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Thin shells in general relativity and cosmology: The lightlike limit

C. Barrabès
Physics Department, UFR Sciences, Université de Tours, 37200 France
and Département d'Astrophysique Relativiste et Cosmologie, Observatoire de Paris, 92190 Meudon, France

W. Israel
Canadian Institute for Advanced Research Cosmology Program, Theoretical Physics Institute, University of Alberta, Edmonton, Canada T6G 2J1

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This paper shows how the structure and dynamics of a thin shell traveling at the speed of light can be obtained from a simple and convenient prescription that is a straightforward extension and continuous limit of the familiar extrinsic-curvature algorithm for subluminal shells. It allows the space-time coordinates to be chosen freely and independently on the two sides of the shell. The prescription is applied to several examples of interest in general relativity and cosmology.

I. INTRODUCTION

The impact of unified gauge theories on cosmology has assigned a key role to phase transitions in the early Universe, for example, in scenarios such as inflation.[1] When two phases coexist, the wall separating them can, as a first approximation, be treated as an infinitely thin bubble or shell[2] whose history is a timelike surface layer. Analogously, in a sudden global phase transition, the transition region can sometimes be idealized as an infinitely thin spacelike surface layer.[3,4]

As a result, the dynamics of bubbles and surface layers in general relativity has been studied extensively, and there are a number of excellent current reviews.[5] The formalism now commonly in use expresses the surface properties in terms of the jump of extrinsic curvature across the shell wall. Its distinctive feature is that the properties are obtained directly as functions of the shell's intrinsic coordinates. Thus the four-dimensional coordinates may be chosen freely and independently on the two sides of the layer. Since the geometry of each of the two adjoining phases is described most naturally and simply in terms of coordinates adapted to its own peculiar symmetries, the practical advantages of such a "coordinate-friendly" formulation are obvious.

However, in its present form this approach has a serious limitation. Its formal scaffolding—intrinsic metric and extrinsic curvature—folds when the surface layer becomes lightlike. The intrinsic metric becomes degenerate, and an extrinsic curvature is no longer definable uniquely, because the normal vector is now tangent to the surface and a distinguished transverse vector no longer exists.

Because of this breakdown, the lightlike case is a relatively neglected area of shell dynamics which remains imperfectly understood. This is regrettable. Far from being an uninteresting oddity, lightlike shells have a structure and dynamics in many respects much simpler than timelike shells. They thus allow the quickest reconnaissance of the kinematical possibilities inherent in the coexistence of two or more cosmological phases. The lightlike idealization is even quantitatively a fair approximation to the behavior of, e.g., sufficiently large bubbles in a sea of false vacuum, which will be rapidly accelerated toward the speed of light by the imbalance of normal pressures.

Existing studies of lightlike surface layers[3,6–11] are generally not well adapted for straightforward cosmological application. In some (Dautcourt,[6] Taub,[8] Redmount,[9] Clarke and Dray[10]) a direct application of the formulas would require preconstruction of space-time coordinates that match continuously at the shell and in which the four-metric is continuous. The elegant geometrical analysis of Penrose[7] is cast in the relatively unfamiliar language of two-spinors (Redmount provides an alternative spin-coefficient formulation). The invariant prescription given by Berezin, Kuzmin, and Tkachev[3] is useful, but confined to spherical symmetry.

Our aim in this paper is to fill what we see as an annoying gap in the literature by providing a simple recipe for obtaining the surface properties of lightlike shells that retains the coordinate-friendly advantages of the familiar timelike case and may be considered a continuous limit of it. The recipe is presented with minimal derivation in Sec. II—details can be found in the Appendix. Section III discusses some features that are peculiar to lightlike shells. In Sec. IV we apply the general recipe to arbitrary spherical shells and to specific examples of cosmological interest. Section V considers a stationary lightlike shell straddling a horizon common to two adjoining phases, a case whose subtleties have led to some confusion in the literature. A generalized form of the remarkable Dray–'t Hooft–Redmount relation, which connects the gravitational masses in the region between any pair of colliding lightlike shells before and after the collision, is the subject of Sec. VI.

A situation peculiar to the lightlike case arises when the shell's history is a future causal boundary of the domain to one side of it. This happens, for instance, if the shell is plane, cylindrical, or spherical, though not necessarily uniform. In such a case the domain on the past side is "not yet aware of," and, hence, unaffected by, the gravitational field of the shell. Its geometry can therefore be chosen arbitrarily, in particular as flat if it is empty. In Sec. VI we examine, following Penrose,[12] Hawking,[13] and Gibbons,[14] some interesting examples of this type.

Conventions. Our metric signature is $(-+++)$, and we follow the now standard curvature conventions of Misner, Thorne, and Wheeler (MTW).[15] However, our sign convention for extrinsic curvature—Eq. (3) below—is the opposite of MTW and Berezin, Kuzmin, aid Tkachev,[3] though in agreement with the other reviews we have cited. It makes the mean extrinsic curvature of a convex closed two-surface positive if the normal is directed outwards.

Greek indices run from 0 to 3, lower-case latin $a,b,\ldots$ from 1 to 3, and upper-case latin $A,B,\ldots=2,3$. (In Sec. VI we depart from this convention: $a,b,\ldots$ take the values 2,3.) The symbols ∇_μ and $\delta/\delta\lambda$ denote covariant and absolute derivatives with respect to the four-dimensional space-time geometry. A semicolon indicates covariant differentiation with respect to a three-metric g_{ab}, when this is nondegenerate. We use square brackets $[\Psi]$ and an overlying tilde $\tilde{\Psi}$, respectively, to denote the jump and arithmetical mean of any quantity Ψ that is discontinuous at a hypersurface Σ.

II. TIMELIKE, SPACELIKE, AND LIGHTLIKE SHELLS: A GENERAL ALGORITHM

In the form in which the problem is often presented to us, the geometry on one or both sides of the shell is given *a priori*, each expressed in terms of coordinates naturally adapted to its symmetries. The task is then to infer the surface dynamics from differences in the way the surface layer is embedded in the two four-geometries.

It is therefore useful to view the general problem in terms of a "cut and paste" approach. We are given two distinct space-time manifolds $\mathcal{M}_+$ and $\mathcal{M}_-$ with metrics $g^+_{\alpha\beta}(x^\mu_+)$ and $g^-_{\alpha\beta}(x^\mu_-)$ in terms of independently defined coordinate systems x^μ_+ and x^μ_-. They are bounded by hypersurfaces Σ_+ and Σ_-, respectively, with induced metrics g^+_{ab} and g^-_{ab} $(a,b,\ldots=1,2,3)$. These hypersurfaces are given to be isometric, i.e., $g^+_{ab}(\xi)=g^-_{ab}(\xi)\equiv g_{ab}(\xi)$ in terms of three-dimensional intrinsic coordinates ξ^a invariant under the isometry. We then glue together $\mathcal{M}_+$ and $\mathcal{M}_-$ at their boundaries to form a single manifold $\mathcal{M}=\mathcal{M}_+\cup\mathcal{M}_-$ by making the natural identification $\Sigma_+=\Sigma_-=\Sigma$.

Parametric equations for the two adjoining imbeddings of Σ will have different functional forms, $x^\alpha_\pm=f^\alpha_\pm(\xi)$. The three holonomic basis vectors $e_{(a)}=\partial/\partial\xi^a$ tangent to Σ have components $e^\alpha_{(a)}|_\pm=\partial x^\alpha_\pm/\partial\xi^a$ with respect to the two four-dimensional coordinate systems. Their scalar products define the metric induced on Σ:

$$g_{ab}=e_{(a)}\cdot e_{(b)}\equiv g_{\alpha\beta}e^\alpha_{(a)}\cdot e^\beta_{(b)}|_\pm \ , \tag{1}$$

the same on both faces Σ_+ and Σ_-. A normal n to Σ, normalized to have constant (but not necessarily unit) length, will have components $n^\alpha_\pm$, satisfying

$$n\cdot n|_+=n\cdot n|_-\equiv\epsilon, \quad n\cdot e_{(a)}|_\pm=0 \ , \tag{2}$$

where ϵ is constant over Σ. We suppose n directed from $\mathcal{M}^-$ to $\mathcal{M}^+$.

To set the stage, let us briefly recall the "normal" extrinsic-curvature prescription for determining the energy and stresses in a layer that is nonlightlike. In this case the normal n is transverse to Σ, and $\varepsilon\neq0$.

In the geometrical sense that both induce the same three-metric (1) on Σ, the join at Σ of the two four-metrics $g^\pm_{\alpha\beta}$ may be said to be "continuous." However, the normal extrinsic curvatures $K^\pm_{ab}$ of the two imbeddings, each defined by an equation of the form

$$K_{ab}=-n\cdot\delta e_{(a)}/\delta\xi^b\equiv-n_\alpha e^\beta_{(b)}\nabla_\beta e^\alpha_{(a)} \ , \tag{3}$$

contain $\Gamma^\mu_{\alpha\beta}|_\pm$, respectively, i.e., first derivatives of $g^\pm_{\alpha\beta}$ transverse to Σ. Thus, in general, $K^+_{ab}\neq K^-_{ab}$. (In Newtonian theory a surface layer is similarly characterized by a discontinuity in the transverse gradient of potential.)

The normal prescription links the surface stress-energy tensor S_{ab} of the layer to the jump $[K_{ab}]=K^+_{ab}-K^-_{ab}$ of normal extrinsic curvature across Σ by a distributional equivalent of Einstein's field equations:

$$-8\pi(S_{ab}-\tfrac{1}{2}g_{ab}S)=[K_{ab}] \ . \tag{4}$$

An inverse three-metric g^{ab} and the trace $S=g^{ab}S_{ab}$ are well defined here, because g_{ab} is nondegenerate for a nonlightlike surface. [Because of our nonstandard normalization (2), the conventionally defined extrinsic curvature and surface energy tensor are actually $|\varepsilon|^{-1/2}K_{ab}$ and $\varepsilon|\varepsilon|^{-1/2}S_{ab}$.]

The conservation law satisfied by S^{ab} is

$$S^b_{a;b}=-[e^\alpha_{(a)}T^\beta_\alpha n_\beta] \ , \tag{5}$$

in which the semicolon denotes covariant differentiation with respect to g_{ab}. This is an immediate consequence of (4) and the Arnowitt-Deser-Misner (ADM) constraint[15]

$$G_{\alpha\beta}e^\alpha_{(a)}n^\beta=K^b_{a;b}-\partial_aK \ , \tag{6}$$

applied to both faces of Σ. The "Hamiltonian constraint"

$$G_{\alpha\beta}n^\alpha n^\beta=\tfrac{1}{2}(K^2-K_{ab}K^{ab}-\varepsilon\,{}^{(3)}R) \tag{7}$$

yields an additional dynamical equation

$$\tilde{K}_{ab}S^{ab}=[T_{\alpha\beta}n^\alpha n^\beta] \ , \tag{8}$$

where the tilde denotes the average of a discontinuous quantity:

$$\tilde{K}_{ab}=\tfrac{1}{2}(K^+_{ab}+K^-_{ab}) \ .$$

Equations (5) and (8) are identities implied by the "field-equations" (4). They have a transparent physical interpretation. In the rest frame of a timelike shell, the time component of (5) equates the net influx of energy from the surroundings to the increase of internal energy plus the work done by the surface stresses in expanding and deforming the shell. Equation (8) is Newton's second law: it equates the net inward normal pressure on the shell to the outward force ($\sim {}^{(2)}KP$, where ${}^{(2)}K$ is extrinsic curvature of the two-surface) due to the surface pressure P, plus an inertial, (mass)$\times$(inward normal acceleration), term. For a (spacelike) transition layer, (5) (read from right to left) determines the change of momentum density in a continuous medium resulting from the momentary appearance of an impulsive stress gradient; (8) equates the change of energy density to minus the work done by the impulsive stress.

Expressed as four-dimensional distribution, the stress energy associated with Σ is given by

$$T^{\alpha\beta}_{\Sigma}=S^{ab}e^{\alpha}_{(a)}e^{\beta}_{(b)}|\alpha|(\mathrm{sgn}\varepsilon)\delta(\Phi) , \tag{9}$$

where $\Phi(x^{\mu})=0$ is the equation of Σ, so that we can set $n_{\mu}=\alpha^{-1}(\xi^{a})\partial_{\mu}\Phi$. Of course, (9) appears in two different forms $T^{\alpha\beta}_{\Sigma}|_{\pm}$, depending on which of the two four-dimensional coordinate systems happens to be in use. The intrinsic formulation (4), (5), and (8) has the advantage that it is independent of this choice.

How must these results be modified in order to extend them to the lightlike case? The answer to this question is remarkable. All of the above equations [with the three-dimensional covariant derivatives in (5) and (6) suitably interpreted] are valid without change for lightlike shells ($\varepsilon=0$). The only snag is that (4), which are the key equations linking the physics and geometry, now assume the unhelpful form $0=0$.

The normal prescription breaks down for lightlike surface layers because the normal extrinsic curvature K_{ab} is disabled as a carrier of transverse geometrical information. As the normal n declines into tangency with Σ, K_{ab} manages to retain only tangential derivatives of the metric, and is now "extrinsic" in name only. (The degeneration of g_{ab} to a matrix of rank 2 is an additional and related—but incidental—complication.)

Thus as extended and unified algorithm, able to encompass the lightlike case, needs recourse to a transverse object decoupled from the normal. Accordingly, returning to (2) and the general case where Σ is arbitrary, we introduce a transversal or "cross vector" N over Σ in addition to (and independently of) the normal n. To ensure "continuity of the transversal," i.e., to be sure that the components N^{α}_{+}, N^{α}_{-} defined on the two faces do in fact represent the "same" vector N, we must require equality of their projections $N_a=N\cdot e_{(a)}$ onto Σ:

$$[N_a]=N_{\alpha}e^{\alpha}_{(a)}|_{+}-N_{\alpha}e^{\alpha}_{(a)}|_{-}=0 , \tag{10}$$

and that they assign the same length to N: $[N\cdot N]=0$.

We denote two frequently occurring scalar products by

$$\eta^{-1}=N\cdot n\neq 0, \quad n\cdot n=\varepsilon . \tag{11}$$

It is convenient [and necessary for the validity of some of our equations, e.g., (5) and (6)] to assume $\varepsilon=$const over Σ. However, the function $\eta(\xi^{a})$ can be specified freely over Σ. For prescribed η, N is still free to the extent of a tangential displacement

$$N\rightarrow N'=N+\lambda^{a}(\xi)e_{(a)} , \tag{12}$$

with arbitrary functions $\lambda^{a}(\xi^{b})$.

In practice, it is often convenient to specify η to be constant and equal to $+1$ for timelike and -1 for non-timelike Σ. In our metric signature $(-+++)$, this amounts to requiring N and n to be directed toward the same side of any hypersurface, e.g., both future directed if Σ is spacelike or lightlike. As for ε, while the discrete values $\pm 1,0$ suffice for any example, retention of a continuous range has an advantage in general arguments: A lightlike hypersurface can then be regarded as a continuous limit of a generic Σ. As Σ leans toward the lightlike, the boost or "tilt" parameter $\varepsilon=n\cdot n$ approaches zero, while the components n_{μ} remain bounded away from zero in a regular coordinate frame x^{μ}.

We now introduce a slight generalization of the concept of extrinsic curvature by defining[11]

$$\mathcal{K}_{ab}=-N_{\mu}\delta e^{\mu}_{(a)}/\delta\xi^{b}=\mathcal{K}_{ba} . \tag{13}$$

$\mathcal{K}_{ab}$ might be called "transverse" or "oblique" extrinsic curvature, to distinguish it from the normal extrinsic curvature K_{ab}.

It is evident that $\mathcal{K}_{ab}$ is not independent of the choice of transversal. Under (12) it transforms as

$$\mathcal{K}_{ab}\rightarrow\mathcal{K}'_{ab}=\mathcal{K}_{ab}-\lambda^{c}\Gamma_{c,ab} . \tag{14}$$

The (three-dimensionally) noncovariant expression (14) also shows that, under a change of intrinsic coordinates ξ^{a}, $\mathcal{K}_{ab}$ would not even transform as a three-tensor. However, the Christoffel symbols $\Gamma_{c,ab}$ contain only tangential derivatives of g_{ab}, which must be continuous across Σ. Hence the *jump* (traversing Σ in the positive sense of N)

$$\tfrac{1}{2}\gamma_{ab}\equiv[\mathcal{K}_{ab}] \tag{15}$$

is both a three-tensor and independent of the choice of transversal direction.

By analogy with the normal field equations (4) for a nonlightlike Σ, we now seek a generalized prescription that will relate the surface stress-energy tensor to γ_{ab} in the case where Σ is arbitrary.

Deferring all proofs, we begin by succinctly stating this prescription in its four-dimensional form:

(i) Select one of the two four-dimensional coordinate systems x^{μ}_{+}, x^{μ}_{-}, and, for simplicity, drop the positive or negative affix.

(ii) Having obtained γ_{ab} from (13) and (15), construct *any* four-tensor $\gamma_{\mu\nu}$ which has γ_{ab} as its projection onto Σ:

$$\gamma_{\mu\nu}e^{\mu}_{(a)}e^{\nu}_{(b)}=\gamma_{ab} . \tag{16}$$

(iii) The surface stress-energy tensor $\eta^{-1}S^{\mu\nu}$ is then given by

$$16\pi\eta^{-1}S^{\mu\nu}=2\gamma^{(\mu}n^{\nu)}-\gamma n^{\mu}n^{\nu}-\gamma^{\dagger}g^{\mu\nu}-\varepsilon(\gamma^{\mu\nu}-\gamma g^{\mu\nu}) , \tag{17}$$

where

$$\gamma^\mu=\gamma^{\mu\nu}n_\nu,\quad \gamma^\dagger=\gamma^\mu n_\mu,\quad \gamma=\gamma_{\mu\nu}g^{\mu\nu}\ . \tag{18}$$

Considered as a distribution, the stress-energy tensor is

$$T^{\mu\nu}_\Sigma=\alpha S^{\mu\nu}\delta(\Phi)\ , \tag{19}$$

where

$$\alpha^{-1}\partial_\mu\Phi=n_\mu\ ,$$

and $\Phi(x^\mu)=0$ is the equation of Σ. This gives $T^{\mu\nu}$ with the correct sign if we adopt the conventions that Φ increases from $\mathcal{M}_-$ to $\mathcal{M}_+$, and that α is positive for timelike, negative for nontimelike, Σ. The surface fluxes $S^{\mu\nu}$ are tangential to Σ as they should be:

$$S^{\mu\nu}n_\nu=0\ . \tag{20}$$

Some explanatory remarks concerning this prescription are in order.

(a) The condition (16) leaves $\gamma_{\mu\nu}$ undetermined to the extent of the transformation

$$\gamma_{\mu\nu}\to\gamma'_{\mu\nu}=\gamma_{\mu\nu}+2\lambda_{(\mu}n_{\nu)}\ , \tag{21}$$

where λ_μ is an arbitrary four-vector field over Σ. However, $S^{\mu\nu}$ is invariant under this transformation.

It is easy to understand this as a coordinate gauge effect. In four-dimensional coordinates which are continuous at Σ, a solution of (16), (15), and (13) is given by the jump in the Lie derivative of the metric along N:

$$\gamma_{\mu\nu}=[\mathcal{L}_N g_{\mu\nu}]=2[\nabla_{(\mu}N_{\nu)}]\quad\text{if } [g_{\mu\nu}]=0\ ,$$

so that (21) can be interpreted in terms of freedom in the choice of transversal off Σ. In coordinates convected along the direction of $N^\mu\partial_\mu=(\alpha/\eta)\partial/\partial\Phi$, we have $\gamma_{\mu\nu}=(\alpha/\eta)[\partial g_{\mu\nu}/\partial\Phi]$, so that (21) corresponds to an arbitrariness in $\partial g^{\pm}_{\mu\phi}/\partial\phi$, i.e., to arbitrariness in the gradients of the ADM lapse and shift functions.

(b) Under (21) the quantities defined in (18) transform as

$$\gamma^{\dagger'}=\gamma^\dagger+2\varepsilon\lambda^\dagger,\quad \gamma^{\mu'}=\gamma^\mu+\varepsilon\lambda^\mu+\lambda^\dagger n^\mu\ ,$$

while $\gamma^\dagger-\varepsilon\gamma$ stays invariant, where $\lambda^\dagger\equiv\lambda^\mu n_\mu$. It follows that γ^μ can be transformed to zero by a gauge transformation (21) for any nonlightlike $\Sigma(\varepsilon\neq 0)$; for lightlike Σ, it contains physically significant information.

(c) The preceding remark is the basis for a simple *proof* of the prescription (17) for an arbitrary surface layer. If we gauge γ^μ to zero for a nonlightlike Σ, (17) projected onto Σ reduces to

$$16\pi S^{ab}=-\eta\varepsilon(\gamma^{ab}-\gamma g^{ab})\ , \tag{22}$$

which is equivalent to the normal prescription (4). Thus (17) is a gauge-invariant expression which reduces to a manifestly correct form (in a particular gauge) for any $\varepsilon\neq 0$. Moreover, it remains well defined when $\varepsilon\to 0$. Therefore, it must be generally correct.

(d) A minor inconvenience of this simple unified treatment is a risk of sign confusion. As (19) makes clear, it is $\alpha\ S^{\mu\nu}$ (rather than $S^{\mu\nu}$) which represents correctly the *sign* of surface energy and pressure, and one needs to keep this in mind for spacelike and lightlike layers.

We pass now to the intrinsic form of the general prescription (17). By virtue of (20), $S^{\mu\nu}$ can be decomposed (uniquely) in terms of the basis $\{e_{(a)}\}$:

$$S^{\mu\nu}=S^{ab}e^\mu_{(a)}e^\nu_{(b)}\ . \tag{23}$$

The intrinsic tensor S^{ab} satisfies the conservation law

$$N_a(\partial_b+\tilde{\Gamma}_b)S^{ab}-S_{ab}\mathcal{K}_{ab}=-[N_\lambda T^{\lambda\mu}n_\mu]\ , \tag{24}$$

where the tilde denotes, as usual, the arithmetical mean at Σ, and

$$\tilde{\Gamma}_b=\tfrac{1}{2}(\Gamma^+_b+\Gamma^-_b),\quad \Gamma_b=\alpha^{-1}\nabla_\mu(\alpha e^\mu_{(b)})\ . \tag{25}$$

For nonlightlike Σ this reduces to the familiar, intrinsically defined expression

$$\tilde{\Gamma}_b=\Gamma^\pm_b=\Gamma^c_{bc}=\partial_b\ln|^3g|^{1/2}\quad(\varepsilon\neq 0)\ . \tag{26}$$

But the more robust definition (25) is needed to anchor the meaning of $\tilde{\Gamma}_b$ in the lightlike limit when the intrinsic metric becomes a degenerate. It retains the essential convenience that the + and − contributions to $\tilde{\Gamma}_b$ can be evaluated separately in the independent charts x^μ_+ and x^μ_-. It is straightforward to verify that Γ_b is invariant under arbitrary rescalings $\Phi(x)\to\mu(x)\Phi(x)$ in (19), if one takes account of the tilt this induces off Σ in the base vectors $e^\mu_{(b)}=(\partial x^\mu/\partial\xi^b)_\Phi$.

Both (5) and (8) may be considered special cases of (24); in fact, (5) is precisely the condition that (24) remain valid under an arbitrary change (12) of transversal N^μ. It is now clear that the three-dimensional covariant derivative in (5) is to be defined for arbitrary Σ as

$$S^b_{a;b}\equiv(\partial_b+\tilde{\Gamma}_b)S^b_a-\tfrac{1}{2}S^{bc}\partial_a g_{bc}\ . \tag{27}$$

According to (17) and (23), $S^{\mu\nu}$, and hence S^{ab}, is fully determined by $\gamma_{\mu\nu}$. Further, the gauge invariance (21) shows that no part of $\gamma_{\mu\nu}$ that is not determined by γ_{ab} contributes to $S^{\mu\nu}$. Hence S^{ab} must be determined uniquely by γ_{ab}. For lightlike Σ, this is not quite a trivial statement: To express S^{ab} in terms of γ_{ab}, machinery is needed for "raising" the indices a,b.

To this end, we decompose the normal n^μ with respect to the oblique basis $\{N^\mu,e^\mu_{(a)}\}$:

$$n=\varepsilon\eta N+l^a e_{(a)}\ . \tag{28}$$

Since $n\cdot e_{(a)}=0$, the three-vector l^a satisfies

$$g_{ab}l^b=-\varepsilon\eta N_a\ . \tag{29}$$

Then a symmetric matrix g^{ab}_* exists such that

$$g^{ac}_*g_{bc}=\delta^a_b-\eta l^a N_b\ . \tag{30}$$

For a nonlightlike Σ, g^{ab}_* would be just the usual contravariant three-metric g^{ab} if N^μ were chosen normal to Σ. In the lightlike case, $g_{ab}l^b=0$, and (30) determines g^{ab}_* only up to the transformation $g^{ab}_*\to g^{ab}_*+2\lambda l^a l^b$, with λ arbitrary. In particular, g^{ab}_* could be chosen as the contravariant two-metric g^{AB} (bordered by zeros) in convected coordinates ($l^a=\delta^a_1$), with the choice $N\cdot e_{(A)}=0$.

The intrinsic stress-energy tensor for an arbitrary layer can now be expressed quite generally in the form

$$16\pi\eta^{-1}S^{ab}=[g_*^{ac}l^b l^d+l^a l^c g_*^{bd}-g_*^{ab}l^c l^d-l^a l^b g_*^{cd}-\varepsilon(g_*^{ac}g_*^{bd}-g_*^{ab}g_*^{cd})]\gamma_{cd}\ . \tag{31}$$

Equations (31) and (17) can be derived straightforwardly from the expression [cf. (22)]

$$16\pi S^{\mu\nu}=-\eta\varepsilon(\Delta^{\mu\alpha}\Delta^{\nu\beta}-\Delta^{\mu\nu}\Delta^{\alpha\beta})\gamma_{\alpha\beta}\ . \tag{32}$$

Here

$$\Delta^{\mu\nu}=g^{\mu\nu}-\varepsilon^{-1}n^\mu n^\nu \tag{33}$$

is the four-tensor that projects onto Σ. It can also be expressed as

$$\Delta^{\mu\nu}=(g_*^{ab}-\varepsilon^{-1}l^a l^b)e^\mu_{(a)}e^\nu_{(b)}\ , \tag{34}$$

by virtue of the completeness relation

$$g^{\mu\nu}=g_*^{ab}e^\mu_{(a)}e^\nu_{(b)}+2\eta l^a e^{(\mu}_{(a)}N^{\nu)}+\eta^2\varepsilon N^\mu N^\nu\ , \tag{35}$$

for the basis $\{N,e_{(a)}\}$.

III. PROPERTIES OF LIGHTLIKE SHELLS

The formulas of Sec. II were valid for arbitrary shells. This section treats some features specific to the lightlike case.

When two space-times with lightlike boundaries Σ_+ and Σ_- are slotted together along corresponding lightlike generators, the only geometrical properties that must match are the degenerate (i.e., effectively two-dimensional) three-surface metrics at corresponding points. The one-metric along the generators, being zero, cannot impose any sort of rigid constraint on the slotting. A residual freedom to slide along the generators may therefore persist, at least to the extent permitted by matching of the two-metrics. [In particular, if $\Sigma_\pm$ are stationary (cf. Sec. V), the generators are orbits of an intrinsic isometry and can be slotted together in a lengthwise-arbitrary fashion.]

One might attempt to remove or reduce such arbitrariness by requiring any soldering of Σ_+ and Σ_- to be "affinely conciliable" in the following sense. The affine parameters λ_+,λ_- along null generators of Σ_+,Σ_- are each arbitrary up to linear transformations with coefficients constant along the rays. A soldering is affinely conciliable if λ_+,λ_- can be made equal at corresponding points by such a linear transformation.

There are important special cases in which affine conciliability is both possible and effective in reducing arbitrariness. In general, however, affine conciliability is incompatible with the overriding requirement of isometry. This can be seen at once from Raychaudhuri's formula[16] for any lightlike three-space:

$$2A^{-1/2}\frac{d^2(A^{1/2})}{d\lambda^2}+\sigma_{\alpha\beta}\sigma^{\alpha\beta}=\kappa\Theta-8\pi T_{\mu\nu}n^\mu n^\nu\ . \tag{36}$$

Here A is an element of two-area convected (Lie transported) along the generators with tangent

$$n^\mu=l^a e^\mu_{(a)}=\frac{dx^\mu}{d\lambda}\quad(\varepsilon=0) \tag{37}$$

(λ not necessarily affine). $\sigma_{\alpha\beta}$ is the shear of the generators, $\Theta=d\ln A/d\lambda$ the dilation rate, and $T_{\mu\nu}$ the stress energy of the ambient medium. The "acceleration" κ is defined by

$$n^\alpha{}_{|\beta}n^\beta=\kappa n^\alpha\ . \tag{38}$$

Now the left-hand side of (36) depends only on the intrinsic metric g_{ab} and must be continuous. It follows that

$$[\kappa]\Theta=8\pi[T_{\mu\nu}n^\mu n^\nu]\quad(\varepsilon=0)\ . \tag{39}$$

This is equivalent to (8) by virtue of

$$[\kappa]=-\tfrac{1}{2}\eta\gamma^\dagger,\quad \tilde{K}_{ab}=K^\pm_{ab}=\tfrac{1}{2}\mathcal{L}_l g_{ab}\quad(\varepsilon=0)\ , \tag{40}$$

in which $\mathcal{L}$ denotes the Lie derivative. According to (40) and (17) $-[\kappa]/8\pi$ is the isotropic surface pressure in a lightlike shell, and (39) expresses the condition that all of the energy absorbed by the shell from its surroundings goes into work done by the surface pressure in dilating the shell. This is to be expected for a lightlike shell, since the "proper" mass of the material must remain fixed at zero.

Thus a lightlike shell is affinely conciliable (i.e., λ can be chosen so that $\kappa_+=\kappa_-=0$) if and only if it is pressureless: $\gamma^\dagger=\gamma_{ab}l^a l^b=0$. Equation (39) shows that this is possible only if $[T_{\mu\nu}n^\mu n^\nu]=0$, i.e., only if there is no net exchange of energy with the surroundings. For a nonstationary shell ($\Theta\neq0$), this condition is also sufficient.

It is, incidentally, useful to note from (36) that if $T_{\mu\nu}n^\mu n^\nu=0$ on one side of a nonstationary spherical or cylindrical (hence shear-free) shell, then $A^{1/2}$, i.e., Schwarzschild's coordinate r, or the square root of the circumferential cylindrical coordinate, is an affine parameters for that side.

Propagation of a surface layer is generally accompanied by a "tidal wave," a distributionlike swell of the Weyl conformal curvature, having the form[17] (see the Appendix)

$$C^{\kappa\lambda}{}_{\mu\nu}=\{2\eta n^{[\kappa}\gamma^{\lambda]}_{[\mu}n_{\nu]}-16\pi\delta^{[\kappa}_{[\mu}S^{\lambda]}_{\nu]}+\tfrac{8}{3}\pi S^\alpha_\alpha\delta^{\kappa\lambda}_{\mu\nu}\}\alpha\delta(\Phi)\ , \tag{41}$$

to be evaluated on either the positive or negative face of the layer.

The tidal wave accompanying a nonlightlike layer carries nothing essentially new. The distributional Weyl curvature is fixed as an algebraic function of the surface stress energy $S^{\mu\nu}$, because (17) and (31) can be solved for γ_{ab} (the gauge-invariant part of $\gamma_{\mu\nu}$) in terms of $S^{\mu\nu}$ when $\varepsilon\neq0$. By contrast, when $\varepsilon=0$, only $\gamma_{\mu\nu}n^\nu$ and γ enter the expression (17) for $S^{\mu\nu}$, and so a specification of $S^{\mu\nu}$ leaves entirely free the traceless part of γ_{AB} (in coordinates convected along the direction of propagation l^a). It is precisely this part of γ^λ_μ that enters the first term of (41) and determines the strength of the purely gravitational part of the shock. Thus, in the lightlike case, material and purely gravitational distributionlike shocks decouple from each other and propagate independently along characteristic hypersurfaces.

The distributional Weyl tensor associated with a lightlike surface layer is algebraically degenerate in the sense of Petrov.[18] From (41) we obtain, in the lightlike case,

$$C^{\kappa\lambda}{}_{\mu\nu}n^{\nu}=-(\tfrac{1}{2}n^{[\kappa}\gamma^{\lambda]}n_{\mu}+\tfrac{1}{6}\delta^{[\kappa}_{\mu}n^{\lambda]}\gamma^{\dagger})\alpha\eta\delta(\phi), \quad (\varepsilon=0) . \tag{42}$$

This implies[17] (i) generally, that n^{ν} is a repeated principal null vector of $C^{\kappa\lambda}{}_{\mu\nu}$ (i.e., that the Weyl tensor is degenerate of Petrov type II, (ii) if the layer is affinely concilable ($\gamma^{\dagger}=0$, then n^{ν} is doubly degenerate (Petrov type III), and finally (iii) for a pure gravitational shock unaccompanied by matter ($S^{\mu\nu}=0 \Longrightarrow \gamma_{\kappa}=\frac{1}{2}\gamma n_{\kappa}$), (42) reduces to $C^{\kappa\lambda}{}_{\mu\nu}=0$, so that n^{ν} is a quadruple principal null vector and the Weyl tensor is of Petrov type N.

IV. SPHERICAL LIGHTLIKE SHELLS

The physical peculiarities of lightlike shells, and the efficacy of the present methods for treating them, are best appreciated in the simple context of spherical symmetry. Spherical shells are in any case of interest in their own right in various cosmological settings.[2–5]

Expressed in terms of Eddington retarded or advanced time u, the metric of a general spherisymmetric geometry is

$$ds^2=-e^{\psi}du(fe^{\psi}du+2\zeta\, dr)+r^2d\Omega^2 , \tag{43}$$

where ψ, f are functions of u and r. The sign factor ζ is $+1$ if r increases toward the future along a ray $u=$const, i.e., if the light cone $u=$const is expanding; if it contracts, $\zeta=-1$.

It proves useful to introduce a local mass function $m(u,r)$ defined by $f=1-2m/r$. The Einstein field equations then take the form

$$\frac{\partial m}{\partial u}=4\pi r^2T^r_u, \quad \frac{\partial m}{\partial r}=-4\pi r^2 T^u_u, \quad \frac{\partial\psi}{\partial r}=4\pi r T_{rr} . \tag{44}$$

We consider a thin shell whose history Σ, a light cone $u=$const, splits spacetime into past and future domains $\mathcal{M}_-$ and $\mathcal{M}_+$. The four-metric has the form (43) in both $\mathcal{M}_-$ and $\mathcal{M}_+$, but with different functions (ψ_-,f_-) and (ψ_+,f_+).

The intrinsic metric of Σ is $(ds^2)_{\Sigma}=r^2d\Omega^2$. It is consistent with the isometry of the positive and negative faces of Σ to choose r as a common parameter along the generators. (Implicit in this choice is the assumption that Σ is nonstationary. The stationary case is dealt with separately in the next section.) With $\xi^a=(r,\theta^A)\equiv(r,\theta,\phi)$ as intrinsic coordinates of Σ, the future-directed lightlike normal generator is

$$n^{\mu}=\zeta e^{\mu}_{(r)}=\zeta\frac{\partial x^{\mu}}{\partial r} . \tag{45}$$

As a future-directed transversal, it is simplest to choose the other radial lightlike vector:

$$N^{\mu}\partial_{\mu}=e^{-\psi}\frac{\partial}{\partial u}-\frac{1}{2}\zeta f\frac{\partial}{\partial r} . \tag{46}$$

Since the scalar products

$$N\cdot N=0, \quad N\cdot n=-1, \quad N\cdot e_{(A)}=0 \tag{47}$$

are the same for $\mathcal{M}_+$ and $\mathcal{M}_-$, this transversal has been correctly chosen as the "same" vector on both faces of Σ.

In our coordinates the transverse extrinsic curvature (13) reduces to $\mathcal{K}_{ab}=-{}^{(4)}\Gamma_{\mu,ab}N^{\mu}$. The nonvanishing components are

$$\mathcal{K}_{rr}=\zeta\partial_r\psi, \quad \mathcal{K}^{\theta}_{\theta}=\mathcal{K}^{\phi}_{\phi}=-\tfrac{1}{2}\zeta f/r . \tag{48}$$

The indices θ,ϕ have been raised with g^{AB}, the inverse of the two-metric $g_{AB}d\theta^A d\theta^B=r^2d\Omega^2$ intrinsic to Σ.

Since $l^a=\zeta\delta^a_r$, $N_b=N\cdot e_{(b)}=-\zeta\delta^r_b$, and $\eta=-1$ according to (28), (45), and (47), the three-tensor g^{ab}_{*} satisfying (30) can be chosen as g^{AB} bordered by zeros, i.e.,

$$g^{ab}_{*}=g^{AB}e^a_{(A)}e^b_{(B)} . \tag{49}$$

The surface energy tensor $(-S^{ab})$ can now be read off from (31) with $\varepsilon=0$ and (15):

$$-S^{ab}=\sigma l^a l^b+Pg^{ab}_{*} , \tag{50}$$

where

$$4\pi r^2\sigma=-\zeta[m], \quad 8\pi P=-\zeta[\partial_r\psi] . \tag{51}$$

From these junction conditions in tandem with the field equations (44), one obtains the balance laws for energy and normal force:

$$\frac{\partial[m]}{\partial r}=4\pi r^2[e^{-\psi}T_{\alpha\beta}n^{\alpha}\xi^{\beta}], \quad \xi^{\beta}\partial_{\beta}=\partial_u , \tag{52}$$

$$2P/r=-\zeta[T_{\alpha\beta}n^{\alpha}n^{\beta}] . \tag{53}$$

Equivalently, these equations can be recovered from the general conservation laws (24), in which we set $\Phi=u$, $\alpha=-e^{-\psi}$, whence $\Gamma_b=\partial_b\ln(r^2\sin\theta)$, and (39).

There is no rest frame for a lightlike shell, and therefore σ and P in (50) cannot be given an absolute operational meaning as *the* surface density and pressure. They nevertheless serve perfectly well to determine the results of measurements by any specified observer. Consider, for example, an observer in radial free fall, having the four-velocity $u^{\alpha}=dx^{\alpha}/d\tau=(\dot u,\dot r,0,0)$. The observer-momentum conjugate to $\dot u$,

$$\mathcal{E}=e^{\psi}(fe^{\psi}\dot u+\zeta\dot r)=\tfrac{1}{2}e^{\psi}(p_{\perp}^{-1}+fp_{\perp}) \tag{54}$$

is conserved along the path segments passing through static (e.g., vacuum) regions, and may be considered a measure of the observer's specific energy. However, $\mathcal{E}$ is discontinuous across the shell. The variable which is continuous at the crossing point is the momentum $p_{\perp}$ normal to the shell:

$$[p_{\perp}]=0, \quad p_{\perp}=-u^{\alpha}n_{\alpha}=\dot u e^{\psi} . \tag{55}$$

As measured by this radially moving observer, the energy density associated with the shell is, according to (19),

$$T^{\mu\nu}_{\Sigma}u_{\mu}u_{\nu}=e^{-\psi}\sigma(n^{\mu}u_{\mu})^2\delta(u)$$
$$=\zeta(u^{\mu}n_{\mu})\delta(\tau)[m]/4\pi r^2 , \tag{56}$$

and is accompanied by an equal energy flux. The total inertial mass energy encountered by observers with normal momentum $p_\perp$ is thus

$$M(p_\perp)=-\zeta p_\perp[m] \ . \tag{57}$$

These observers also register an impulsive transverse pressure

$$e^{-\psi}P\delta(u)=p_\perp^{-1}\delta(\tau)P \ , \tag{58}$$

where we set $\tau=0$ as the proper time of crossing the shell, and P is given by (51).

Particularly simple is the case where the geometry on both sides of the shell is static, and

$$T_{\alpha\beta}n^\alpha n^\beta=T_{rr}=0 \ , \tag{59}$$

so that we can set $\psi=0$, $f=f(r)$ in (43). This allows the Schwarzschild, Reissner-Nordström, and de Sitter geometries, or any superposition of these, as backgrounds. It follows from (53) and (39) that the shell is pressureless and that the parameter r is affine. The mass of the shell is determined by

$$4\pi r^2\sigma=\tfrac{1}{2}r\zeta(f_+-f_-) \ . \tag{60}$$

As a very specific example, consider a lightlike bubble whose interior consists of a false vacuum with density ρ_0 expanding ($\zeta=+1$) into Schwarzschild exterior spacetime. The exterior lies to the past if the shell is expanding, and so we set

$$f_-=1-2m_0/r, \quad f_+=1-\tfrac{8}{3}\pi\rho_0 r^2 \ . \tag{61}$$

The shell's mass

$$4\pi r^2\sigma=m_0-\tfrac{4}{3}\pi r^3\rho_0 \tag{62}$$

is steadily expended to create the energy of the expanding false vacuum. A subluminal bubble wall would slow down and begin to fall back when all of its kinetic energy was used up in this way. But a lightlike wall is constrained to remain on its geodesic path, with the consequence that the shell's inertial mass (62) decreases and inevitably becomes negative. This situation is quite unphysical, and it is more natural, as Dray[19] has suggested, to suppose that the shell negotiates a hairpin bend at the moment when its inertial mass is reduced to zero and becomes an infalling lightlike shell. This convention preserves the parallelism of the lightlike and timelike cases.

It is instructive to trace explicitly how this discontinuous lightlike behavior arises as the limit of the continuous history of a subliminal shell. Consider a spherical shell moving through a background in which the exterior and interior metrics have the form

$$ds^2=\frac{dr^2}{f(r)}+r^2d\Omega^2-f(r)dt^2$$

with different functions $f_-(r)$ and $f_+(r)$. The equation of motion for the shell radius $r=r(\tau)$ is[5]

$$[\mathrm{sgn}(n^\alpha\partial_\alpha r)(f+\dot r^2)^{1/2}]=-M/r \ .$$

The square brackets indicate, as usual, the jump across the shell and the overdot means differentiation with respect to τ, the proper time. The sign factor is $+1$ or -1 accordingly as r increases or decreases along the normal n^α directed from the negative to the positive side. The proper inertial mass $M(\tau)$ satisfies the conservation law

$$dM+Pd(4\pi r^2)=0 \ ,$$

where P is the surface pressure.

For a dust shell in a de Sitter–cum–Schwarzschild background, we set $P=0$ so that M is a positive constant, and choose $f_\pm$ as in (61). The equation of motion can be expressed in the form

$$\gamma M=m_0-\tfrac{4}{3}\pi r^3\rho_0 \ , \tag{63}$$

where

$$2\gamma=(f_++\dot r^2)^{1/2}+(f_-+\dot r^2)^{1/2} \ .$$

It is clear that a shell which is expanding initially must reverse its motion before the right-hand side of (63) becomes negative. For a rapidly moving shell of small proper mass ($\dot r \gg 1, M/m_0 \ll 1$), (63) simplifies to

$$M|\dot r|\approx m_0-\tfrac{4}{3}\pi r^3\rho_0 \ ,$$

showing that the turning point is very close to the zero of the right-hand side.

The lightlike limit is obtained by replacing τ by another monotonic parameter λ according to $Md\lambda=m_S(\lambda)d\tau$, then letting $M\to 0$ with $m_S(\lambda)$ (an unspecified positive function) held fixed. (In particular, we could choose $|d\lambda/dr|=1$, $d\lambda/d\tau>0$, so that λ is affine in the lightlike limit according to the remarks in Sec. III.) The lightlike limit of (63),

$$m_S(\lambda)\left|\frac{dr}{d\lambda}\right|=m_0-\frac{4}{3}\pi r^3\rho_0 \ ,$$

is compatible with (62) and displays explicitly an abrupt reversal of motion at the point where the right-hand side becomes zero.

V. HORIZON-STRADDLING LIGHTLIKE SHELLS IN SPHERICAL GEOMETRIES

For a nonstationary shell, as we saw in the previous section, the requirement that the intrinsic geometry be unique determines how the two faces Σ_+,Σ_- have to be soldered. In particular, this requirement designates Schwarzschild's radial coordinate r as the soldering parameter to be identified along the generators in the spherisymmetric case.

We turn now to the study of stationary lightlike shells, for which this requirement is merely an initial condition that does not otherwise constrain the soldering. "Affine" and "static" solderings—respectively defined by identification of corresponding affine parameters and (advanced or retarded) static time coordinates over Σ_+ and Σ_-—are only two of many soldering possibilities that now arise and are permitted geometrically. Nor are any of these possibilities physically inconsistent. Rather, different solderings correspond to shells with different physical characteristics. A statically soldered shell has a surface pressure and density which are time independent. This object appears simple to a stationary observer, who views the shell as held in static equilibrium by a balance between the surface pressure and external forces.

If, on the other hand, the shell is affinely soldered, then it has vanishing surface pressure and a time-dependent surface density. This would be considered a simple object by a local inertial observer, who sees the shell in free fall, moving radially with the speed of light, its inertial mass energy changing as the result of work done by environmental forces.

To give concrete form to these remarks, let us consider a stationary shell $r=r_0=2m_\pm(r_0,u)$ straddling a horizon $(f_\pm=0)$ common to two geometries of the form (43). According to (36), stationarity implies that, on the shell,

$$0=T_{\alpha\beta}n^\alpha n^\beta|_\pm=-\zeta e^{\psi}T^r_u|_\pm ,$$

a result that follows equivalently from $\partial f_\pm/\partial u=0$ for $r=r_0$ and the first of (44). Thus, there is no energy flux incident on the shell from either side.

We select u as a parameter along the generators, continuous across Σ, but place no restrictions on the functions $\psi_\pm(u,r)$, so that the arbitrariness in the soldering is entirely reflected in the arbitrariness of the u dependence of $\psi_\pm$. With u,θ,φ as intrinsic coordinates of Σ, the vectors

$$n^\alpha=\frac{\partial x^\alpha}{\partial u},\quad N^\alpha=\zeta e^{-\psi}\frac{\partial x^\alpha}{\partial r} \tag{64}$$

are correctly identified over $r=r_0$ as normal and transverse radial lightlike vectors, the "same" on both faces, satisfying $N\cdot n=-1$.

The second of Eq. (44) yields

$$[\kappa_0]/4\pi r_0=-[T^\alpha_\beta n_\alpha N^\beta]=[T^r_r]=[T^u_u] . \tag{65}$$

This expresses the net ambient radial pressure on the shell in terms of the jump of "surface gravity":

$$\kappa_0=\left.\frac{\frac{1}{2}\partial f(r,u)}{\partial r}\right|_{r=r_0} . \tag{66}$$

The transverse extrinsic curvature (13) reduces to

$$\mathcal{K}_{ab}=-N_\mu\Gamma^\mu_{ab}=-\zeta e^{-\psi}\,{}^{(4)}\Gamma_{r,ab} .$$

Hence

$$\mathcal{K}_{ab}l^a l^a=\mathcal{K}_{uu}=\kappa,\quad \mathcal{K}^\theta_\theta=\mathcal{K}^\varphi_\varphi=\zeta r^{-1}e^{-\psi} ,$$

where the "acceleration" $\kappa(u)$, defined by (38), is given explicitly by

$$\kappa={}^{(4)}\Gamma^u_{uu}=-\zeta\kappa_0 e^{\psi}+\partial_u\psi . \tag{67}$$

From (15) and (31) we immediately find that the surface stress energy $-S^{ab}$ has the perfect-fluid form (50), with

$$8\pi P=-[\kappa],\quad 4\pi r_0\sigma=-[\zeta e^{-\psi}] . \tag{68}$$

We note that, while the shell's inertial mass $4\pi r_0^2\sigma$ does not vanish in general, its gravitational mass $[m]=0$ always, since $2m_\pm(r_0)=r_0$. The gravitational mass may be interpreted in an asymptotically flat space as the mass energy calibrated for an observer at infinity. It vanishes in this instance because of the infinite redshift associated with matter placed statically at a horizon.

A radially moving inertial observer who crosses the shell with four-velocity u^α has a continuous normal momentum $p_\perp$:

$$[p_\perp]=0,\quad p_\perp=-u^\alpha n_\alpha|_{r=r_0}=\zeta e^{\psi}\dot r . \tag{69}$$

His inward radial velocity is therefore abruptly boosted by an amount proportional to the surface density:

$$[-\dot r]=p_\perp 4\pi r_0\sigma . \tag{70}$$

This is a manifestation of Raychaudhuri focusing[16] by the shell material of a bundle of radial timelike geodesics.

All of the foregoing results apply to an arbitrary soldering. For simplicity, we now specialize to an everywhere-static geometry without radial energy flow $(T_{rr}=0)$, so that, by (44), $\psi=\psi_\pm(u)$, $f=f_\pm(r)$ everywhere.

The shell faces are *statically soldered* if we choose $\psi_\pm(u)=0$, so that u is the standard advanced or retarded time along the shell's history. From (67) and (68) the surface pressure and density are given by

$$8\pi P=[\zeta\kappa_0],\quad 4\pi r_0\sigma=-[\zeta] , \tag{71}$$

and are independent of time.

If an initially expanding $(\zeta_-=+1)$ bundle of radial light rays $u=\text{const}$ is focused by the shell so as to become contracting $(\zeta_+=-1)$, (71) shows that the surface density $\sigma=2r_0/4\pi r_0^2$ is positive, in agreement with the Raychaudhuri effect. In the more conventional situation where r varies monotonically across the shell, ζ will be continuous and $\sigma=0$. Since the material has no inertial mass, the net external force on it must vanish: The equation

$$2\zeta P/r_0=[T^r_r] , \tag{72}$$

obtained from (71) and (65), expresses the balance between the radial force $2P/r_0$ due to surface pressure, acting "outward" (in the direction of increasing r), and the net ambient radial pressure, acting from $\mathcal{M}_+$ toward $\mathcal{M}_-$.

Let us turn now to the case where the shell faces are affinely soldered. The soldering parameter u is affine if the acceleration κ, given by (67), vanishes. This yields

$$e^{-\psi}=-\zeta\kappa_0(u-u_0) , \tag{73}$$

as the appropriate choice for ψ, where u_0 is an arbitrary constant. (The constants ζ, κ_0, and u_0 all have different values on the two faces in general.) From (68) we thus obtain, for an *affinely soldered* shell,

$$8\pi P=0,\quad 4\pi r_0\sigma=[\kappa_0(u-u_0)] . \tag{74}$$

Recalling (65), we derive

$$\frac{d\sigma}{du}=[T^r_r] . \tag{75}$$

This shows, from the viewpoint of a local inertial observer, how the inertial mass varies with time as the result of work done by the ambient radial pressure on the shell wall traveling with the speed of light. To clarify the issue of signs, we remark that, for a shell moving outward relative to an inertial observer, the future side $\mathcal{M}_+$ corresponds to the interior of the shell, so that $[T^r_r]$ represents the net outward radial force.

It is worthwhile to look at a concrete example. We consider a stationary lightlike bubble, with a de Sitter interior geometry, given by (43) with

$$f_{dS}=1-\tfrac{8}{3}\pi\rho_0 r^2=1-r^2/a^2 ,$$

and a Reissner-Nordström exterior:

$$f_{\rm RN}=1-2m/r+e^2/r^2 .$$

The ambient stress-energy tensors have, respectively, the false-vacuum and Maxwellian forms

$$T^u_u=T^r_r=T^\theta_\theta=T^\varphi_\varphi=-\rho_0 \quad ({\rm dS})$$

and

$$-T^u_u=-T^r_r=T^\theta_\theta=T^\varphi_\varphi=(8\pi)^{-1}(e/r^2)^2 \quad ({\rm RN}) .$$

Demanding that the shell be located on a horizon $r=r_0=a$ common to the two geometries yields the relation

$$2ma=a^2+e^2 , \tag{76}$$

and, from (66), the surface gravities

$$\kappa_0|_{\rm dS}=-a^{-1}, \quad \kappa_0|_{\rm RN}=(m-e^2/r_0)/r_0^2 . \tag{77}$$

The tug between the false-vacuum and Maxwell tensions $(-T^r_r)$ produces a net inward radial force on unit area:

$$\rho_0-e^2/8\pi r_0^4=(2a-m)/4\pi a^3=-[\kappa_0]^{\rm dS}_{\rm RN}/4\pi a . \tag{78}$$

We now fix attention on an episode Σ of the shell's history, represented by segment AB in Fig. 1, when the shell is moving outward relative to an observer in free fall, and u thus has the character of an advanced time. Light rays $u=$const incident upon Σ from the past (RN) side $\mathcal{M}_-$ have r increasing toward the future, but after traversal of the shell, r becomes decreasing on the de Sitter side $\mathcal{M}_+$. Hence $\zeta_-=+1$, $\zeta_+=-1$ along segment AB. Note, however, that these signs would be reversed on BD if the boundary were extended to D.

If Σ is statically soldered, we find from (71), (76), and (77),

$$8\pi P=m/a^2, \quad \sigma=(2\pi a)^{-1} . \tag{79}$$

If Σ were extended toward D, σ and P would abruptly reverse sign at B. Such development of negative energies can be forestalled by Dray's stratagem[19] of introducing a kink in the shell history at B, with subsequent infall along BC. The density is now everywhere positive. However, the sudden reversal of motion at B requires *ad hoc* intervention by a momentary infinite surface tension. This statically soldered singular configuration ABC might be

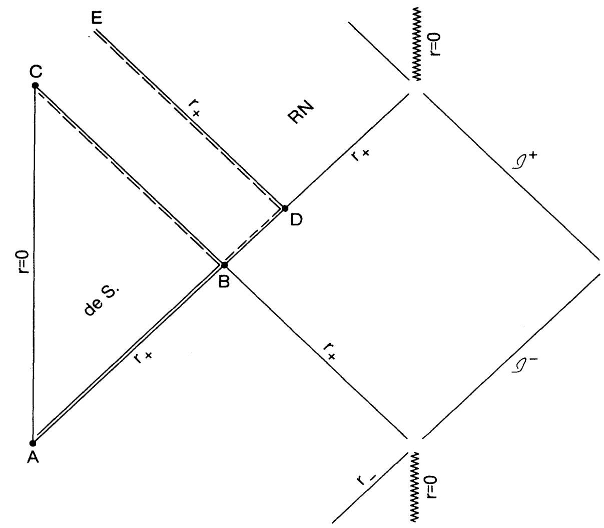

FIG. 1. Reissner-Nordström black-hole exterior space-time joined to an interior de Sitter geometry by a stationary lightlike shell ABC or ADE located on segments of the outer RN horizon. If the shell is statically soldered, a horizon $r=r_0$ on one side must extend to a horizon on the other, since both correspond to infinite values of the static time coordinate. But for an affinely soldered shell, this is not necessary.

considered the limit of a sequence of subluminal static shells whose radii approach the horizon radius $r=a$. It must, however, be considered unphysical.

There is a trail of attempts in the recent literature to construct a classical elementary-particle model in the form of a ball of false vacuum bounded by a de Sitter horizon. Even when the boundary is permitted to be shell-like, static models of this type (commonly treated in the current literature as lightlike limits of subluminal static shells) all exhibit singular behavior (infinite surface pressure).[20] These pathologies are essentially due to the kink necessarily encountered on every surface of constant Schwarzschild time at the Schwarzschild throat B.

Parenthetically, it is amusing to observe that for the special choice of parameters $a^2=\frac{1}{3}e^2$, the surface gravities (77) become equal on a shell placed at the *inner RN* horizon. Hence, from (71), $\sigma=P=0$. This means that a perfectly smooth transition is now possible, yielding a de Sitter-like model of a charged particle which would be entirely free of geometrical singularities and discontinuities having a massless charge distribution balanced at the inner horizon, if the charge fully occupies both sheets of the inner horizon. The gravitational mass of this "electron" is on the order of the Planck mass.

We have been discussing shells that are soldered statically. Affine soldering, on the other hand, opens up a variety of possibilities for avoiding both negative energies and singular impulses at a kink. Choosing $u_0^+=0$, $u_0^-=(2a-m)/u_D a$, with B as the origin for the affine soldering parameter u, we obtain, from (74),

$$P=0, \quad 4\pi a^3\sigma=(2a-m)(u_D-u) ,$$

along the affinely soldered shell segment *ABD*. In striving outward against the force imbalance (78), the shell steadily loses its inertial mass. At the (arbitrary) point *D*, $\sigma=0$, the shell has effectively disappeared and it is possible to introduce a kink without invoking any extraneous force. The complete shell history *ADE* has $P=0$, $\sigma\geq 0$ everywhere and may be regarded as the regular lightlike limit of the history of a subluminal dust shell which expands to a maximal size and then falls back.

If an affinely soldered shell of this type occupies the past and future *outer* horizons of the exterior RN geometry, then it is easily seen to have a negative inertial mass. Even as a nonstatic model for a particle, it therefore fails to satisfy reasonable criteria.

VI. WHEN SHELLS COLLIDE

One of the useful features of a thin shell as a gravitational source is that it can effectively localize interactions that are ordinarily nonlocal by bringing into close proximity two regions $\mathcal{M}_+$, $\mathcal{M}_-$ in which the gravitational fields may be different. A much greater variety of such effects becomes accessible when one considers a pair of shells in collision. Five years ago Dray and 't Hooft[21] and, independently, Redmount[9] derived a simple formula—the "DTR relation"—which connects the gravitational masses in the vacuum region between two spherical lightlike shells before and after they collide. This formula is quite remarkable. In the absence of gravity, it would take a trivial linear form, expressing the conservation of material energy in the collision. Its actual, nonlinear, form encapsulates algebraically a number of surprising nonlocal and nonlinear effects hidden in the Einstein field equations. The most dramatic of these—"mass inflation"—occurs when opposing streams of matter collide near the past horizon of a white hole,[22] or the Cauchy (inner) horizon of a black hole,[23] releasing arbitrarily large amounts of gravitational binding energy as material forms of energy.

There is a straightforward generalization[24] of the DTR spherical formula, governing the collision of a pair of arbitrary lightlike shells.

We briefly sketch the derivation. Let the spacelike two-surface *S*, parametrized by $\theta^a=(\theta,\varphi)$, with associated tangential base vectors $e^{\alpha}_{(a)}$, be the intersection of two lightlike shell histories, labeled Σ_3,Σ_4. Let Σ_1,Σ_2 be two other lightlike shell histories that reemerge from *S*. The normal to Σ_i ($i=1,\ldots,4$) is denoted $n^{\alpha}_{(i)}\partial_\alpha=\partial/\partial u_i$ with u_i a (generally nonaffine) parameter along the generators, the same on both faces. The normal "extrinsic" curvature of Σ_i has components

$$K_{iab}=-n_{(i)}\cdot\delta e_{(a)}/\delta\xi^b \quad (a,b=\theta,\varphi)\ ,$$

tangent to *S*. As we noted in Sec. II, for a lightlike hypersurface, K_{iab} is really a measure of purely intrinsic properties—the dilation and shear rates, given by its trace K_i and the magnitude σ_i of its trace-free part—and, as such, must have the same value on both faces.

The hypersurfaces $\Sigma_1,\ldots,\Sigma_4$ divide the spacetime near *S* into four sectors which we label (clockwise from noon, as in Fig. 2) $12=A$, $23=C$, $34=B$, $41=D$.

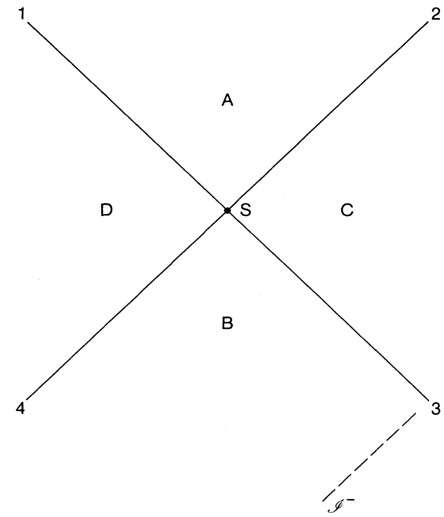

FIG. 2. Collision of two lightlike shells, whose three-dimensional histories are represented by lines 3 and 4 in the figure, to form outgoing lightlike shells represented by 1 and 2. In the application to mass inflation in a black hole, sector *B* extends out to the exterior of the hole, and shell 3, representing infall from the radiative tail of the collapse, falls close to the inner horizon which lies just beyond it in the extension of sector *B*.

The basic assumption which underlies the generalized DTR relations (at least in their simplest form) is that each point of *S* has a neighborhood that can be covered by an "admissible" chart (for instance, Gaussian coordinates anchored to geodesics orthogonal to *S*) in which the components of the four-metric are continuous and piecewise continuously differentiable. In other words, it is assumed that the points at which the two thin layers interpenetrate constitute singularities which are not qualitatively "worse" than those of a single thin layer. We thus exclude the possibility that points of *S* have conical (or worse) singularities of the four-geometry, a possibility in principle conceivable for the collision of two coherent streams of nonlinearly interactive fluids or fields.

The assumption gives unambiguous meaning to the equality or parallelism of a pair of vectors transverse to *S*. It follows that

$$(n_{(1)}\cdot n_{(2)})(n_{(3)}\cdot n_{(4)})=(n_{(1)}\cdot n_{(4)})(n_{(2)}\cdot n_{(3)})\ , \tag{80}$$

at each point of *S*, since all four lightlike generators are orthogonal to *S* and there are only two lightlike directions orthogonal to a spacelike two-surface.

We now define eight scalar functions $F_A,\ldots,F_d$ and $D_A,\ldots,D_D$ over *S* by, e.g., $F_A=F_{12}=K_1K_2/(n_{(1)}\cdot n_{(2)})$, $D_A=D_{12}=\sigma_1\sigma_2/(n_{(1)}\cdot n_{(2)})$. They are clearly independent of the choice of parameters u_i.

From these definitions and (80) it follows by inspection that, at each point (θ,φ) of *S*,

$$|F_AF_B|=|F_CF_D|\ , \tag{81a}$$

$$|D_AD_B|=|D_CD_D|\ . \tag{81b}$$

These are the generalized DTR relations.

For spherical shells, $K_i = 2r^{-1} n^{\alpha}_{(i)} \partial_\alpha r$. By virtue of the completeness relation

$$g^{\alpha\beta} = g^{ab} e^{\alpha}_{(a)} e^{\alpha}_{(a)} e^{\beta}_{(b)} + 2n^{(\alpha}_{(1)} n^{\beta)}_{(2)} / n_{(1)} \cdot n_{(2)} ,$$

we obtain $F_A = 2r^{-2} f_A$, where

$$f = g^{\alpha\beta} r_{,\alpha} r_{,\beta} = g^{rr} .$$

Hence (81a) reduces to the single, angle-independent condition at the point of collision,

$$|f_A f_B| = |f_C f_D| , \tag{82}$$

which is the formula originally found by Dray and 't Hooft and Redmount. We may define, as in Sec. IV, a quasilocal Schwarzschild mass function $m(x^\alpha)$ by $f = 1 - 2m/r$. Then (82) relates the values of the masses $m_A, \ldots, m_D$ at the collision point. For weak fields, to linear order (neglecting quadratic potential-energy terms), it is seen to express conservation of gravitational mass in the collision.

For nonspherical shells the generalized DTR relations (81a) similarly connect the "Hawking quasilocal mass aspects" $m_A(\theta,\varphi), \ldots, m_D(\theta,\varphi)$ over S, where m_A, for example, is defined by

$$m_A(\theta,\varphi) = (\mathcal{A}/16\pi)^{1/2} [1 - (\mathcal{A}/8\pi) F_A] , \tag{83}$$

with $\mathcal{A}$ denoting the area of S. The quasilocal mass originally introduced by Hawking[25] was defined as the mean value of (83) over S and is known to give reasonable answers in simple instances. In spherisymmetric fields it reproduces the Schwarzschild mass function, as we have in effect just shown.

A typical situation where (81a) or (82) predicts that "mass-inflation" occurs when shell 3, schematically representing infall from the gravitational wave tail of a collapse, falls close to the inner horizon of the resulting black hole, so that F_B is an arbitrarily small positive number. Because of the (finite) jump in mass across shells 3 and 4, the value of F_C, F_D are bounded away from zero at S. Then, after the collision with shell 4 (representing outflow from the collapsing star), $|F_A|$, and hence m_A, must be correspondingly large. Relations (81b) can similarly be used to place upper bounds on the growth of nonspherical deformations in this process. For more detailed discussion, the reader is referred to the papers cited.[22–24]

VII. GRAVITATIONAL RADIATION FROM THE COLLAPSE OF LIGHTLIKE SHELLS AND COSMIC-STRING LOOPS

There is an interesting class of problems which exploits the ability of an appropriately shaped lightlike shell to act as causal boundary for the spacetime region on its past side. Such models were first considered in 1973 by Penrose[12] in his search for counterexamples to the cosmic-censorship hypothesis. Similar arguments were employed by Hawking[13] in 1987 to obtain an upper bound for the amount of gravitational radiation that can be emitted by a circular loop of string collapsing with the speed of light. Since none of this work is widely known (that of Hawking remains unpublished at the date of writing), it may be useful to give a brief account of it here.

The type of lightlike shell Σ that we consider has the property that the past side $\mathcal{M}_-$ of Σ is contained in the complement of its domain of influence. This means that if a point p lies to the past of a point of Σ, then no point of Σ lies to the past of p. Thus Σ cannot influence $\mathcal{M}_-$ gravitationally and the geometry of $\mathcal{M}_-$ can be chosen independently of Σ, in particular as flat if it is empty. The lightlike histories of plane layers and of collapsing spheres and cylinders are casual boundaries in this sense. Precisely because of its causal disconnection from $\mathcal{M}_-$, the matter distribution over Σ need not share these geometrical symmetries, but can be arbitrarily nonuniform.

As an example, we consider, following Penrose,[12] a nonuniform spherical shell of coherently moving photons falling radially inwards from infinity. The interior of the shell is assumed flat. In this domain we can introduce spherical coordinates $x^\alpha = (t, r, \theta, \varphi)$ in terms of which the shell has the equation $t = -r$ and its stress energy is

$$T^{\alpha\beta} = \sigma n^\alpha n^\beta \delta(t + r) . \tag{84}$$

Energy conservation demands that $4\pi r^2 \sigma(r,\theta,\varphi)$ be conserved along the flow lines of $n^\alpha \partial_\alpha = \partial_t - \partial_r$.

Our objective is to relate the gravitational mass of the shell to the area of the apparent horizon formed in the collapse.

For our purposes the apparent horizon S is conveniently defined as a subspace of the light cone $t = r$, a closed two-surface on which light rays beamed perpendicularly outwards are marginally trapped, i.e., expansionless. If the collapsing shell is nonuniform, S will not be a sphere; in terms of intrinsic coordinates $\theta^A = (\theta, \varphi)$, we write its parametric equation as

$$-t = r = h(\theta^A) .$$

The associated base vectors $e_{(A)}$ have four-dimensional components

$$e^\alpha_{(A)} = \frac{\partial x^\alpha}{\partial \theta^A} = (-\partial_A h, \partial_A h, \delta^2_A, \delta^3_A) ,$$

and the intrinsic metric is

$$g_{AB} d\theta^A d\theta^B = h^2(\theta,\varphi)(d\theta^2 + \sin^2\theta \, d\varphi^2) .$$

The outgoing lightlike vector N orthogonal to S has components

$$N_\alpha = (-\lambda, (1-\lambda), -\partial_\theta h, -\partial_\varphi h) ,$$

on the inner face, where $\lambda = \frac{1}{2}(1 + g^{AB} \partial_A h \cdot \partial_B h)$. One easily verifies that

$$N \cdot N = N \cdot e_{A)} = 0, \quad N \cdot n = -1 .$$

A slightly tedious but straightforward calculation from (13), using the interior (flat) affine connection, gives, for the expansion rate of outgoing light rays on the inner face of S,

$$g^{AB} \mathcal{K}^-_{AB} = h^{-1}(1 - \nabla^2 \ln h) ,$$

in which the Laplacian refers to the spherical two-metric:

$$\nabla^2 \equiv \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left[\sin\theta\frac{\partial}{\partial\theta}\right]+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\ .$$

Since the exterior geometry of the shell is unknown, a corresponding calculation for the outer face cannot be carried through in general. However, in the particular case of a marginally trapped S, $g^{AB}\mathcal{K}^{+}_{AB}=0$ by hypothesis. Then the surface density σ over S follows at once from (15) and (17), in which we set $\gamma_\mu=0$ for a pressureless shell of radially moving material. The result is

$$8\pi h\sigma=(1-\nabla^2\ln h)\ . \tag{85}$$

(Alternatively, this can be derived by integration of Raychaudhuri's equation through the layer.)

This yields the shell's gravitational mass as an integral over a complete solid angle,

$$M=\int \sigma h^2 d\Omega=(8\pi)^{-1}\int h(1-\nabla^2\ln h)d\Omega\ . \tag{86}$$

since, by the conservation law, the first integral is equal to a corresponding integral taken over a sphere at infinity on the past light cone.

There is a well-known argument[26] which concludes that, if cosmic censorship is valid, the difference

$$E_{\max}=M-(\mathcal{A}/16\pi)^{1/2} \tag{87}$$

(where $\mathcal{A}=\int h^2 d\Omega$ is the area of the apparent horizon) must be non-negative. It then represents an upper bound for the energy emitted as gravitational radiation in the collapse, since by the Hawking area theorem, $\mathcal{A}$ cannot be larger than the area of the final stationary black hole.

Penrose's efforts in 1973 were directed toward a search for functions $h(\theta,\varphi)$ that violate the inequality $E_{\max}\geq 0$. It is now established[27] that no initial data can violate the Penrose inequality. Today, (87) is chiefly of interest as a rigorous upper bound on gravitational energy emission, *assuming* that a black hole is formed in the collapse.

An example of special interest, considered by Hawking[13] is the collapse of a circular loop of cosmic string, moving at the speed of light. This idealization is reasonable, since the tension in the loop will rapidly accelerate it to relativistic velocities. The dustlike stress energy (84) is an accurate representation in the lightlike limit for a cosmic string, since the (transverse) tension is invariant under boosts in the radial direction and thus becomes negligible compared with the energy density.

To locate the apparent horizon, we focus on the critical moment $t=-a$ (say), when light rays emitted radially outward in the plane of the loop (assumed to be the equatorial plane) become marginally trapped. At this moment the loop occupies a circle C whose equation is

$$-t=r=a,\quad z=0\ .$$

The pair of lightlike hyperplanes

$$\Sigma_\pm:\quad t=-a+|z|,\quad z\equiv r\cos\theta$$

intersect on C, forming a cusp (Fig. 3). It is now easy to see that the apparent horizon S is the cusped two-surface

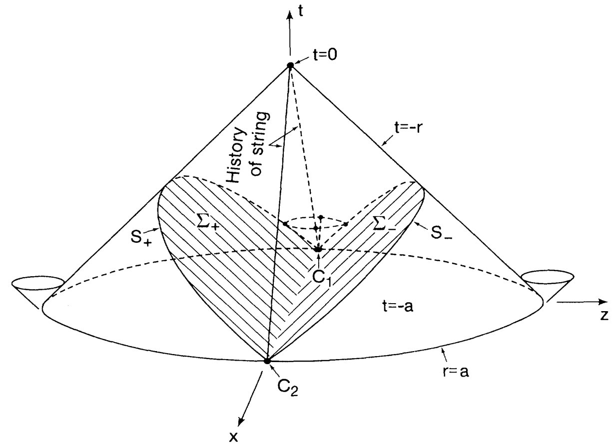

FIG. 3. Azimuthal three-section $y=0$ of the flat interior of the past light cone $t+r=0$. This is the region that has "not yet become aware of" a circular loop of cosmic string collapsing along the cone. At a given moment, $t=-a$, say, the string loop is represented in the figure by the two points C_1, C_2 in which it intersects the x axis. The apparent horizon is the cusped two-surface formed by the pair of surfaces whose sections are the parabolic arcs S_-, S_+.

formed as the union of the intersections $S_\pm$ of $\Sigma_\pm$ with the light cone $t=-r$. This is because $S_\pm$ encounter the string's history only on C, and hence $[\mathcal{K}_{ab}]=0$ elsewhere on $S_\pm$. The outward lightlike normals to $S_\pm$ are the null generators that rule the hyperplanes $\Sigma_\pm$ and are expansionless. From this, it follows that $\mathcal{K}^{+}=\mathcal{K}^{-}=0$ everywhere on $S_\pm$ except at the cusp C, which, however, is marginally trapped by hypothesis. Thus the surface $S=S_+\cup S_-$, with equation

$$r=h(\theta)=a/(1+|\cos\theta|)\ , \tag{88}$$

is the apparent horizon. Substitution in (85) shows that σ is a Dirac δ function with support on the equatorial circle $r=a$, $\theta=\pi/2$, as expected (we are here following a slight variant of Hawking's argument due to Gibbons[14]).

From (86) the gravitational mass is found to be $M=\frac{1}{2}a$. On the other hand, $\mathcal{A}=2\pi a^2$, since the intrinsic geometry of $S_\pm$ is that of a pair of disks of radius a. (In cylindrical coordinates, the intrinsic metrics of $\Sigma_\pm$ are $ds^2=d\rho^2+\rho^2 d\varphi^2$, with $\rho\leq a$ on $S_\pm$.)

Thus, from (87),

$$E_{\max}=(1-2^{-1/2})M\ .$$

This is Hawking's result that at most 29.3% of the initial mass of a collapsing lightlike circular loop can be radiated gravitationally before it forms a black hole, assuming cosmic censorship valid in this instance.

VIII. CONCLUDING REMARKS

A fair amount of work has been done on lightlike surface layers over the past 25 years, but the lack of a convenient intrinsic description of their dynamics, similar to the extrinsic curvature formalism for subluminal layers, has proved an impediment which has left this as one of the murkier areas of general relativity. We have been able to show in this paper that an intrinsic description can indeed be formulated in the lightlike case and that it constitutes a natural limit of the subluminal description.

Because they require matching of a pair of metrics that are only two-dimensional, lightlike shells offer the simplest and most plastic means for joining two four-geometries in a dynamically consistent way. Once their idiosyncrasies have become familiar, they emerge as a versatile and useful resource in the relativist's arsenal. We hope that this presentation will help to make lightlike shell dynamics a working tool for practioners in both standard and (by straightforward extension) higher-dimensional cosmologies, where (as characteristic surfaces) they form a natural bridge between phases or domains with different material content, dimensionality, metric signature, or topology.

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APPENDIX: THIN SHELLS AS DIRAC DISTRIBUTIONS

For completeness we here present a straightforward derivation, based on distribution theory, of the jump conditions and conservation laws stated in Secs. II and III, some of them justified there by an indirect limiting procedure. Our approach here is similar to that of Taub[8] and Clarke and Dray.[10]

As in Sec. II our formulas here apply to arbitrary (lightlike or nonlightlike) surface layers. But by contrast to that treatment, where, mindful of the flexibility needed in practical applications, we admitted a pair of independent, disconnected charts $x^\mu_\pm$ in the two domains abutting on the shell, here we lay down a single chart x^μ (for instance, skew-Gaussian coordinates attached to geodesics) that reaches into both domains and is maximally smooth.

In more detail, our assumptions are that we are given the following.

(a) A manifold $\mathcal{M}$ consisting of overlapping domains $\mathcal{M}_+, \mathcal{M}_-$.

(b) A hypersurface Σ contained in the overlap $\mathcal{M}_+ \cap \mathcal{M}_-$. The equation of Σ is $\Phi(x^\mu)=0$, where Φ is a smooth function, and x^μ a chart that covers the overlap. We assume that the domains of $\mathcal{M}$ in which Φ is positive and negative are (properly) contained in $\mathcal{M}_+$ and $\mathcal{M}_-$, respectively.

(c) A pair of metrics $g^+_{\alpha\beta}(x)$ and $g^-_{\alpha\beta}(x)$, defined over $\mathcal{M}_+$ and $\mathcal{M}_-$, respectively, each at least three times continuously differentiable.

(d) Equality of $g^+_{\alpha\beta}$ and $g^-_{\alpha\beta}$ on Σ:

$$[g_{\alpha\beta}]=0 \tag{A1}$$

We employ the following notation. If $F^+(x), F^-(x)$ are functions defined over $\mathcal{M}_+, \mathcal{M}_-$ respectively, then we define the "hybrid" function $\widetilde{F}$ and the jump on Σ by

$$\begin{aligned}\widetilde{F}(x)&=F^+(x)\Theta(\Phi)+F^-(x)\Theta(-\Phi)\ ,\\ [F]&=[F^+(x)-F^-(x)]|_\Sigma\ ,\end{aligned} \tag{A2}$$

where the Heaviside step function $\Theta(\Phi)=1, \frac{1}{2}$, or 0 accordingly as Φ is positive, zero, or negative. [The convention $\Theta(0)=\frac{1}{2}$ makes $\Theta(\Phi)+\Theta(-\Phi)=1$ into a pointwise identity.] From these definitions it follows that

$$\partial_\mu\widetilde{F}(x)=(\partial_\mu F)^\sim-[F]\delta(\Phi)(\partial_\mu\Phi)\ , \tag{A3}$$

$$\widetilde{F}\widetilde{G}(x)=(FG)^\sim-[F][G]\Theta(\Phi)\Theta(-\Phi)\ , \tag{A4}$$

which hold pointwise. [In (A4) the last term of course vanishes when considered as a distribution.]

We assume (if necessary, by restriction to a subregion of Σ) that the intrinsic metric does not change signature over Σ. Then it proves convenient to set

$$g^{\alpha\beta}_\pm(\partial_\alpha\Phi)(\partial_\beta\Phi)=\varepsilon_\pm(\Phi)\alpha^2_\pm(x^\mu)\ ,$$

where $\varepsilon_+, \varepsilon_-$ are (nonunique) smooth functions of Φ which reproduce the sign of the left-hand side and, in particular, vanish if the hypersurface $\Phi=$const is lightlike. There is then no loss of generality in requiring $\alpha(x)\neq 0$ everywhere. The normals n^+_μ, defined on $\mathcal{M}_+$ and $\mathcal{M}_-$ by $n^+_\mu=\alpha^{-1}_\pm\partial_\mu\Phi$, satisfy

$$g^{\mu\nu}n_\mu n_\nu|_\pm=\varepsilon_\pm(\Phi)\ .$$

On Σ the values of these functions are unambiguous:

$$[\varepsilon]=[\alpha]=[n_\mu]=0\ .$$

Because tangential derivatives of $g^+_{\alpha\beta}-g^-_{\alpha\beta}$ vanish over Σ by virtue of (A1),

$$[\partial_\mu g_{\alpha\beta}]=\partial_\mu(g^+-g^-)_{\alpha\beta}|_\Sigma=\eta\gamma_{\alpha\beta}n_\mu\ . \tag{A5}$$

Here $\gamma_{\alpha\beta}$ is the jump in the transverse derivative:

$$\gamma_{\alpha\beta}=\alpha N^\mu[\partial_\mu g_{\alpha\beta}]\ .$$

$N^\mu(x)$ is an arbitrary vector field (the same over $\mathcal{M}_+$ and $\mathcal{M}_-$) transverse to hypersurfaces of constant Φ, so that we can define functions $\eta_\pm(x)$ by

$$\alpha/\eta|_\pm=N^\mu\partial_\mu\Phi\neq 0\ .$$

From (A5)

$$[\Gamma^\lambda_{\alpha\beta}]=\eta(\gamma^\lambda_{(\alpha}n_{\beta)}-\tfrac{1}{2}\gamma_{\alpha\beta}n^\lambda)\ . \tag{A6}$$

The key step is the introduction of a "hybrid" metric $g_{\alpha\beta}$ over $\mathcal{M}$ which solders the metrics $g^+_{\alpha\beta}, g^-_{\alpha\beta}$ together (continuously) over Σ. Recalling the notation (A2), we define

$$g_{\alpha\beta}=\tilde{g}_{\alpha\beta}\Longrightarrow g^{\alpha\beta}=\tilde{g}^{\alpha\beta} . \tag{A7}$$

It follows from (A3) and (A1) that the Christoffel symbols derived from the hybrid metric $g_{\alpha\beta}$ are the hybrid of $\Gamma^{\lambda}_{\pm\mu\nu}$, i.e.,

$$\Gamma^{\lambda}_{\mu\nu}=\tilde{\Gamma}^{\lambda}_{\mu\nu} . \tag{A8}$$

The curvature associated with $\Gamma^{\lambda}_{\mu\nu}$ can be written down at once by applying (A3) and (A4),

$$R^{\kappa}_{\ \lambda\mu\nu}=\tilde{R}^{\kappa}_{\ \lambda\mu\nu}-2[\Gamma^{\kappa}_{\lambda[\mu}]n_{\nu]}\alpha\delta(\Phi) -2[\Gamma^{\alpha}_{\lambda[\mu}][\Gamma^{\kappa}_{\nu]\alpha}]\Theta(\Phi)\Theta(-\Phi) , \tag{A9}$$

and it satisfies the Bianchi identity

$$\nabla_{[\sigma}R^{\kappa\lambda}{}_{\mu\nu]}=0 . \tag{A10}$$

Here ∇_σ is the covariant derivative associated with the hybrid affine connection (A8), so that $\nabla_\lambda g_{\mu\nu}=0$, and $\tilde{R}^{\kappa}_{\ \lambda\mu\nu}$ is, of course, the hybrid of $R^{\kappa}_{\pm\lambda\mu\nu}$. In view of (A6), (A9) takes the explicit form

$$R^{\kappa}_{\ \lambda\mu\nu}=\tilde{R}^{\kappa}_{\ \lambda\mu\nu}-(n^{\kappa}\gamma_{\lambda[\mu}-n_{\lambda}\gamma^{\kappa}_{[\mu})n_{\nu]}\alpha\eta\delta(\Phi) . \tag{A11}$$

(Terms that vanish distributionally have been discarded.)

Contraction of (A11) and use of the Einstein field equation yields

$$T^{\lambda\mu}=\tilde{T}^{\lambda\mu}+S^{\lambda\mu}\alpha\delta(\Phi) , \tag{A12}$$

with $S^{\lambda\mu}$ given by Eq. (17).

The conservation identity $\nabla_\mu T^{\lambda\mu}=0$, implied by the contraction of (A10), leads to

$$\nabla_\mu(\alpha S^{\lambda\mu})\delta(\Phi)=-\nabla_\mu\tilde{T}^{\lambda\mu} ,$$

by virtue of (20). Using (A3) to extract the distributional part of the right-hand side, we find

$$\nabla_\mu(\alpha S^{\lambda\mu})=-\alpha[T^{\lambda\mu}n_\mu] . \tag{A13}$$

Transvecting (A13) with N_λ and noting that

$$N_\lambda S^{\lambda\mu}=N_a S^{ab}e^{\mu}_{(b)} ,$$

$$S^{\lambda\mu}\nabla_\mu N_\lambda=S^{ab}e^{\lambda}_{(a)}(\delta N_\lambda/\delta\xi^b)^{\sim}=S^{ab}(\partial_b N_a+\tilde{\mathcal{K}}_{ab}) ,$$

where we have made use of (23), (A8), and (13), we finally arrive at the conservation law for the layer in its intrinsic form (24).

Returning to (A11), we can obtain the expression (41) for the Weyl conformal curvature, defined by

$$C^{\kappa\lambda}{}_{\mu\nu}=R^{\kappa\lambda}{}_{\mu\nu}-2\delta^{[\kappa}_{[\mu}R^{\lambda]}_{\nu]}+\tfrac{1}{6}R\,\delta^{\kappa\lambda}_{\mu\nu} ,$$

if we use (A12) to express the distributional part of R^{λ}_{ν} in terms of S^{λ}_{ν}.

This completes the derivation of the main formulas quoted without proof in the text.

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