde{h}_{mn}\tilde{G}^{mnrs}\tilde{h}_{rs}\ , \tag{3.9}$$

where

$$\tilde{G}^{mnrs}=\tfrac{1}{2}(\eta^{mr}\eta^{ns}+\eta^{ms}\eta^{nr}+C\eta^{mn}\eta^{rs}) \tag{3.10}$$

is independent of x.

Then, by analogy with the invariant volume form on a pseudo-Riemannian manifold, $\prod_m d\tilde{x}^m=\det(e_\mu^m)\prod_\mu dx^\mu$, we might try to define the invariant volume form on the function space of metrics $\mathcal{M}$ by

$$[\mathcal{D}h_{\mu\nu}]\equiv\mathrm{const}\times\prod_{x,m\le n}d\tilde{h}_{mn}$$
$$=\mathrm{const}\times\prod_x e^{(D-4)(D+1)/4}\prod_{\mu\le\nu}dh_{\mu\nu}(x)\ , \tag{3.11}$$

in D spacetime dimensions. Although useful for some formal manipulations, this "definition" leads to ambiguities for continuum functional integrals, because of the ill-defined nature of the product at each spacetime point, which also leaves the normalization of the measure undefined. Instead, it is preferable to define the measure by the Gaussian normalization condition

$$\int[\mathcal{D}h_{\mu\nu}]\exp\left[-\frac{i}{2}\langle h,h\rangle_T\right]=1\ , \tag{3.12}$$

which is formally satisfied by (3.11). This Gaussian integral is well defined in the Feynman-Kac sense.

The overall constant in front of the supermetric (3.5) is irrelevant, since it may always be reabsorbed into the normalization integral (3.12). The constant C is *not* irrelevant, since it determines the signature of the metric on $\mathcal{M}$. This may be seen by decomposing the arbitrary tangent space tensor into its trace-free and trace parts,

$$h_{\mu\nu}=h_{\mu\nu}^{\mathrm{TF}}+\frac{hg_{\mu\nu}}{4}\ , \tag{3.13}$$

and operating on this decomposition with the metric G. The $(D+2)(D-1)/2$ trace-free parts of $h_{\mu\nu}$ in D dimensions are mapped onto $h^{\mathrm{TF}\mu\nu}$, independent of C. However, on the scalar trace mode, G has the eigenvalue $1+CD/2$. Hence the signature of G depends on the value of C: for $C>-2/D$ the signature of G in the scalar trace sector is positive, while for $C<-2/D$ the signature is negative. If $C=-2/D$, the metric is noninvertible and becomes a projector onto the trace-free subspace. We leave the value of C undetermined for now and return to this issue in Ref. 9 in connection with the conformal factor problem.

For any value of C it is clear that the functional measure defined with reference to the inner product $\langle h,h\rangle_T$ is invariant under the infinitesimal general coordinate transformation (3.1). Under (3.3),

$$\delta_\xi\tilde{h}_{rs}=[\xi^\lambda\partial_\lambda+\tfrac{1}{2}(\partial_\lambda\xi^\lambda)]\tilde{h}_{rs}\ , \tag{3.14}$$

so that

$$\int d^4x\,\tilde{h}_{mn}\tilde{G}^{mnrs}(\delta_\xi\tilde{h}_{rs})=-\int d^4x(\delta_\xi\tilde{h}_{mn})\tilde{G}^{mnrs}\tilde{h}_{rs}\ , \tag{3.15}$$

by a simple integration by parts. This shows that the operator

$$Y_{rs}^{mn}\equiv\frac{\partial}{\partial\tilde{h}_{mn}}\delta_\xi\tilde{h}_{rs} \tag{3.16}$$

is anti-Hermitian and therefore traceless with respect to the inner product (3.2). This is exactly the property needed to prove that the Jacobian of transformation for the measure under (3.3) is unity; i.e., the measure (3.11) is invariant under infinitesimal coordinate transformations:

$$\delta_\xi[\mathcal{D}h_{\mu\nu}]=\det(1+Y)[\mathcal{D}h_{\mu\nu}]$$
$$=(1+\mathrm{tr}Y)[\mathcal{D}h_{\mu\nu}]=[\mathcal{D}h_{\mu\nu}]\ . \tag{3.17}$$

Once we have a coordinate-invariant functional measure, we must extract the infinite gauge-orbit volume in an invariant way as well. To this end we introduce a change of coordinates in the tangent space of $\mathcal{M}$ at $g_{\mu\nu}$:

$$h_{\mu\nu}=h^{\perp}_{\mu\nu}+(L\xi)_{\mu\nu}+(2\sigma+\tfrac{1}{2}\nabla_\lambda\xi^\lambda)g_{\mu\nu}\ , \tag{3.18}$$

where L, the "conformal Killing form," maps vectors into traceless symmetric tensors:

$$(L\xi)_{\mu\nu}\equiv\nabla_\mu\xi_\nu+\nabla_\mu\xi_\nu-\tfrac{1}{2}(\nabla_\lambda\xi^\lambda)g_{\mu\nu}\ . \tag{3.19}$$

Thus $L\xi$ spans all symmetric tensors which are gauge transforms of $h^{\rm TF}_{\mu\nu}$, the traceless part of $h_{\mu\nu}$. The scalar σ is the gauge-invariant piece of the trace, and $h^{\perp}_{\mu\nu}$ is the gauge-invariant piece of $h^{\rm TF}_{\mu\nu}$. In Ref. 6, $h^{\perp}_{\mu\nu}$ was chosen to lie in the orthogonal complement to L, with respect to the inner product (3.2), which required $(L^\dagger h^\perp)_\mu=-2\nabla^\nu h^{\perp}_{\mu\nu}=0$. Indeed, this is the simplest choice for doing one-loop calculations, which involve only Gaussian integrals, and justifies the notation $h^\perp$. However, the choice of orthogonal coordinates on the tangent space of $\mathcal{M}$ is by no means necessary, and $h^{\perp}_{\mu\nu}$ may be required to satisfy an arbitrary coordinate (gauge) condition:

$$(F\cdot h^\perp)_\mu\equiv F^\nu h^{\perp}_{\mu\nu}=0\ . \tag{3.20}$$

The only condition on F is that the operator $F\circ L$ be locally invertible, so that (3.18) can be solved uniquely for ξ:

$$\xi_\mu=(F\circ L)_\mu^{-1\ \nu}(F\circ h^{\rm TF})_\nu\ . \tag{3.21}$$

Otherwise, the local coordinate chart (3.18) is singular at the point $g_{\mu\nu}$.

Following the discussion in Sec. II for the case of the non-Abelian gauge field, to extract the infinite gauge-orbit volume generated by the gauge direction, ξ_μ, we must find the Jacobian of the transformation to the new field coordinates $(h^{\perp}_{\mu\nu},\xi_\mu,\sigma)$:

$$[\mathcal{D}h_{\mu\nu}]=J[\mathcal{D}h^{\perp}_{\mu\nu}][\mathcal{D}\xi_\mu][\mathcal{D}\sigma]\ . \tag{3.22}$$

This is accomplished by substituting the decomposition (3.18) into the inner product (3.2), completing the square of the term quadratic in ξ_μ, and computing the Gaussian integrals over each of the components, respectively:

$$\begin{aligned}1&=\int[\mathcal{D}h_{\mu\nu}]\exp\left[-\frac{i}{2}\langle h,h\rangle_T\right]\\&=J\int[\mathcal{D}h^{\perp}_{\mu\nu}]\exp\left[-\frac{i}{2}\langle h^\perp,(1-M)h^\perp\rangle_T\right]\int[\mathcal{D}\xi_\mu]\exp\left[-\frac{i}{2}\langle\xi',\Delta_1\xi'\rangle_V\right]\int[\mathcal{D}\sigma]\exp[-8i(1+2C)\langle\sigma,\sigma\rangle_S]\ ,\end{aligned} \tag{3.23}$$

where the vector Laplacian Δ_1 is defined by

$$(\Delta_1)_\mu{}^\nu\equiv(L^\dagger L)_\mu{}^\nu=-2(\delta_\mu{}^\nu\nabla^2+\tfrac{1}{2}\nabla_\mu\nabla^\nu+R_\mu{}^\nu)\ , \tag{3.24}$$

the tensor operator M is given by

$$M_{\mu\nu}{}^{\rho\sigma}\equiv[L(\Delta_1)^{-1}L^\dagger]_{\mu\nu}{}^{\rho\sigma}=-2\nabla_\mu(\Delta_1^{-1})_\nu{}^\rho\nabla^\sigma-2\nabla_\nu(\Delta_1^{-1})_\mu{}^\rho\nabla^\sigma+g_{\mu\nu}\nabla^\lambda(\Delta_1^{-1})_\lambda{}^\rho\nabla^\sigma\ , \tag{3.25}$$

and $\xi'_\mu=\xi_\mu+(\Delta_1^{-1}L^\dagger h^\perp)_\mu$ is the shifted vector obtained by completing the square. The notations $\langle\,,\rangle_V$ and $\langle\,,\rangle_S$ denote the covariant inner products on vectors and scalars, respectively:

$$\begin{aligned}\langle\xi,\xi\rangle_V&=\int d^4x\sqrt{-g}\,\xi_\mu g^{\mu\nu}\xi_\nu\ ,\\ \langle\sigma,\sigma\rangle_S&=\int d^4x\sqrt{-g}\,\sigma^2\ .\end{aligned} \tag{3.26}$$

The remaining tensor and vector Gaussian functional integrals in (3.23) yield, for the Jacobian,

$$J=[\det{}_{\perp T}(1-M)]^{1/2}(\det{}_V\Delta_1)^{1/2}\ , \tag{3.27}$$

where we have used

$$\begin{aligned}&\int[\mathcal{D}h^{\perp}_{\mu\nu}]\exp\left[-\frac{i}{2}\langle h^\perp,h^\perp\rangle_T\right]=1\ ,\\&\int[\mathcal{D}\xi_\mu]\exp\left[-\frac{i}{2}\langle\xi,\xi\rangle_V\right]=1\ ,\end{aligned} \tag{3.28}$$

and discarded a (C-dependent) constant. The notation $\det_{\perp T}$ denotes the determinant over tensor modes satisfying the condition (3.20). The tensor determinant may be simplified by introducing the projector onto the subspace obeying (3.20):

$$P^{\perp\rho\sigma}_{\mu\nu}(x,x')=[\delta^{\rm TF}-\delta^{\rm TF}F^\dagger(F\circ F^\dagger)^{-1}F\delta^{\rm TF}]_{\mu\nu}{}^{\rho\sigma}\delta^4(x,x')\ , \tag{3.29}$$

where

$$\delta^{\mathrm{TF}\rho\sigma}_{\mu\nu} \equiv \tfrac{1}{2}(\delta^{\rho}_{\mu}\delta^{\sigma}_{\nu}+\delta^{\sigma}_{\mu}\delta^{\rho}_{\nu}-\tfrac{1}{2}g_{\mu\nu}g^{\rho\sigma}) \tag{3.30}$$

is the trace-free part of the tensor Kronecker δ symbol in $D=4$ dimensions. Then the tensor determinant in (3.27) may be written in the alternative forms

$$\begin{aligned}\det_{\perp T}(1-M) &= \exp\left[-\sum_{k=1}^{\infty}\frac{\mathrm{Tr}_{\perp T}(M)^k}{k}\right] \\ &= \exp\left[-\sum_{k=1}^{\infty}\frac{\mathrm{Tr}_{T}(MP^{\perp})^k}{k}\right] \\ &= \exp\left[-\sum_{k=1}^{\infty}\frac{\mathrm{Tr}_{V}(1-W)^k}{k}\right] \\ &= \det_V W \ ,\end{aligned} \tag{3.31}$$

where the cyclic property of the trace has been used, and W is the vector operator:

$$W=(\Delta_1)^{-1}(F\circ L)^{\dagger}(F\circ F^{\dagger})^{-1}(F\circ L) \ . \tag{3.32}$$

Combining the results of (3.27), (3.31), and (3.32) yields the Jacobian in the more transparent form

$$J=[\det_V(F\circ F^{\dagger})]^{-1/2}\det_V(F\circ L) \ , \tag{3.33}$$

in complete analogy with the gauge theory result (2.30). The second factor is the Faddeev-Popov determinant for the gauge (3.20) on the trace-free components of $h_{\mu\nu}$, while the first factor is an $h_{\mu\nu}$-independent normalization factor that makes no contribution to the Feynman rules. Note that the Jacobian factor has been derived by tangent-space methods, involving only Gaussian functional integrals, but that this involves no restriction to one-loop order. The result (3.33) is valid to *all* orders of perturbation theory.

With the Jacobian (3.33), we now know how to factor the infinite diffeomorphism gauge group volume out of the covariant quantum measure (3.22) in a manifestly covariant way. If the action is independent of the vector gauge-orbit parameter ξ_μ, integration over ξ_μ would simply yield the infinite volume of the diffeomorphism group, and so

$$\begin{aligned}[\mathrm{Vol}(\mathcal{G})]^{-1}\int[\mathcal{D}h_{\mu\nu}] &= [\mathrm{Vol}(\mathcal{G})]^{-1}\int[\mathcal{D}\xi_\mu]\int J[\mathcal{D}h^{\perp}_{\mu\nu}][\mathcal{D}\sigma] \\ &= \int J[\mathcal{D}h^{\perp}_{\mu\nu}][\mathcal{D}\sigma] \ ,\end{aligned} \tag{3.34}$$

when integrated over functions independent of ξ_μ.

The final steps in constructing the covariant path integral for quantum gravity involves extending the integration measure defined on the tangent space to a measure on the full metric. The extension of the coordinates on the tangent space (3.18) to coordinates on $\mathcal{M}$ is straightforward. We write

$$\begin{aligned} g_{\mu\nu}(x) &= \frac{\partial X^{\rho}}{\partial x^{\mu}}\frac{\partial X^{\sigma}}{\partial x^{\nu}}e^{2\sigma(X)}g^{\perp}_{\rho\sigma}(X) \ , \\ (F\cdot g^{\perp})_\mu &= 0 \ ,\end{aligned} \tag{3.35}$$

where σ may be fixed by the requirement that $g^{\perp}_{\mu\nu}$ have fixed constant scalar curvature $R^{\perp}$. This implies that σ must satisfy the Yamabe condition

$$R[g_{\mu\nu}]=e^{-2\sigma}R^{\perp}-6e^{-2\sigma}[\nabla^{\perp}\cdot\nabla^{\perp}\sigma+(\nabla^{\perp}\sigma)\cdot(\nabla^{\perp}\sigma)] \ , \tag{3.36}$$

where $\nabla^{\perp}\cdot\nabla^{\perp}$ is the scalar Laplace-Beltrami operator of the metric $g^{\perp}$. The path integral for quantum gravity may be written then in the succinct generally covariant form

$$\begin{aligned} Z &= [\mathrm{Vol}(\mathcal{G})]^{-1}\int[\mathcal{D}g_{\mu\nu}]\exp(iS_{\mathrm{inv}}[g]) \\ &= \int J[\mathcal{D}g^{\perp}_{\mu\nu}][\mathcal{D}\sigma]\exp(iS_{\mathrm{inv}}[e^{2\sigma}g^{\perp}]) \\ &= [\det_V(F\circ F^{\dagger})]^{-1/2}\int[\mathcal{D}\sigma][\mathcal{D}g^{\perp}_{\mu\nu}]\det_V(F\circ L)\big|_{g=e^{2\sigma}g^{\perp}}\exp(iS_{\mathrm{inv}}[e^{2\sigma}g^{\perp}]) \ ,\end{aligned} \tag{3.37}$$

where (3.33) has been used.

If we set

$$g^{\perp}_{\mu\nu}=\bar{g}_{\mu\nu}+h^{\perp}_{\mu\nu} \ , \tag{3.38}$$

the analogy with the gauge theory result (2.31) will be evident. In both cases the $a^{\perp}$- or $h^{\perp}$-independent determinant makes no contribution to the Feynman rules. It arises only because

$$[\mathcal{D}h_{\mu\nu}]\delta(F\circ h)=[\det{}_V(F\circ F^{\dagger})]^{-1/2}[\mathcal{D}h^{\perp}_{\mu\nu}] \ , \tag{3.39}$$

when both the $[\mathcal{D}h_{\mu\nu}]$ and $[\mathcal{D}h^{\perp}_{\mu\nu}]$ measures are normalized by Gaussian conditions (3.12) and (3.28) in their respective spaces.

At one-loop order (around an arbitrary background metric $\bar{g}$), (3.37) reproduces the results of Ref. 6. An interesting aspect of the general formula for the path integral (3.37) is that it requires that the classical Yamabe problem (3.36) be solvable for each g and $g^{\perp}$.[11] This problem of classical Riemannian geometry thus takes on an added importance at the quantum level, particularly in the nonperturbative domain.

IV. BRST INVARIANCE OF THE MEASURE

The construction of the path integral in the last section is manifestly invariant under coordinate transformations of the underlying four-geometries contributing to the Feynman path integral. In particular, the infinite gauge-orbit volume has been divided out in a geometric fashion. Let us compare this result to the more standard introduction of a gauge-fixing condition and Faddeev-Popov determinant into the path integral. The comparison is not quite as immediate as that for non-Abelian gauge theory because the metric on the field space $\mathcal{M}$ is nontrivial.

The gauge-fixed form of the tangent-space path integral for quantum gravity is

$$Z=\int[\mathcal{D}h_{\mu\nu}][\mathcal{D}c^{\mu}][\mathcal{D}\bar{c}^{\mu}][\mathcal{D}b^{\mu}]\exp[i(S_{\rm inv}+S_{\rm gh}+S_{\rm gf})] \ . \tag{4.1}$$

As in Sec. II, the identity

$$\int[\mathcal{D}\xi_{\mu}]\delta[(F\cdot h^{(\xi)}_{\mu})]\Delta_{\rm FP}=1 \ , \tag{4.2}$$

with

$$\Delta_{\rm FP}=\det\left|\frac{\delta}{\delta\xi}(F\cdot h^{(\xi)})\right| \ , \tag{4.3}$$

has been inserted into the path integral, and the volume of the diffeomorphism group $\int[\mathcal{D}\xi_{\mu}]$ been divided out. The anticommuting ghost fields c^{μ} and $\bar{c}_{\nu}$ have been introduced to express the Faddeev-Popov vector determinant in the form

$$\begin{aligned}\Delta_{\rm FP}&=\int[\mathcal{D}c^{\mu}][\mathcal{D}\bar{c}_{\nu}]\exp(iS_{\rm gh}[h,c,\bar{c}]) \ ,\\ S_{\rm gh}&=\int d^4x\sqrt{-g}\,\bar{c}^{\nu}(g_{\nu\lambda}F^{\lambda}\nabla_{\mu}+g_{\mu\nu}F^{\lambda}\nabla_{\lambda})c^{\mu} \ .\end{aligned} \tag{4.4}$$

The gauge-fixing δ function has been enforced with the aid of a local auxiliary field as well:

$$\begin{aligned}\delta[(F\cdot h)_{\mu}]&=\int[\mathcal{D}b^{\mu}]\exp(iS_{\rm gf}[h,b]) \ ,\\ S_{\rm gf}&=\int d^4x\sqrt{-g}\,b^{\mu}(F\cdot h)_{\mu} \ .\end{aligned} \tag{4.5}$$

To show that this form for the path integral is BRST invariant, introduce the BRST supersymmetry operator for quantum gravity, defined by

$$\begin{aligned}Qh_{\mu\nu}&=c^{\lambda}\partial_{\lambda}g_{\mu\nu}+g_{\mu\nu}\partial_{\nu}c^{\lambda}+g_{\nu\lambda}\partial_{\mu}c^{\lambda} \ ,\\ Qc^{\mu}&=c^{\lambda}\partial_{\lambda}c^{\mu} \ ,\\ Q[(-g)^{p}\bar{c}^{\mu}]&=(-g)^{p}b^{\mu} \ ,\\ Q[(-g)^{p}b^{\mu}]&=0 \ .\end{aligned} \tag{4.6}$$

This operator is nilpotent for *arbitrary* weighting power p:

$$Q^2=0 \ . \tag{4.7}$$

This differs from the standard definition of the BRST transformation ($p=0$), but is related to it by a redefinition of fields. The covariant ghost plus gauge-fixing terms in the quantum action may be written as an exact form,

$$S_{\rm gh}+S_{\rm gf}=Q\int d^4x\sqrt{-g}\,\bar{c}^{\mu}(F\cdot h)_{\mu} \ , \tag{4.8}$$

provided that p is chosen to have the value $\frac{1}{2}$. Since $S_{\rm inv}[g]$ is automatically BRST invariant if it is coordinate invariant, the full quantum action is invariant under the BRST supersymmetry (4.6), with this choice of p. Therefore, the vacuum amplitude (4.1) will be BRST invariant provided the product of the measure factors is invariant.

Now Fujikawa determined the measure by requiring BRST invariance at the outset. However, we already know that the coordinate-invariant measure on the space of metrics $\mathcal{M}$ is determined by the inner product (3.2) and (3.5) and the Gaussian condition (3.12). Formally, this measure is equivalent to (3.11), which is the same factor found by Fujikawa by requiring BRST invariance. Of course, this is no accident: The factor in (3.11) required for BRST invariance is the same as that required for general coordinate invariance because the BRST transformation of $h_{\mu\nu}$ is just an infinitesimal coordinate transformation with c^{μ} replacing ξ^{μ}. Hence the $[\mathcal{D}h_{\mu\nu}]$ measure in (4.1) is the same as that defined previously by (3.12).

To prove the BRST invariance of the full functional measure, the strategy will be as follows. We know from our previous demonstration of the coordinate invariance of measure $[\mathcal{D}h_{\mu\nu}]$ that it was convenient to introduce densities with "internal" Lorentz indices. So we define

$$\begin{aligned}\tilde{c}^{m}&\equiv\sqrt{e}\,e^{m}_{\mu}c^{\mu} \ ,\\ \tilde{\bar{c}}^{m}&\equiv\sqrt{e}\,e^{m}_{\mu}\bar{c}^{\mu} \ ,\\ \tilde{b}^{m}&\equiv\sqrt{e}\,e^{m}_{\mu}b^{\mu} \ ,\end{aligned} \tag{4.9}$$

with $\tilde{h}_{mn}$ defined in Eq. (3.6). Then we regard the BRST transformation as an infinitesimal transformation on the set of field *densities*, $\tilde{\Psi}\equiv(\tilde{h}_{mn},\tilde{c}^{m},\tilde{\bar{c}}^{m},\tilde{b}^{m})$. It is the Jacobian of the transformation $\tilde{\Psi}\rightarrow\tilde{\Psi}+Q\tilde{\Psi}$ which we wish to show is equal to unity:

$$J_Q=\text{sdet}\left[1+\frac{\partial Q\tilde{\Psi}}{\partial\tilde{\Psi}}\right]=1+\text{str}\left[\frac{\partial Q\tilde{\Psi}}{\partial\tilde{\Psi}}\right] . \tag{4.10}$$

From the trivial antighost $\bar{c}^{\mu}$ and the auxiliary field b^{μ} transformation (4.6), the measure factors $[\mathcal{D}\bar{c}^{\mu}]$ and $[\mathcal{D}b^{\mu}]$ are invariant. Thus we need be concerned only with the submatrix of transformation with respect to $\tilde{h}_{mn}$ and $\tilde{c}^{m}$.

The subtlety now is that this submatrix must be evaluated by holding one of these field *densities* fixed while the other is varied, whereas the coordinate invariance of $[\mathcal{D}h_{\mu\nu}]$ made use of a variation of the density $\tilde{h}_{mn}$ with the *vector* ξ^{μ} (*not* its corresponding density) held fixed. This is easily handled by carrying out the transformation from $(\tilde{h}_{mn},\tilde{c}^{m})$ to $(\tilde{h}'_{mn},\tilde{c}'^{m})=(1+Q)(\tilde{h}_{mn},\tilde{c}^{m})$ in the following three steps:

$$(\tilde{h}_{mn},\tilde{c}^{m})\to(\tilde{h}_{mn},c^{\mu})\to(\tilde{h}'_{mn},c^{\mu})\to(\tilde{h}'_{mn},\tilde{c}'^{m}) . \tag{4.11}$$

In step 1, $\tilde{c}^{m}$ is changed to c^{μ} with $\tilde{h}_{mn}$ (and consequently $h_{\mu\nu}$) held fixed. In step 2, the BRST transformation is applied to $\tilde{h}_{mn}$ with the vector c^{μ} held fixed. The determinant of *this* transformation is precisely the one we have already computed to verify the coordinate invariance of $[\mathcal{D}h_{\mu\nu}]$. Provided we replace the gauge function ξ^{μ} by the ghost field c^{μ} in Eq. (3.14), it follows that this second transformation has unit Jacobian. Finally, we change variables from c^{μ} into its BRST transformed density $\tilde{c}'^{m}=\tilde{c}^{m}+Q\tilde{c}^{m}$, keeping $\tilde{h}'_{mn}=\tilde{h}_{mn}+Q\tilde{h}_{mn}$ fixed. Since the Jacobian of step 2 is unity, the Jacobian of the full transformation J_Q is given by the determinant of the product of steps 3 and 1:

$$J_Q=\text{sdet}\left[\left[\frac{\partial(\tilde{c}^{n}+Q\tilde{c}^{n})}{\partial c^{\mu}}\right]_{\tilde{h}'_{mn}}\left[\frac{\partial c^{\mu}}{\partial\tilde{c}^{m}}\right]_{\tilde{h}_{mn}}\right]\equiv\text{sdet}_{mn}(M^{m}_{\mu}N^{\mu}_{n}) , \tag{4.12}$$

where

$$M^{m}_{\mu}(x,y)=(e^{1/2}e_{\nu}{}^{m}-\epsilon\{c^{\rho}[\partial_{\rho}(e^{1/2}e_{\nu}{}^{m})]+[\partial_{\nu}c^{\rho}]e_{\rho}{}^{m}e^{1/2}+\tfrac{1}{2}[\partial_{\rho}c^{\rho}]e^{1/2}e_{\nu}{}^{m}\})\{\delta^{\nu}_{\mu}-\epsilon(c^{\rho}\partial^{x}_{\rho}\delta^{\nu}_{\mu}-[\partial_{\mu}c^{\nu}])\}\delta^{4}(x,y) , \tag{4.13}$$

and

$$N^{\mu}_{n}(y,z)=\left[\frac{\partial c^{\mu}(y)}{\partial\tilde{c}^{n}(z)}\right]_{h_{mn}}=e^{-1/2}(y)e_{n}{}^{\mu}(y)\delta^{4}(y,z) , \tag{4.14}$$

and we have introduced the Grassmannian constant ϵ in the BRST transformation to help keep track of signs. Following Ref. 5, we may simplify the first matrix somewhat by rewriting it in the form

$$M^{m}_{\mu}(x,y)=\{e^{1/2}e_{\mu}{}^{m}-\epsilon(c^{\nu}\partial_{\nu}e^{1/2}e_{\mu}{}^{m}+\tfrac{1}{2}e^{1/2}e_{\mu}{}^{m}[\partial_{\kappa}c^{\kappa}])\}\delta^{4}(x,y) . \tag{4.15}$$

Multiplying the two matrices together and using the identity

$$e^{1/2}(x)e_{\mu}{}^{m}(x)\partial^{x}_{\nu}[e^{-1/2}(x)e^{\mu}_{n}(x)\delta^{4}(x,z)]+\{\partial^{x}_{\nu}[e^{1/2}(x)e_{\mu}{}^{m}(x)]\}e^{-1/2}(z)e^{\mu}_{n}(z)\delta^{4}(x,z)=\delta^{m}_{n}\partial^{x}_{\nu}\delta^{4}(x,z) \tag{4.16}$$

gives

$$J_Q=\text{sdet}\{\delta^{m}_{n}\delta^{4}(x,z)-\epsilon[c^{\nu}(x)\partial^{x}_{\nu}\delta^{m}_{n}+\tfrac{1}{2}\delta^{m}_{n}(\partial_{\nu}c^{\nu})]\delta^{4}(x,z)\} ,$$
$$=\text{sdet}(1+Y) \tag{4.17}$$

with Y given by

$$Y^{m}_{n}=\delta^{m}_{n}[c^{\lambda}\partial_{\lambda}+\tfrac{1}{2}(\partial_{\lambda}c^{\lambda})] , \tag{4.18}$$

which is exactly of the same form as the operator appearing in Eq. (3.14). Therefore, it is anti-Hermitian, its trace vanishes by an integration by parts, and we have secured the desired result: viz.,

$$J_Q=1 . \tag{4.19}$$

The authors of Ref. 5 point out the difficulties with regularization that arise if one tries to write the determinant of the product of transformations in Eq. (4.12) as a product of determinants. However, such a separation is unnecessary from the present point of view since we are interested only in the Jacobian of the full transformation. Indeed, formal manipulations of the determinants of infinite-dimensional matrices may lead to incorrect (i.e., noninvariant) results, unless an invariant regulator is used during all intermediate steps and then removed only at the very end of the calculation. Since "invariant" means invariant under general coordinate and/or BRST transformations, this requirement means that one can never encounter the noncovariant factors of g^{00} obtained by Fradkin and Vilkovisky, and that infinite factors such as δ functions of zero are automatically absorbed into the normalization of the path-integral measure. Much of the technical difficulty of the earlier literature on the path-integral measure for the gravitational field is thereby avoided.

Finally, we remark that the gauge-fixed amplitude (4.1) is not quite in the same form as (3.37) because the covariant construction necessarily involves the separation of the metric into its conformal and conformally equivalent parts (σ and $g^{\perp}$, respectively), while no such decomposition has been introduced in (4.1). Thus the gauge-fixing condition and corresponding Faddeev-Popov determinant refer to the metric $\exp(2\sigma)g^{\perp}$ rather than just $g^{\perp}$ of the geometric approach. If the conformal factor is separately defined by (3.36) and the gauge condition F, then, applied to the class of conformally equivalent metrics $g^{\perp}$, (4.1) and (3.37) will agree completely.

To summarize, the general coordinate-invariant measure defined by the normalization condition (3.12), with respect to the ultralocal metric on the space of metrics $G^{\mu\nu\rho\sigma}$, defines a BRST-invariant measure for quantum gravity after gauge fixing. Our construction of the measure is guaranteed to be covariant, provided a strictly covariant regularization technique is used systematically. The densities appearing in the formal proofs of BRST invariance in Refs. 4 and 5 are seen to be precisely those required by general covariance with respect to the underlying spacetime geometry. In addition, the geometric construction of the tangent-space measure lifts immediately to a measure on the full space of metrics in (3.37). Thus the semiclassical aspects of quantum gravity may be addressed in this approach without explicit reference to the cumbersome perturbative formalism of gauge fixing and ghosts.

In order to remove all doubt of the correctness of this measure for quantum gravity, the connection to the manifestly unitary Hamiltonian form of the path integral should be made, and the constant C appearing in $G^{\mu\nu\rho\sigma}$ should be determined by canonical methods. These issues and their relation to the conformal factor problem of the Einstein-Hilbert action we take up in a separate publication.[9]

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