Incompleteness theorems for the spherically symmetric spacetimes

Gregory A. Burnett*
Enrico Fermi Institute and Department of Physics, 5640 S. Ellis Avenue, University of Chicago, Chicago, Illinois 60637
(Received 29 May 1990)

The closed-universe recollapse conjecture is studied for the spherically symmetric spacetimes. It is proven that there exists an upper bound to the lengths of timelike curves in any spherically symmetric spacetime that possesses $S^1\times S^2$ Cauchy surfaces and that satisfies the non-negative-pressures and dominant-energy conditions. Further, an explicit bound is obtained that is determined by the initial data for the spacetime on any Cauchy surface. The conjecture is further studied for the spherically symmetric spacetimes possessing an extra spatial symmetry—the Kantowski-Sachs spacetimes. It is proven, for example, that there exists an upper bound to the lengths of timelike curves in any Kantowski-Sachs spacetime that possesses compact Cauchy surfaces and that satisfies the non-negative-sum-pressures condition.

I. INTRODUCTION

One of the most striking possibilities raised by general relativity is that our Universe may exist for only a finite length of time. Indeed, if our Universe has S^3 spatial topology, is well represented as being spatially homogeneous and isotropic, and the matter content is ordinary, then the lifetime of the Universe is known to be finite.[1] But, what if the Universe is not well represented as being spatially homogeneous and isotropic: Must its lifetime still be finite? The closed-universe recollapse conjecture[1–3] asserts that it must.

One version of the closed-universe recollapse conjecture states, roughly, that all spacetimes that possess compact Cauchy surfaces of an appropriate topology and that satisfy an appropriate energy condition "expand from an all-encompassing initial singularity to a maximal hypersurface and recollapse to an all-encompassing final singularity."[1] However, there is a difficulty in formulating a viable form of this conjecture: It is a simple matter to construct spacetimes that possess S^3 Cauchy surfaces, contain ordinary matter, and yet do not possess maximal hypersurfaces. For instance, take the past of any expanding spatially homogeneous hypersurface of a $k=+1$ Friedmann-Robertson-Walker spacetime with positive energy density and pressure. Although one might think that this spacetime, when continued to the future, must eventually possess a maximal hypersurface, this is not the case. The spacetime can become singular before attaining a maximal hypersurface.[1]

Rather than attempting to place further conditions on the spacetimes being considered to overcome this difficulty, we choose to investigate the following weak version of the closed-universe recollapse conjecture.

Conjecture. There exists an upper bound to the lengths of timelike curves in any spacetime that possesses compact Cauchy surfaces of an appropriate topology and that satisfies an appropriate energy condition.

It should be noted that the existence of an upper bound to the lengths of timelike curves does not necessarily mean the existence of an "all-encompassing final singularity"—the spacetime may be extendible. However, the weak version of the closed-universe recollapse conjecture would follow from the prior with the further demand that the spacetime satisfy the timelike-convergence condition ($R_{ab}t^at^b\geq 0$ for all timelike t^a) and a genericity requirement.[2,3]

What Cauchy-surface topologies are appropriate for the closed-universe recollapse conjecture? For the version asserting the existence of a maximal hypersurface, the appropriate topologies are[2,4]

$$\Xi_1\#\Xi_2\#\cdots\#\Xi_j\#k(S^1\times S^2)\ , \tag{1.1}$$

where # denotes the connected sum,[5] each Ξ_i is a compact orientable three-manifold with a finite fundamental group,[6] and $k(S^1\times S^2)$ denotes the connected sum of k copies of $S^1\times S^2$. So, for example, S^3 and $S^1\times S^2$ are appropriate. That Eq. (1.1) is the appropriate topologies is seen as follows. First, any three-manifold Σ that admits a flat Riemannian metric q_{ab} must not be allowed: The flat spacetime $(\mathbb{R}\times\Sigma,-(dt)_a(dt)_b+q_{ab})$ clearly admits infinite-length timelike curves. Second, the scalar curvature R associated with the induced metric on a maximal hypersurface must, by the initial-value constraint equation and the non-negative-energy condition ($G_{ab}t^at^b\geq 0$ for all timelike t^a), be non-negative. But, it is known that the compact three-manifolds that admit Riemannian metrics with $R\geq 0$ are rare among the compact three-manifolds.[7] In fact, any compact orientable three-manifold with a metric having $R\geq 0$, must be either[8] (i) a three-manifold with a flat metric or (ii) one of those in Eq. (1.1). Combining these two observations, we see that the appropriate three-manifolds are those in Eq. (1.1).

Although the weak version of the closed-universe collapse conjecture above does not require the existence of a maximal hypersurface, we shall nonetheless regard the three-manifolds, given in Eq. (1.1), as appropriate Cauchy-surface topologies for this form of the conjecture.

What energy conditions are appropriate for the closed-universe recollapse conjecture? It is known that the non-negative-energy, timelike-convergence, and dominant-energy conditions together are not sufficient: The $k=+1$ Friedmann-Robertson-Walker spacetime with scale factor $a(t)=t$ satisfies all three conditions and yet expands indefinitely. Although what energy condition is appropriate remains an issue to be resolved in the study of this conjecture, three energy conditions which have played a prominent role are the following.

(i) *The dominant-energy condition:*

$$G_{ab}t^a u^b \geq 0 \tag{1.2}$$

for all future-directed timelike t^a and u^b. For those G_{ab} possessing a timelike eigenvector (with eigenvalue $-\rho$) and, hence, three spacelike eigenvectors (with eigenvalues p_k, $k=1,2,3$), Eq. (1.2) is equivalent to the condition that $\rho \geq |p_k|$, $k=1,2,3$.

(ii) *The non-negative-pressures condition:*

$$G_{ab}x^a x^b \geq 0 \tag{1.3}$$

for all spacelike x^a. For those G_{ab} as before Eq. (1.3) is equivalent to the condition that $p_k \geq 0$, $k=1,2,3$ (non-negative principal pressures) and $\rho + p_k \geq 0$, $k=1,2,3$.

(iii) *The non-negative-sum-pressures condition,* which is the special case of the one-parameter family of energy conditions:

$$G_{ab}(t^a t^b + \lambda g^{ab}) \geq 0 \tag{1.4}$$

for all unit-timelike t^a, when $\lambda=1$. Note that for $\lambda=0$ and $\lambda=\frac{1}{2}$, Eq. (1.4) is the non-negative-energy and timelike-convergence condition, respectively. For those G_{ab} as before, Eq. (1.4) is equivalent to the condition that $(1-\lambda)\rho + \lambda\sum_{k=1}^{3} p_k \geq 0$ and $\rho + p_k \geq 0$, $k=1,2,3$. Note that the null-convergence condition ($G_{ab}k^a k^b \geq 0$ for all null k^a) follows, by continuity, from any of the above conditions.

Both versions of the closed-universe recollapse conjecture provide interesting possible extensions of the singularity theorems. It is known, for example, that spacetimes that possess compact Cauchy surfaces and that satisfy the timelike-convergence and generic conditions cannot be timelike and null geodesically complete.[9] The closed-universe recollapse conjecture asserts that with the further restrictions on the topology of the Cauchy surfaces and the material content, not only are we guaranteed that one timelike or null geodesic is incomplete, but, in fact, that all timelike curves are incomplete.

Here, we study the above conjecture for the spherically symmetric spacetimes. In Sec. II, we prove the following theorem.

Theorem 1.1. There exists an upper bound to the lengths of timelike curves in any spherically symmetric spacetime that possesses $S^1 \times S^2$ Cauchy surfaces and that satisfies the non-negative-pressures and dominant-energy conditions.

Further, we obtain an explicit expression for an upper bound on the lengths of timelike curves in the spacetime that is determined by the initial data for the spacetime on any Cauchy surface. It should be noted that although the recollapse of the dust-filled spherically symmetric spacetimes (i.e., the Tolman spacetimes) have been studied by a number of authors, none have provided a proof of theorem 1.1 for these spacetimes with either $S^1 \times S^2$ or S^3 spatial topology.[10–12] Theorem 1.1 resolves the issue of recollapse for the Tolman spacetimes with $S^1 \times S^2$ spatial topology and non-negative energy density.

In Sec. III, the above conjecture is further studied for the spherically symmetric spacetimes possessing an extra spatial symmetry—the Kantowski-Sachs spacetimes. There the following theorem is proven.

Theorem 1.2. Only for $\lambda \geq 1/\sqrt{3}$ does there exist an upper bound to the lengths of timelike curves in any Kantowski-Sachs spacetime that possesses compact Cauchy surfaces and that satisfies Eq. (1.4).

In particular, this implies that the non-negative-sum-pressures condition ($\lambda=1$) is sufficient to ensure the existence of an upper bound to the lengths of timelike curves in any Kantowski-Sachs spacetime with compact Cauchy surfaces.

Lastly, in Sec. IV we make a few remarks regarding possible extensions of these results.

The conventions used herein are those of Ref. 9. In particular, our metrics are such that timelike vectors have negative norm. Further, all metrics are to be C^2.

II. SPHERICALLY SYMMETRIC SPACETIMES

We begin by reviewing a few of the basics of the spherically symmetric spacetimes. A spacetime (M, g_{ab}) is said to be *spherically symmetric* if it admits the group $G \approx SO(3)$ of isometries, acting effectively on M, each of whose orbits is either a two-sphere or a point. Introduce the non-negative scalar field r whose value at $p \in M$ is such that $4\pi r^2$ is the area of the orbit through p. As shown in the Appendix, the Cauchy surfaces of any globally hyperbolic spherically symmetric spacetime must be diffeomorphic to either $\mathbb{R}^3$, S^3, $\mathbb{R} \times S^2$, or $S^1 \times S^2$, with r strictly positive in and only in the latter two cases.[13] Thus, the spherically symmetric spacetimes have only two possible compact Cauchy-surface topologies: S^3 and $S^1 \times S^2$.

For any spherically symmetric spacetime (M, g_{ab}) with r strictly positive, we can construct a two-dimensional spacetime (B, h_{ab}) where $B = M/G$ and $h^{ab} = (\pi_1^* g)^{ab}$ (π_1 is the natural map from M to M/G.) It is straightforward to show that (B, h_{ab}) is globally hyperbolic iff (M, g_{ab}) is globally hyperbolic. Further, if (M, g_{ab}) is globally hyperbolic, then from (B, h_{ab}) and r (viewed as a field on B) we can reconstruct the full spacetime by setting

$$M = B \times S^2 , \tag{2.1}$$

$$g_{ab} = (\pi_1^* h)_{ab} + (r \circ \pi_1)^2 (\pi_2^* \Omega)_{ab} , \tag{2.2}$$

where Ω_{ab} is a unit-metric on S^2 and π_1 and π_2 are the natural maps from M to B and S^2, respectively. By noting that (M, B, π_1, S^2) is a fiber bundle with total space M, base space B, projection map π_1, and typical fiber S^2, we see that Eq. (2.1) follows from the fact that all orientable S^2 bundles over B (which must have topology $\mathbb{R}^2$ or $\mathbb{R} \times S^1$) are trivial.[14,15] That Eq. (2.2) holds locally about any orbit is well known.[16] The decomposition of g_{ab} must hold globally since we have demanded that the spherical symmetry be a global symmetry—trying to put a twist in the metric is not allowed.

Denote by D_a that derivative operator compatible with h_{ab}, on B, and set

$$2m = r(1 - D_m r D^m r) \ . \tag{2.3}$$

The Einstein tensor G_{ab} associated with the metric g_{ab} is then given by

$$G_{ab} = \frac{2}{r}\left[D_m D^m r - \frac{m}{r^2}\right] h_{ab} - \frac{2}{r} D_a D_b r + (D_m D^m r - \tfrac{1}{2} R r) r \Omega_{ab} \ , \tag{2.4}$$

where R is the scalar curvature associated with h_{ab} and we have used the above decomposition to make an identification between fields on M and fields on B and S^2. From Eq. (2.4), we have

$$D_a D_b r = \frac{m}{r^2} h_{ab} - \frac{r}{2} G^{mn} \epsilon_{ma} \epsilon_{nb} \ , \tag{2.5}$$

where ϵ_{ab} is either of the two volume elements on B constructed from h_{ab}. Further, from Eq. (2.5), we have

$$D_a(2m) = r^2 G^{mn} \epsilon_{ma} \epsilon_{nb} D^b r \ . \tag{2.6}$$

Notice that, in the vacuum case, m is a constant. It is the mass parameter of that extended Schwarzschild spacetime to which this spacetime is, by Birkhoff's theorem,[16] locally isometric. While it is tempting to interpret m as a sort of quasilocal gravitational mass, since there does not exist, currently, such a notion in general relativity, it is difficult to justify such an identification.[17] Nevertheless, the field m will be very useful in what follows.

We now begin the proof of theorem 1.1. Fix any spherically symmetric spacetime (M, g_{ab}) that possesses $S^1 \times S^2$ Cauchy surfaces and that satisfies the non-negative-pressures and dominant-energy conditions. Construct the two-dimensional spacetime (B, h_{ab}), as described above. This spacetime is globally hyperbolic with compact S^1 Cauchy surfaces.

Denote by $L(M, g_{ab})$ the (possibly infinite) least upper bound to the lengths of timelike curves in a spacetime (M, g_{ab}). Using the decomposition above, it is not difficult to show that $L(M, g_{ab}) = L(B, h_{ab})$, and so theorem 1.1 will be proven if we can show that $L(B, h_{ab})$ is finite. The strategy is to take advantage of the following theorem.

Theorem 2.1. Fix any two-dimensional globally hyperbolic spacetime (B, h_{ab}) and any positive scalar field r that satisfies Eq. (2.5), with G_{ab} satisfying the non-negative-pressures condition. If $\sup_B(r)$ is finite and $\inf_B(2m)$ is positive, then

$$L(B, h_{ab}) \le \pi \left[\frac{\sup_B(r)^3}{\inf_B(2m)}\right]^{1/2} \ . \tag{2.7}$$

Proof. Denote by T the right-hand side of Eq. (2.7). Supposing, for contradiction, that Eq. (2.7) were false, then there would exist, in B, a timelike geodesic having a length greater than T. Along this geodesic, using Eq. (2.5) and the non-negative-pressures condition, we have

$$\frac{d^2 r}{dt^2} = t^a t^b D_a D_b r \le -\frac{m}{r^2} \le -\frac{\inf_B(2m)}{2r^2} \ , \tag{2.8}$$

where t^a is the unit tangent vector along the curve and t its parameter. But, it is a straightforward exercise to show that it is impossible for r to satisfy Eq. (2.8) and the inequalities $0 < r \le \sup_B(r)$, for a time T or greater.[18] This establishes theorem 2.1.□

Remarkably, we can, in fact, show that r is bounded from above and that $2m$ is everywhere bounded away from zero by a positive constant. We have the following theorem.

Theorem 2.2. Fix any two-dimensional spacetime (B, h_{ab}) that possesses a compact Cauchy surface S and any positive scalar field r that satisfies Eq. (2.5), with G_{ab} satisfying the non-negative-pressures and dominant-energy conditions. Then

$$r \le \max_S(2m) \ , \tag{2.9}$$

$$2m \ge \min_S(r) > 0 \ . \tag{2.10}$$

In fact, it can be shown that Eqs. (2.9) and (2.10) hold equally well when S is any Cauchy surface in the full four-dimensional spacetime. Thus, we have the following explicit bound on $L(M, g_{ab})$ that is determined by the induced initial data on any Cauchy surface S, in M:

$$L(M, g_{ab}) \le \pi \left[\frac{\max_S(2m)^3}{\min_S(r)}\right]^{1/2} \ . \tag{2.11}$$

Theorem 2.2 follows at once from the following two lemmas. Notice that the first does not require compactness of the Cauchy surfaces.

Lemma 2.1. Fix any two-dimensional spacetime (B, h_{ab}) with a Cauchy surface, S, and any positive scalar field r that satisfies Eq. (2.5), with G_{ab} satisfying the non-negative-pressures and dominant-energy conditions. Then

$$r \le \max(\sup_S(r), \sup_S(2m)) \ , \tag{2.12}$$

$$2m \ge \min(\inf_S(r), \inf_S(2m)) \ . \tag{2.13}$$

Proof. It suffices, since we can always reverse the roles of past and future, to establish these bounds for any $p \in D^+(S)$.

For the proof of Eq. (2.12), consider any point q, where r reaches its maximum value on the compact set $C = J^-(p) \cap D^+(S)$. If $q \in C \cap S$, then $r(p) \le r(q) \le \sup_S(r)$. If $q \notin C \cap S$, then $D^a r$ must be either past-directed timelike, zero, or past-directed null, at q, for otherwise there would exist a past-directed timelike direction along which r would increase. We now show, in all three cases, that $r(q) \le \sup_S(2m)$.

First, if $D^a r$ is past-directed timelike at q, then, by Eq. (2.3), $r(q) < 2m(q)$. Consider the integral curve τ of $D^a r$

starting from q. Using Eq. (2.5) and the fact that G_{ab} satisfies the non-negative-pressures condition, we have $(D^a r)D_a(-D^b r D_b r)>0$, at q, showing that τ is a past-directed timelike curve without a past end point. So, by global hyperbolicity, τ must intersect S. Again using the non-negative-pressures condition, $(D^a r)D_a(2m)\geq 0$, along τ, so that $2m(q)\leq 2m(\tau\cap S)\leq \sup_S(2m)$.

Second, if $D^a r$ vanishes at q, then so does $D_a(-D^m r D_m r)$. Using Eq. (2.5) we find that, at q, for unit past-directed timelike t^a,

$$t^a t^b D_a D_b(-D^m r D_m r)=\frac{1}{2r^2}+G^{mn}\epsilon_{ma}\epsilon_{nb}t^a t^b + \frac{r^2}{2}(G^{mn}\epsilon_{ma}t^a)(G^{pq}\epsilon_{pb}t^b)h_{nq}\ . \tag{2.14}$$

The first term is manifestly positive; the second term is non-negative by the non-negative-pressures condition; and by the dominant-energy condition, there exist t^a for which the last term in non-negative. (Sketch of proof: If $G_a{}^b$ vanishes, any t^a will do. Otherwise, take t^a to be in the image of the past-directed timelike vectors under $G_a{}^b$. Then, use the fact that $h_{ab}=-2k_{(a}1_{b)}$ where k_a and 1_b are two linearly independent past-directed null vectors.) Consider a past-directed timelike geodesic starting at q, with initial tangent vector t^a such that the last term, in Eq. (2.14), is non-negative. Then, at q, $t^a D_a(t^b D_b(-D^m r D_m r))>0$ and, by the non-negative-pressures condition, $t^a D_a(t^b D_b r)<0$. Hence, $D^a r$ immediately becomes past-directed timelike along the curve. From our analysis of such points, we conclude that $r(q)\leq \sup_S(2m)$.

Third, and last, if $D^a r$ is past-directed null at q, then, by Eq. (2.3), $r(q)=2m(q)$. For $D^a r$ past-directed null, $(D^a r)D_a(2m)\geq 0$, by the non-negative-pressures condition. There are three cases as we move along the integral curve τ of $D^a r$: (i) $D^a r$ becomes past-directed timelike; (ii) we reach a point at which $D^a r$ vanishes; (iii) $D^a r$ remains past-directed null and so, by global hyperbolicity, τ intersects S. In all three cases we conclude that $r(q)\leq \sup_S(2m)$.

Combining these results, Eq. (2.12) is established. The proof of Eq. (2.13) breaks up into three cases.

First, if $D^a r$ is zero, null, or past-directed timelike at p, then, by Eq. (2.3), $2m(p)\geq r(p)$. Consider the past-directed null geodesic λ from p to S with null tangent vector k^a, such that $k^a D_a r\leq 0$, at p. Using Eq. (2.5), by the null-convergence condition, $k^a D_a r\leq 0$ along λ, so we conclude that $r(p)\geq \inf_S(r)$. Hence, $2m(p)\geq \inf_S(r)$.

Second, if $D^a r$ is future-directed timelike at p, consider the integral curve of $(-D^a r)$ starting from p. By the non-negative-pressures condition, $(-D^a r)D_a(2m)\leq 0$ as long as $D^a r$ is timelike or null. There are two possibilities as we move along this curve: (i) $D^a r$ remains future-directed timelike along the curve, in which case it must, by global hyperbolicity, intersect S, for which we conclude that $2m(p)\geq \inf_S(2m)$; (ii) we reach a point q at which $D^a r$ becomes null or zero, for which we conclude from the above that $2m(p)\geq 2m(q)\geq \inf_S(r)$. Hence, in this case

$$2m(p)\geq \min(\inf_S(2m),\inf_S(r))\ .$$

Third, and last, if $D^a r$ is spacelike at p, consider the past-directed null geodesic λ from p to S with tangent vector k^a such that $k^a D_a r<0$, at p. There are two possibilities as we move down thus curve: (i) $D^a r$ remains spacelike so that, by the dominant-energy condition, $k^a D_a(2m)\leq 0$ along λ, allowing us to conclude that $2m(p)\geq 2m(\lambda\cap S)\geq \inf_S(2m)$; (ii) at some point q along λ, $D^a r$ becomes null or zero for which we conclude, from the above, that $2m(p)\geq 2m(q)\geq \inf_S(r)$. Hence, in this case

$$2m(p)\geq \min(\inf_S(2m),\inf_S(r))\ .$$

Combining these results, Eq. (2.13) is also established.□

Lemma 2.2 Fix any two-dimensional spacetime (B,h_{ab}) with a compact Cauchy surface S and any positive scalar field r that satisfies Eq. (2.5), with G_{ab} satisfying the dominant-energy condition. Then

$$\max_S(r)\leq \max_S(2m)\ , \tag{2.15}$$

$$\min_S(r)\leq \min_S(2m)\ . \tag{2.16}$$

Proof. To establish Eq. (2.15), consider a point p where r reaches its maximum value on S. At such a point $D^a r$ is necessarily timelike or zero. Hence,

$$\max_S(r)=r(p)\leq 2m(p)\leq \max_S(2m)\ ,$$

where the first inequality is by Eq. (2.3).

To establish Eq. (2.16), consider the open subset U of S defined by $U=\{s\in S|2m(s)<\min_S(r)\}$. On U, $D^a r$ is necessarily spacelike. Denote, by s^a, the unit vector field on each connected component of U, tangent to the surface S, such that $s^a D_a r>0$. Then, by Eq. (2.6) and the fact that G_{ab} satisfies the dominant-energy condition, we find that, on U, $s^a D_a(2m)\geq 0$. Using this fact, and noting that $2m=\min_S(r)$ on the boundary of U, we conclude that $2m=\min_S(r)$ on U. But, this is possible only if U is empty. Hence, on S, $2m$ is everywhere no less than $\min_S(r)$.□

III. KANTOWSKI-SACHS SPACETIMES

In this section, we investigate the conjecture of Sec. I for the globally hyperbolic spherically symmetric spacetimes that further admit the group $G'\approx SO(2)$ of isometries whose orbits are one-spheres. These are Kantowski-Sachs spacetimes,[19] with orbits of $G\times G'$ being spatially homogeneous Cauchy surfaces with topology $S^1\times S^2$. We prove theorem 1.2 by first showing that, for $\lambda\geq 1/\sqrt{3}$, there exists an upper bound to the lengths of timelike curves in any Kantowski-Sachs spacetime that possesses compact Cauchy surfaces and that satisfies Eq. (1.4). We then present a Kantowski-Sachs spacetime that possesses compact Cauchy surfaces, satisfies Eq. (1.4) for all $\lambda<1/\sqrt{3}$, and admits infinite-length timelike curves.

Theorem 1.2 states that the Kantowski-Sachs spacetimes recollapse under far weaker energy conditions than those needed for the more general spherically symmetric case. In particular, the non-negative-sum-pressures condition ($\lambda=1$) is sufficient to ensure the existence of an upper bound to the lengths of timelike curves in these spacetimes.

Fix such a Kantowski-Sachs spacetime (M,g_{ab}) that satisfies Eq. (1.4) for some $\lambda \geq 1/\sqrt{3}$. Fix, once and for all, a spatially homogeneous Cauchy surface Σ therein. To show that $L(M,g_{ab})$ must be finite, we argue that $L(D^+(\Sigma),g_{ab})$, and hence by a similar argument $L(D^-(\Sigma),g_{ab})$, must be finite. The strategy is to first assume, for contradiction, that $L(D^+(\Sigma),g_{ab})$ is infinite and to then show the existence, in $D^+(\Sigma)$, of a spatially homogeneous Cauchy surface S with negative trace extrinsic curvature. Then, since the timelike-convergence condition holds,[20] it follows[21] that $L(D^+(S),g_{ab})$, and hence $L(D^+(\Sigma),g_{ab})$, must be finite.

So, assume, for contradiction, that $L(D^+(\Sigma),g_{ab})$ is infinite. Introduce the time function t, on $D^+(\Sigma)$, by setting $t(p)=L(D^+(\Sigma)\cap J^-(p),g_{ab})$ for each $p\in D^+(\Sigma)$. By construction, surfaces of constant t are surfaces of homogeneity with t vanishing on Σ and t unbounded from above on $D^+(\Sigma)$. Using our decomposition of the globally hyperbolic spherically symmetric spacetimes in Sec. II, and performing a similar decomposition for the additional symmetry, the metric g_{ab} for our spacetime is given by

$$g_{ab}=-(dt)_a(dt)_b+\frac{e^{2\omega}}{r^4}\theta_a\theta_b+r^2\Omega_{ab} \ , \tag{3.1}$$

where ω and r are functions of t alone, θ_a is a nonvanishing one-form on S^1, and Ω_{ab} is a unit-metric on S^2. The trace of the extrinsic curvature of the surfaces of homogeneity is then given by $\dot\omega$, overdots denoting derivatives with respect to t. The Einstein tensor for this metric is given by

$$G_{ab}=\rho(dt)_a(dt)_b+p_1\frac{e^{2\omega}}{r^4}\theta_a\theta_b+p_2r^2\Omega_{ab} \ , \tag{3.2}$$

where we have set

$$\rho r^2=-3\dot r^{\,2}+2\dot r(r\dot\omega)+1 \ , \tag{3.3}$$

$$p_1r^2=-2r\ddot r-\dot r^{\,2}-1 \ , \tag{3.4}$$

$$p_2r^2=r\ddot r-r^2\ddot\omega-4\dot r^{\,2}+3\dot r(r\dot\omega)-(r\dot\omega)^2 \ . \tag{3.5}$$

From Eq. (1.4), we have $(1-\lambda)\rho+\lambda(p_1+2p_2)\geq 0$. Using Eqs. (3.3)–(3.5) this becomes

$$r\frac{d}{dt}\left[r\frac{d}{dt}\omega\right]+Q+P\leq 0 \ , \tag{3.6}$$

where we have set $P=(1-1/2\lambda)$ and

$$Q=\frac{1}{2\lambda}[(6\lambda+3)\dot r^{\,2}-2(3\lambda+1)\dot r(r\dot\omega)+2\lambda(r\dot\omega)^2] \ . \tag{3.7}$$

For $\lambda\geq 1/\sqrt{3}$, Q is non-negative and P is positive. Setting

$$\chi(t)=\int_0^t\frac{dt'}{r(t')} \ , \tag{3.8}$$

we have, from Eq. (3.6), the inequalities

$$r\frac{d}{dt}\omega\leq c-P\chi \ , \tag{3.9}$$

$$\omega\leq\omega(0)+c\chi-\tfrac{1}{2}P\chi^2 \ , \tag{3.10}$$

where c is a constant. We see, from Eq. (3.10), that ω is bounded from above.

Further, from Eq. (1.4), we have $\rho+p_1\geq 0$. Using Eqs. (3.3) and (3.4) this becomes

$$\frac{d}{dt}\left[e^{-\omega}\frac{d}{dt}r^3\right]\leq 0 \ . \tag{3.11}$$

From this and the fact that ω is bounded from above we find that

$$r^3\leq a+bt \ , \tag{3.12}$$

where a and b are constants. From Eqs. (3.8) and (3.12) we immediately conclude that χ is unbounded from above. Thus, by Eq. (3.9), ω must become negative, establishing the existence of a spatially homogeneous Cauchy surface with negative trace extrinsic curvature. Hence, $L(M,g_{ab})$ must be finite.

To complete the proof of theorem 1.2, consider the Kantowski-Sachs spacetime with $\omega=\frac{1}{2}(3+\sqrt{3})\ln t$ and $r=t\ln t$. It is clear that this spacetime contains infinite-length timelike curves and it is straightforward to show that Eq. (1.4) is satisfied for all $\lambda<1/\sqrt{3}$ and t sufficiently large.

IV. DISCUSSION

Theorem 1.1 resolves the weak version of the closed-universe recollapse conjecture for the spherically symmetric spacetimes with $S^1\times S^2$ Cauchy surfaces under the requirement that the matter satisfy the non-negative-pressures and dominant-energy conditions. Unfortunately, the methods used to prove theorem 1.1 do not work for the spherically symmetric spacetimes with S^3 Cauchy surfaces. Although we still learn that r is everywhere bounded above by the maximum of $2m$ on any Cauchy surface, because r vanishes at the "poles," we learn only that $2m$ is everywhere non-negative. Hence, we are unable to take advantage of theorem 2.1. Thus, it appears that a more subtle argument is necessary for the S^3 case.

We have seen, from theorem 1.2, that those spherically symmetric spacetimes that admit an extra spatial symmetry "recollapse" under the non-negative-sum-pressures condition. Although it seems quite difficult to prove, one might conjecture that all spherically symmetric spacetimes with compact Cauchy surfaces "recollapse" under the non-negative-sum-pressures and dominant-energy conditions. If so, one could include electromagnetic fields as a source meeting the requirements of the theorem. Further, we note that the energy conditions imposed in theorem 1.1 need only hold for those vectors orthogonal to the surfaces of spherical symmetry. This relaxation, although slight, does enlarge the set of source meeting the requirements of the theorem: e.g., this now includes the massless scalar field.

If the weak version of the closed-universe recollapse conjecture is false, then any attempt towards its proof must fail. One line of attack towards the construction of a possible counterexample is the following. Fix any spacetime that possesses a Cauchy surface S of appropriate topology and that satisfies an appropriate energy condition. Fix any subset, $C \subset S$, that is diffeomorphic to the three-ball. If the weak version of the closed-universe recollapse conjecture is true, then the lengths of timelike curves in $D(C)$ must be bounded from above. Hence, we will have a counterexample to the weak version of the closed-universe recollapse conjecture if we can construct an initial data set (C, q_{ab}, K_{ab}) where C is a three-manifold, with boundary, that is diffeomorphic to the three-ball, q_{ab} is a Riemannian metric on C (the induced metric), and K_{ab} is a symmetric tensor on C (the extrinsic curvature), and an evolution such that (i) $D(C)$ contains infinite-length timelike curves; (ii) the evolved spacetime satisfies the appropriate energy condition; and (iii) the initial data set on C can be extended smoothly as an initial data set on a three-manifold of appropriate topology and so that it satisfies the appropriate energy condition (e.g., non-negative energy density.) Unfortunately, completing this task or proving that it must fail seems at least as difficult as proving the weak version of the closed-universe recollapse conjecture itself. However, there is at least one indication that the task must fail: In the two-dimensional case, where C is diffeomorphic to the closed interval, and the energy condition is $R \leq 0$, being essentially the only energy condition in two dimensions, we can prove that the above construction cannot be carried out, i.e., the lengths of timelike curves in $D(C)$ are bounded from above. However, there seems to be no simple way to extend this result to higher dimensions.

Lastly, we note that theorem 1.2, along with the recent result that the Bianchi IX spacetimes "recollapse" under the non-negative-sum-pressures and dominant-energy conditions,[22] establishes that the weak form of the closed-universe recollapse conjecture holds, under the stated energy conditions, for all spatially homogeneous spacetimes with $S^1 \times S^2$ or S^3 Cauchy surfaces.

ACKNOWLEDGMENTS

I wish to thank Robert Geroch and Robert Wald for the helpful remarks and encouragement they gave me in the course of this work, which was presented as a thesis to the Department of Physics, The University of Chicago, in partial fulfillment of the requirements for the Ph.D. degree. This research was supported by National Science Foundation Grant No. PHY-8918388 to the University of Chicago.

APPENDIX: CAUCHY SURFACE TOPOLOGIES

In this appendix we show that the Cauchy surfaces for any globally hyperbolic spherically symmetric spacetime must have one of four possible topologies: $\mathbb{R}^3$, S^3, $\mathbb{R} \times S^2$, or $S^1 \times S^2$. Further, we show that r, as defined in Sec. II, is strictly positive in and only in the latter two cases.

Fix any globally hyperbolic spherically symmetric spacetime and consider the pullback q_{ab} of the metric onto a spherically symmetric Cauchy surface Σ (of which the theorem below guarantees existence.) The pair (Σ, q_{ab}) is then a three-dimensional, orientable, connected, spherically symmetric Riemannian space. Set $\Sigma' = \{p \in M \mid r(p) > 0\}$. Then, each connected component of Σ'/G is a one-dimensional manifold and hence is diffeomorphic to either $\mathbb{R}$ or S^1. From this, and the fact that all orientable S^2 bundles over $\mathbb{R}$ or S^1 are trivial, it follows that each connected component of Σ' is diffeomorphic to either $\mathbb{R} \times S^2$ or $S^1 \times S^2$. If $\Sigma = \Sigma'$, then it follows that Σ must be diffeomorphic to either $\mathbb{R} \times S^2$ or $S^1 \times S^2$. If $\Sigma \neq \Sigma'$, then it follows that Σ must be diffeomorphic to either $\mathbb{R}^3$ or S^3 (in the first case there being a single point in Σ where r vanishes and in the second case two such points).

Theorem. Any globally hyperbolic spacetime with an orthochronos compact isometry group G can be foliated by Cauchy surfaces each of which is invariant under G.

Proof. Fix any globally hyperbolic spacetime (M, g_{ab}) with an orthochronos compact isometry group G. By global hyperbolicity, there exists a function $f: M \to \mathbb{R}$ such that $(df)_a$ is everywhere past-directed timelike and each surface of constant f is a Cauchy surface.[23] Set

$$t = \int_G f \circ \phi \ , \tag{A1}$$

where the integral is over G (using that invariant volume element that gives G unit volume) and ϕ denotes a typical element of G. By construction, t is invariant under G. Further, using the fact that G is orthochronos, it is not difficult to show that $(dt)_a$ is everywhere past-directed timelike. Hence, to establish the theorem, we need only show that surfaces of constant t are Cauchy surfaces.

For any $p \in M$, consider the surface Σ defined by $t = t(p)$. To show that Σ is indeed a Cauchy surface we need to show that all inextendible-directed (future or past) causal curves intersect Σ. Fix any such curve $\gamma: \mathbb{R} \to M$ and consider the map $\mu: G \to \mathbb{R}$ defined, for each $\phi \in G$, by setting $\mu(\phi)$ to be the solution to

$$f(\phi(\gamma(\mu(\phi)))) = t(p) \ . \tag{A2}$$

That is, $\mu(\phi)$ is that parameter of the curve $\phi \circ \gamma$ where $\phi \circ \gamma$ intersects the Cauchy surface defined by $f = t(p)$. Then, by construction,

$$f(\phi(\gamma(\min_G \mu))) \leq t(p) \leq f(\phi(\gamma(\max_G \mu))) \tag{A3}$$

for all $\phi \in G$. Integrating this relation over G we obtain the inequalities

$$t(\gamma(\min_G \mu)) \leq t(p) \leq t(\gamma(\max_G \mu)) \ . \tag{A4}$$

So, by continuity, there exists λ so that $t(\gamma(\lambda)) = t(p)$, showing that γ must intersect Σ.□

*Current address: Department of Physics, University of Utah, Salt Lake City, Utah 84112.

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[5]Given two three-manifolds Σ_1 and Σ_2, their connected sum $\Sigma_1 \# \Sigma_2$ is the three-manifold constructed by removing a three-ball from Σ_1 and Σ_2 and identifying the two manifolds along the resulting boundaries. See, e.g., J. Hempel, *3-Manifolds* (Princeton University Press, Princeton, 1976).

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[13]The three-sphere with antipodal points identified is not allowed since there will be one orbit that is diffeomorphic to the two-sphere with antipodal points identified. Of course, if the definition of spherical symmetry were relaxed to include such orbits, the three-sphere with antipodal points identified would be allowed.

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[18]This is essentially the problem of finding the longest time a test particle, moving radially in an attractive central field everywhere no weaker than some attractive inverse-square law, can remain between the center and a fixed outer radius.

[19]The Kantowski-Sachs spacetimes are those that possess these symmetries locally (i.e., admits local Killing vector fields that obey the same Lie algebra). For a nice overview of many of the basics of the Kantowski-Sachs spacetimes see C. B. Collins, J. Math. Phys. **18**, 2116 (1977).

[20]It is not difficult to show that if the energy condition given by Eq. (1.4) holds for some λ_0, then it holds for all λ satisfying $\frac{1}{4} \le \lambda \le \lambda_0$. So, since Eq. (1.4) holds for some $\lambda_0 \ge 1/\sqrt{3}$, it holds for $\lambda = \frac{1}{2}$; i.e., the timelike-convergence condition is satisfied.

[21]See theorem 9.5.1 of Ref. 9.

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[23]See theorem 8.3.14 of Ref. 9.