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Cosmic string dynamics with friction

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Cosmic-string equations of motion are derived with the force of friction due to string-particle scattering taken into account. For strings in a Robertson-Walker space, the additional terms in the equations of motion have the same functional form as the terms due to the expansion of the Universe. As a result, a computer simulation of strings with friction would require only a trivial modification of the existing programs.

I. INTRODUCTION

Cosmic strings are topological defects that could be formed at a phase transition in the early Universe. They could play an important role in the formation of structure in the Universe and can lead to various observational effects at the present time.[1] The cosmological evolution of strings has been a subject of extensive research in the last several years (see, e.g., Refs. 2–4, and references therein). In most of this research it was assumed that friction due to the interaction of strings with matter can be neglected. For superheavy strings this assumption is justified, except at very early times, soon after string formation. However, for lighter strings, friction can be significant during most of their evolution.

The purpose of this paper is to derive the string equations of motion with friction taken into account. In the next section the force of friction is calculated in the local rest frame of the string. The general string equations of motion with friction are derived in Sec. III, and the conclusions are briefly stated in Sec. IV.

II. FRICTIONAL FORCE

Let us consider a straight segment of string moving at speed $-\mathbf{v}$ through a gas of massless particles in radiation era. It is convenient to do the calculation in the rest frame of the string, where the gas is moving with velocity $\mathbf{v}$ and the phase-space distribution function of the particles is given by

$$n_{\mathbf{k}}=n[\gamma(k-\mathbf{k}\cdot\mathbf{v})/T_0] . \tag{2.1}$$

Here, $\mathbf{k}$ is the particle momentum, $k=|\mathbf{k}|$, $\gamma=(1-v^2)^{-1/2}$, T_0 is the temperature (in the frame of the gas), and $n(k/T)$ is a Fermi or Bose distribution function. The force per unit length of string can be written as

$$\mathbf{F}=\int \frac{d^3k}{(2\pi)^3} n_{\mathbf{k}} \frac{q}{k}\mathbf{k}\int_{-\pi}^{\pi} d\theta \frac{d\sigma}{d\theta}(1-\cos\theta) . \tag{2.2}$$

Here, q is the particle momentum in the plane perpendicular to the string, $n_{\mathbf{k}}(q/k)d^3k/(2\pi)^3$ is the flux of particles with momenta in the interval d^3k, $\mathbf{k}(1-\cos\theta)$ is the momentum transferred to the string by a particle scattered by an angle θ, and $d\sigma/d\theta$ is the differential cross section of scattering per unit length of string.

The cross section $d\sigma/d\theta$ depends on the type of interaction of particles with the string. Everett[5] studied the scattering of particles belonging to a multiplet ψ_a with a mass matrix $M^2_{ab}(\theta)$ changing around the string. He found that the low-energy scattering cross section for the light members of the multiplet is

$$\frac{d\sigma}{d\theta}=\frac{\pi}{2q[\ln(q\delta)]^2} , \tag{2.3}$$

where, like before, q is the transverse momentum of the particle and δ is the string thickness. A different and potentially more important effect is the Aharonov-Bohm-type interaction of charged particles with the pure gauge field outside the string.[6–8] The phase change experienced by a particle as it is transported around the string is

$$2\pi\nu=e\Phi , \tag{2.4}$$

where e is the particle charge relative to the gauge field of the string and Φ is the string magnetic flux. Aharonov-Bohm scattering is present when ν has a noninteger value. The corresponding cross section is

$$\frac{d\sigma}{d\theta}=\frac{\sin^2(\pi\nu)}{2\pi q\sin^2(\theta/2)} . \tag{2.5}$$

Alford and Wilczek[8] gave an example of a realistic SO (10) model in which light fermions have noninteger values of ν (e.g., $\nu=\frac{1}{4}$ for electrons and $\nu=\frac{1}{2}$ for electron neutrinos). This appears to be a generic phenomenon. We shall assume ν to be a noninteger (at least for some particles) and neglect the effect of (2.3), which is suppressed by a large logarithmic factor. Then the angular integration in (2.2) gives

$$\int_{-\pi}^{\pi} d\theta\frac{d\sigma}{d\theta}(1-\cos\theta)=2q^{-1}\sin^2(\pi\nu) , \tag{2.6}$$

and Eq. (2.2) takes the form

$$F=2\sin^2(\pi\nu)\int_0^\infty \frac{k^2dk}{(2\pi)^3} \times\int_0^\pi d\xi\sin\xi\cos\xi\, n\left[\frac{\gamma k(1-v\cos\xi)}{T_0}\right] , \tag{2.7}$$

where ξ is the angle between $\mathbf{k}$ and $\mathbf{v}$.

The integration over k is easily done using the relation

$$n_0=\int\frac{d^3k}{(2\pi)^3}n\left[\frac{k}{T_0}\right]=b\pi^{-2}\zeta(3)T_0^3\ , \tag{2.8}$$

where n_0 is the number density of particles in the rest frame of the gas, $b=\frac{3}{4}$ for fermions and $b=1$ for bosons, and we have assumed that particles have a zero chemical potential. The remaining angular integration is also easily performed; the result is

$$\mathbf{F}=2\sin^2(\pi\nu)n_0\mathbf{v}(1-v^2)^{-1/2}\ . \tag{2.9}$$

Adding up the contributions of different particles, we obtain

$$\mathbf{F}=\beta T_0^3\mathbf{v}(1-v^2)^{-1/2}\ , \tag{2.10}$$

where

$$\beta=2\pi^{-2}\zeta(3)\sum_a b_a\sin^2(\pi\nu_a) \tag{2.11}$$

and the summation is over the spin states of light particles ($m\ll T_0$). Equation (2.10) agrees with an order-of-magnitude estimate obtained earlier by Everett.[5]

III. STRING EQUATIONS OF MOTION WITH FRICTION

The world history of a string can be represented by two-dimensional surface in spacetime,

$$x^\mu=x^\mu(\zeta^a)\ , \tag{3.1}$$

which is called the string world sheet. Here, ζ^a with $a=0,1$ are two arbitrary parameters on the surface. In the absence of friction, the string dynamics is described by the Nambu action

$$S=-\mu\int d^2\zeta\sqrt{-\gamma}\ , \tag{3.2}$$

where $\gamma=\det(\gamma_{ab})$,

$$\gamma_{ab}=g_{\mu\nu}x^\mu{}_{,a}x^\nu{}_{,b} \tag{3.3}$$

is the two-dimensional world sheet metric, and $g_{\mu\nu}$ is the four-dimensional space-time metric. Variation of (3.2) with respect to $x^\mu(\zeta)$ gives

$$x^\mu{}_{,a}{}^{;a}+\Gamma^\mu_{\nu\sigma}\gamma^{ab}x^\nu{}_{,a}x^\sigma{}_{,b}=0\ , \tag{3.4}$$

where $\Gamma^\mu_{\nu\sigma}$ is the four-dimensional Christoffel symbol. From the two-dimensional point of view, $x^\mu(\zeta)$ is a set of four scalar fields, and the covariant Laplacian in Eq. (3.4) is given by

$$x^\mu{}_{,a}{}^{;a}=\frac{1}{\sqrt{-\gamma}}\partial_a(\sqrt{-\gamma}\gamma^{ab}x^\mu{}_{,b})\ . \tag{3.5}$$

Let us now see how the frictional force (2.20) modifies the string equations of motion. We shall start with motion of strings in flat spacetime with the Minkowski metric. The general form of string equations of motion is

$$\mu x^\nu{}_{,a}{}^{;a}=F^\nu \tag{3.6}$$

where F^ν is a four-vector representing the force of friction. F^ν should be expressed in terms of $x^\nu{}_{,a}$, temperature T_0, and the four-velocity of radiation u^ν. In the local rest frame of the string, the spatial components of F^ν should be given by (2.10),

$$\mathbf{F}=\beta T_0^3\mathbf{u}\ , \tag{3.7}$$

with $\mathbf{u}=\mathbf{v}(1-v^2)^{-1/2}$, while the time component of F^ν, representing the rate of energy change, should be equal to zero:

$$F^0=0\ . \tag{3.8}$$

To complete the derivation, we have to find a covariant expression for F^ν which reduces to (3.7); (3.8) in the string rest frame. This is not difficult to do:

$$F^\nu=\beta T_0^3(u^\nu-x^\nu{}_{,a}x^{\sigma,a}u_\sigma)\ . \tag{3.9}$$

To verify that (3.9) is the desired expression, we can locally introduce world-sheet coordinates such that $t=\zeta^0$ and $\mathbf{x}=\mathbf{n}\zeta^1$, where $\mathbf{n}$ is a unit vector in the direction of the string. Then $\gamma_{ab}=\eta_{ab}$ and

$$\mathbf{F}=\beta T_0^3[\mathbf{u}-\mathbf{n}(\mathbf{n}\cdot\mathbf{u})]=\beta T_0^3\mathbf{u}\ ,$$

$$F^0=0\ .$$

Substitution of (3.9) into (3.6) gives

$$x^\nu{}_{,a}{}^{;a}=(\beta T_0^3/\mu)(u^\nu-x^\nu{}_{,a}x^{\sigma,a}u_\sigma)\ . \tag{3.10}$$

A generalization of this equation to arbitrary curved spacetime is

$$x^\nu{}_{,a}{}^{;a}+\Gamma^\nu_{\sigma\tau}x^\sigma{}_{,a}x^{\tau,a}=(\beta T_0^3/\mu)(u^\nu-x^\nu{}_{,a}x^{\sigma,a}u_\sigma)\ . \tag{3.11}$$

We are particularly interested in the dynamics of strings in a Robertson-Walker universe:

$$ds^2=a^2(\tau)(d\tau^2-d\mathbf{x}^2)\ , \tag{3.12}$$

with the four-velocity of radiation given by

$$u^\mu=(a^{-1},0,0,0)\ . \tag{3.13}$$

The string equations of motion (3.11) considerably simplify if we choose the gauge in which

$$\zeta^0=\tau,\quad \dot{\mathbf{x}}\cdot\mathbf{x}'=0\ . \tag{3.14}$$

With this choice, the spatial components of (3.11) take the form

$$\ddot{\mathbf{x}}-\varepsilon^{-1}(\mathbf{x}'/\varepsilon)'+\left[2\frac{\dot a}{a}+\frac{\beta T_0^3}{\mu}a\right]\dot{\mathbf{x}}=0\ , \tag{3.15}$$

and the time component becomes

$$\dot\varepsilon+\left[2\frac{\dot a}{a}+\frac{\beta T_0^3}{\mu}a\right]\dot{\mathbf{x}}^2\varepsilon=0\ . \tag{3.16}$$

Here,

$$\varepsilon=\left[\frac{x'2}{1-\dot{\mathbf{x}}^2}\right]^{1/2} \tag{3.17}$$

and overdots and primes stand for derivatives with respect to ζ^0 and ζ^1, respectively. Equation (3.16) is not an independent equation; it follows from (3.15) and (3.17). We see that, somewhat surprisingly, the only difference introduced by the force of friction is to replace the factors $2\dot{a}/a$ in equations of motion for a free string in the expanding Universe[1] by the factor

$$[2(\dot{a}/a)+(\beta T_0^3/\mu)a] \ . \tag{3.18}$$

The importance of the force of friction is determined by the relative magnitude of the second term in (3.18):

$$r=\frac{\beta T_0^3}{2\mu\dot{a}/a^2} \ . \tag{3.19}$$

Disregarding numerical factors and using Einstein's equation,

$$\left[\frac{\dot{a}}{a^2}\right]^2=\frac{8\pi G}{3}\rho\sim G T_0^4 \ , \tag{3.20}$$

we find that friction is negligible ($r\ll 1$) when $T_0 \ll G\mu m_P$ or

$$t \gg (G\mu)^{-2} t_P \ , \tag{3.21}$$

in agreement with previous estimates.[1] Here, t_P and m_P are Planck time and mass, respectively, and t is the cosmic time, which is related to the conformal time τ by $dt=a\,d\tau$.

IV. CONCLUSIONS

The main result of this paper is Eq. (3.15), which is the equation of motion for a string in a Robertson-Walker universe (3.12) with friction taken into account. A remarkable fact about this equation is that it is identical to a free string equation of motion with a factor $2\dot{a}/a$ replaced by the factor

$$[2(\dot{a}/a)+(\beta T_0^3/\mu)a] \ .$$

As a result, computer programs developed to study the free string evolution in the expanding Universe require only a trivial modification to include the effects of friction.

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