Importance of boundary conditions for topological production of textures and Skyrmions

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The Kibble mechanism has recently been used to estimate the production of topological objects such as Skyrmions in hadronic events and textures in the context of the Universe. These objects correspond to a nontrivial third homotopy group of an appropriate group manifold. We discuss the importance of boundary conditions required for a topological description of such objects. These considerations show that textures do not have any topological meaning in the context of a homogeneous and isotropic universe and actually are homotopic to trivial configurations (as long as the universe is much bigger than the horizon size). We point out (and as has been noted by others) that a scale-invariant distribution of density fluctuations is expected for the spontaneous breaking of any global symmetry as long as the connected component of the vacuum manifold is degenerate. It does not matter whether the vacuum manifold has any nontrivial homotopy group. For the case of Skyrmion production in hadronic events, our considerations lead to a strong suppression of the Skyrmion production. A recent numerical simulation of the texture formation found that textures rarely occur. Our results provide a simple explanation of this result.

I. INTRODUCTION

Production of topological objects in various physical systems has been of great interest from theoretical as well as experimental points of view. There are numerous examples of such objects in condensed-matter systems such as flux tubes in type-II superconductors, vortices in superfluid helium, and point defects in certain liquid-crystal systems. In the context of particle physics, familiar examples are domain walls, cosmic strings, and monopoles, which are expected to form in a phase transition in the early Universe if the vacuum manifold of the Higgs field has, respectively, zeroth, first, or second homotopy groups which are nontrivial. In the effective-Lagrangian approach to QCD it has been shown that baryons can be thought of as certain topological objects, called Skyrmions, which owe their existence to the third homotopy group (of an appropriate group manifold) being nontrivial.[1] Similar objects corresponding to the third homotopy group have also been discussed in the context of the Universe under the name texture.[2–6]

Production of domain walls, cosmic strings, and monopoles in the early universe has been extensively discussed[7] in the framework of the Kibble mechanism.[2] Davis[3] studied the case when the vacuum manifold has a nontrivial third homotopy group. He considered the case when our Universe is topologically S^3 and discussed the formation of textures. The study of textures has recently been revived by Turok[4] who considered the case when textures may form in a given region enclosing certain horizon volumes. He studied the evolution of a texture with asymptotically fixed boundary conditions and found that the texture collapses with the speed of light.[4] The implications for the density fluctuations and anisotropy in microwave-background radiation have been further investigated.[4–6]

Another interesting application of the Kibble mechanism has been recently proposed by Ellis and Kowalski *et al.*[8,9] who use it to estimate the production of baryons in jet events[8] and in quark-gluon plasma.[9] A baryon is viewed as a Skyrmion which is a soliton in the pion condensate field U valued in the group SU(2) for the two-flavor case. The region under consideration (for jet events this region is the volume of the jet containing initial partons) is divided in small cells of size roughly 1 fm^3 each and arbitrary directions for the U field are assumed in each cell by assuming that the hadronization in these cells happens independently. An algorithm[10] is then used to estimate if a given region encloses a configuration which has a nontrivial winding number in the group SU(2), identifying such regions with baryons.

When the Kibble mechanism is applied to estimate the production of strings or monopoles in space R^3, one considers respectively a circle S^1 or a two-sphere S^2 embedded in R^3. By considering field configurations on S^1 (or on S^2), one explicitly constructs various winding-number maps associated with the corresponding homotopy groups of the vacuum manifold (first homotopy group for the case of strings and second homotopy group for monopoles). The situation is quite different if we are interested in configurations corresponding to the third homotopy group of the vacuum manifold. A three-sphere S^3 can be embedded in R^4 but not in R^3, which is our physical space. The only way to construct an explicit representation of a nontrivial winding-number map in R^3 corresponding to the third homotopy group is by viewing R^3 as a stereographic projection of an S^3. Since in such a projection the north pole of the S^3 is mapped to the infinity of R^3, this representation is meaningful only by implementation of appropriate boundary conditions.

In the context of the Universe, for texture formation, and in the interior of a hadronic jet, for the topological production of baryons, this requirement of boundary conditions poses important questions. We address these questions in this paper.

We first describe, in Sec. II, the Kibble mechanism to estimate the production of domain walls, strings, and monopoles in the early Universe. The scenario of Turok[4] for the formation of textures is discussed in Sec III. We argue that due to the absence of any asymptotic boundary conditions in the context of the Universe (for the case when the universe is much bigger than the horizon scale) textures do not have any topologically invariant meaning in the sense that they are homotopic to trivial configurations. We point out that scale-invariant distribution of density fluctuations will be expected for the spontaneous breaking of any global symmetry as long as the connected component of the vacuum manifold is degenerate. It does not matter whether the vacuum manifold has any nonvanishing homotopy group. Such conclusions were first obtained by Press.[11]

We next consider, in Sec. IV, the topological mechanism for the baryon production in jet events proposed by Ellis *et al.*[8] In this case we show that the considerations of suitable boundary conditions lead to a strong suppression (by a factor $\sim 10^{-7}$) in the production probabilities, making it virtually impossible to find an exactly integer winding-number Skyrmion configuration.

In order to identify configurations which may later evolve into integer-winging-number configurations, we relax these requirements of boundary conditions (needed for integer-winding-number configurations) and consider, in Sec. IV B, the case when a field configuration in a given region closely approximates a Skyrmion configuration, even though it may have only fractional winding number. Such a configuration may be able to separate out as an integer-winding-number Skyrmion configuration. We find that such a configuration necessarily extends over at least four elementary tetrahedra in R^3 (as opposed to only one needed in Ref. 8). The production probability of such configurations in that region is about 0.01 (when all Skyrmions are to have the same value of the field at their centers, the case relevant for hadronic events) and is about 0.05 (when different Skyrmions can have different value of the field at their centers). This leads to the production probability per tetrahedron ~ 0.003 which is greatly suppressed as compared to the value $\frac{1}{16}$ as obtained in Ref. 8.

After a preliminary version of this paper was written, we became aware of a paper by Spergel, Turok, Press, and Ryden[6] who have done a numerical simulation of texture production and evolution in the Universe. These results show that any texture with winding number $\geq \frac{1}{2}$ collapses. The formation probability of such configurations per horizon volume at the time of their collapse was numerically found to be about 0.04. Thus, as far as one is concerned with the number of collapsing textures, there is no reason to single out configurations with integer winding number (i.e., those with constant boundary conditions outside a given region). (It is important to note here that integer-winding-number textures are of importance for the model of generation of baryons discussed by Turok and Zadrozny, Ref. 5.) In fact these textures with fractional winding number are similar to fractional-winding-number Skyrmions (with no special value of the field at the center) as discussed in Sec. IV B. As we have mentioned above, we find in Sec. IV B that the formation probability of such configurations is about 0.05, thus providing an analytical understanding of the numerical results of Spergel *et al.*[6] We further predict almost an absence of any integer-winding-number texture configurations which are suppressed at least by a factor of 10^{-7} or so.

II. FORMATION OF TOPOLOGICAL OBJECTS AND KIBBLE MECHANISM

Topological objects are generally associated with certain phase transitions, where the order parameter space (the vacuum manifold) has nontrivial topology. We will limit our considerations to fields defined in flat space R^3 (for topologically nontrivial spaces we consider small regions which are topologically equivalent to a ball B^3). In this context we should mention that Davis has considered the case when there is only one texture in the whole Universe S^3 (Ref. 3). Since in any horizon volume the field will tend to be uniform to minimize the energy, such a configuration is physically interesting only when the horizon is of the same size as the Universe (which was the case in Ref. 3). We here consider the case when the horizon is much smaller than the Universe.

Let us first consider the case when the order parameter space has nontrivial first homotopy group. This means that it contains loops which cannot be smoothly contracted to a point. For any space of dimension ≥ 2 we can consider loops embedded in the space and check if the field varies along a nontrivial loop in the vacuum manifold as we trace this loop. If it does, then the loop encloses a string. Similar considerations tell us that to get domain walls and monopoles we have to embed respectively, two points and S^2 in R^3.

Now if the vacuum manifold has a nontrivial third homotopy group (so there are S^3's in the vacuum manifold which cannot be contracted to a point) then in order to form a topological object in the above discussed manner we will need to consider an S^3 embedded in the space which is not possible unless number of spatial dimensions ≥ 4.

As we know, one usually prescribes certain *fixed* boundary conditions on the fields at large distances which lead to the compactification of R^3 into S^3 (equivalently, R^3 is regarded as a stereographic projection of S^3 with the north pole of S^3 mapped to the infinity of R^3). Solitonic configurations are then constructed by the usual winding-number maps from S^3 to S^3. This is what happens for the case of Skyrmions where the finiteness of energy forces the boundary condition that the pion condensate field U go to 1 at large distances. Since there is no need for fixing any such boundary conditions (and identifying a suitable asymptotic region) for the formation of other topological objects like domain walls, strings, and monopoles, there is a qualitative difference between their production and the production of Skyrmions and textures.

III. TEXTURE FORMATION

Let us now see if we can apply the Kibble mechanism to the case of textures in the early Universe. Textures arise when the third homotopy group of the vacuum manifold is nontrivial. However, textures in the context of the early Universe differ from (say) Skyrmions in the laboratory in one very important aspect. Whereas in the context of Skyrmions in the laboratory there is a well-defined asymptotic region where the field must go to a constant value to maintain the finiteness of energy, there is no such region in the context of the Universe. In fact any such condition will be inconsistent with a homogeneous and isotropic universe.

In order to apply the Kibble mechanism, we divide the space into elementary cells (let us take these elementary cells to be the horizon volumes) where all horizon volumes are equivalent and the field varies randomly from one cell to another. We can consider a few horizon volumes adjacent to each other and then calculate if there is a texture in this region. A region comprising a few adjacent horizon volumes will be topologically equivalent to a ball B^3 (since there is no reason to expect that at the surface of the ball B^3 the field goes to a constant value).

As we know, there is no topologically nontrivial map from B^3 to S^3. Any field configuration on B^3 is homotopic to any other field configuration on B^3. Thus texture configurations in the context of the Universe have no topologically invariant meaning and are actually homotopic to trivial configurations (except for the case when the horizon is about the same size as the universe S^3, then a collection of horizon volumes may actually comprise an S^3 instead of B^3). This is different from the unstable nature of texture discussed in Ref. 4 where a texture could change its winding number only by going over the top of the potential barrier when it has collapsed to extremely small size. We find that, even without collapsing, a texture can smoothly change to a trivial configuration by a suitable evolution of the field at the boundary of the texture. Of course it may happen, due to random variations of the field, that at the surface of some ball B^3 the field happens to have a constant value. The field configuration on such a ball will have integer winding number. However, as shown in Sec. IV, the probability of such configurations is at most 10^{-7}. (We should mention here that in Ref. 4 the probability of a full knot was estimated to be about $\frac{1}{25}$, though it is not clear what a full knot means in the absence of constant boundary conditions.) Thus the random occurrence of such configurations is highly suppressed. Also, there is no reason to expect that as the field smoothens out, it will maintain such a boundary condition. In general, the field at different points of the surface of B^3 will change depending upon the value of the field at neighboring points and after a little bit of smoothening, there may not remain any surface in that region such that the field has some constant value on that surface. Thus all the field configurations in space (now some of them may have fractional winding number on the ball B^3 under consideration) have equal topological significance.

An important point brought out in Ref. 4 was that there will be density fluctuations associated with the formation (and later collapse) of global textures. One will still expect this feature of course now for any spatially varying field configuration. In fact the density fluctuation will always be present whenever any global symmetry is spontaneously broken as long as the vacuum manifold is degenerate; it does not matter whether it has any nonvanishing homotopy group. (This should be clear from the consideration of nontopological textures as discussed by Turok,[4] see also Ref. 6. Similar conclusions were obtained earlier by Press.[11]) We illustrate this point in the following by considering a case of U(1) symmetry breaking.

Consider a model in which a global U(1) symmetry is spontaneously broken such that the vacuum manifold for the Higgs field is U(1). In any causally connected region the Higgs field will tend to have the same phase in order to minimize the gradient energy but the phases will vary randomly for points separated by distances much larger than the horizon scale. [Note that this model has global strings and in general a horizon volume may be expected to have a string stretching across it (see Ref. 7). However, these strings are not relevant for the point we want to make, which is to only consider the degeneracy of the vacuum manifold irrespective of its topology. It may be that the density fluctuations we get in this specific model can be associated with the formation of strings (after all string formation is also calculated by assuming random variation of field from one horizon volume to the next).]

Now consider a region of space at some time t, which is just entering into the horizon. On the average, the phase change (due to random variation of the phase of the Higgs field Φ) across the region will be $\pi/2$. This is because if the phase at one edge of the region is θ then the largest phase variation will correspond to the phase at the other edge being $\theta+\pi$, whereas the smallest variation is zero, thus giving average variation to be $\pi/2$. Gradient energy density coming from a term such as $(\nabla\Phi)^2$ in that region from this phase variation will then be

$$\delta\rho \simeq \text{const} \times \frac{\pi^2}{4c^2t^2} , \tag{3.1}$$

where the const contains the parameters of the Lagrangian and ct gives the horizon length at time t. Since the background energy density of the Universe decreases as $\rho_b \sim t^{-2}$ we get

$$\frac{\delta\rho}{\rho_b} = \text{const} . \tag{3.2}$$

We thus get a scale-invariant distribution of the density fluctuations. In fact this conclusion holds for any degenerate connected vacuum manifold M. One just needs to replace $\pi^2/4$ in Eq. (3.1) by d^2 where d is the expected separation between two points of M. This only changes the amplitude of the distribution of density fluctuations in Eq. (3.2) without affecting its scale-invariant character.

IV. TOPOLOGICAL PRODUCTION OF SKYRMIONS

The situation is somewhat different for the case of Skyrmion production in jets (and in quark-gluon plasma) studied by Ellis *et al.*[8,9] There, since one is attempting to estimate the production of baryons, one has to consider those configurations which give an integer winding number and hence can be identified with Skyrmion configurations. Any other field configuration which is varying in space must be thought of as (most probably) ultimately decaying into pions. In the following, we briefly recall the model discussed by Ellis *et al.*[8]

A. Skyrmion production and Kibble mechanism

In jet events the initial partons are contained in roughly cylindrical shaped regions. By assuming that the hadronization in various portions of a jet happens independently and by assuming that the pion condensate field U [valued in group SU(2) for the two-flavor case] assumes arbitrary values in the group manifold in these far portions of the jet, one can map this picture to the one employed in the Kibble mechanism. One divides the region in the jet into small elementary cells (of size roughly 1 fm^3; see Ref. 8) and then attaches random values of U at the vertices of these cells. Skyrmions are topological defects in the configurations of pion condensate field U which arise due to the fact that the third homotopy group of SU(2) is nontrivial. One then attempts to find, using a generalization of the Kibble mechanism, if there are Skyrmions (which are identified with baryons) in any given region.[8] (It should be clear from this construction that the problem of formation of Skyrmions in the interior of a hadronic jet is identical to the problem of formation of textures in the Universe.)

This generalization of the Kibble mechanism was achieved in Ref. 8 by considering the three-dimensional lattice (obtained by dividing the region of the jet in elementary cells) as part of a hypercubic lattice in four-dimensional space. Values of the U field were then randomly attached to the vertices of this hypercubic lattice (by suitably discretizing the vacuum manifold S^3 using five points) and by considering surfaces (homotopic to S^3) embedded in this R^4, the distribution of Skyrmions was calculated. A certain algorithm[10] was employed to calculate the winding number of a given field configuration in a region. It was emphasized in Ref. 8 that the fourth dimension is hypothetical and certain conditions were derived from this (leading to correlations between the production of baryons and antibaryons).

Even though this fourth dimension is introduced only for mathematical simplicity in Ref. 8, it plays an absolutely crucial role from the point of view of topology. Thus if we do not assume the existence of such a fourth dimension, we realize that any region inside the jet consisting of few elementary cells is homotopic to a ball B^3 and we are again considering the mapping of B^3 into S^3, all such mappings being topologically trivial. A simple way to illustrate this point is by going to two spatial dimensions and trying to construct a two-dimensional Skyrmion configuration. The field is now valued in the vacuum manifold S^2. If we consider a third (hypothetical) dimension and embed an S^2 in this R^3, we can construct a Skyrmion as shown in Fig. 1(a). Four distinct values of the field, U_1, U_2, U_3, and U_4, are attached at four points of S^2 (in order to triangulate S^2 one needs four points). The x-y plane is the physical space and the z axis represents the hypothetical third dimension. However, from this one cannot conclude that we have a Skyrmion in the physical space because, as Figs. 1(b) and 1(c) show, in physical space R^2 one can simply deform the field to a constant value (say) U_1. Here we may mention that the algorithm employed in Ref. 10 to compute instanton numbers for S^3 embedded in R^4 is perfectly fine since an instanton is a topological configuration in four-dimensional physical space-time R^4.

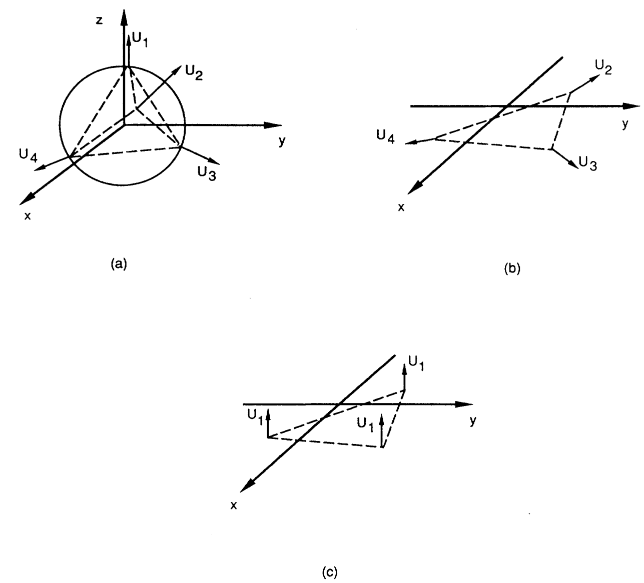

FIG. 1. (a) A winding-number-1 Skyrmion configuration (shown here by U_1, U_2, U_3, and U_4) on S^2 embedded in R^3. The x-y plane is the two-dimensional physical space and the z axis represents the hypothetical third dimension. (b) The field in the physical two-dimensional space. (c) The field in Fig. 1 (b) is homotopic to the constant field (arbitrarily chosen to be U_1) shown here.

As we mentioned earlier, Skyrmion configurations can be constructed only with a suitable boundary condition. Figure 2 shows a Skyrmion in two spatial dimensions, with the boundary condition that $U=1$ beyond a certain localized region (enclosed by the dashed circle) in R^2. Nontriviality of the field configuration is contained inside the dashed circle (later in Sec. IV B, we will consider such a configuration in more detail). For an isolated Skyrmion such boundary conditions follow from the requirement that the energy be finite. In the case of jet events such boundary conditions are imposed by assuming that U goes to 1 outside the jet (see Ref. 8). However, for very small regions (such as regions comprised of few elementary cells) inside the jet there is no natural way in which one can impose such a boundary condition.

In the absence of any topologically invariant description of a baryon inside the jet and by realizing that any baryon produced in the jet will be strongly coupled with other nearby baryons as well as with other field

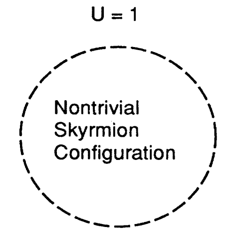

FIG. 2. A localized Skyrmion in two-dimensional space. $U=1$ outside the dashed circle.

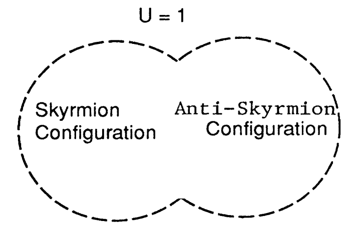

FIG. 3. An interacting Skyrmion–anti-Skyrmion pair. $U=1$ outside the dashed loop.

configurations (which may later decay into pions), it becomes very difficult to identify field configurations which may later separate as baryons. For this purpose we suggest the following modification of the method proposed by Ellis *et al.* in Ref. 8.

First let us attempt to identify those configurations inside the jet which represent single, integer-winding-number Skyrmions. We again divide the three-dimensional region inside the jet into elementary cells which are taken to be tetrahedrons with unit edges (in fm; see Ref. 8). The vacuum manifold S^3 is triangulated using five points and these five values of U are randomly attached at the vertices of the lattice (though we will see in Sec. IV B that, for getting integer winding number, it is more appropriate to consider six points on the vacuum manifold S^3). We want to see if the field configuration on a given tetrahedron corresponds to an integer-winding-number Skyrmion configuration. Consider a surface enclosing this tetrahedron such that the points on this surface are about a unit distance away from the vertices of the tetrahedron. Such a surface will have at least about 10–12 points which will be separated from each other by a unit distance. (Remember that the directions of U are supposed to be independent for such points. We will not attempt to be more careful about these estimates since we are only interested in order of magnitudes for this case.)

Now consider the probability that all such points happen to have a fixed value (say U_0) attached to them. Since there are five values of U available, this probability is $\simeq(\frac{1}{5})^{10}\simeq 10^{-7}$ which is exceedingly small as compared to the probability of about $\frac{1}{16}$ expected for integer-winding-number Skyrmions in Ref. 8. (Of course this probability then has to be multiplied by the probability that on the enclosed tetrahedron one has the appropriate field configuration for a Skyrmion.) Later, in Sec. IV B, we sill present these considerations in more detail. Note here that we are not requiring that U_0 be the same as the constant value of U outside the jet. At present we want the find any configuration which may have integer winding number.

Of course one may consider configurations which represent a pair (or more) of interacting Skyrmions. As should be clear from Fig. 3 for the case of a Skyrmion–anti-Skyrmion pair in two spatial dimensions (see Ref. 12 for details of such a configuration), U still needs to assume some constant value at a suitable surface enclosing the Skyrmion–anti-Skyrmion pair giving roughly the same suppression in production probability (for the three-dimensional case) as discussed above. The same remains true for the case of say three interacting Skyrmions. For a very large number of interacting Skyrmions there may not be any such surface for the Skyrmions in the interior. But then these configurations may also have no chance of separating out as Skyrmions. In any case, at least from the topological point of view, these kinds of configurations do not seem any different from any other spatially varying field configurations which are generally expected to decay into pions.

It is possible that when some suitable dynamical consideration is employed then certain fractional-winding-number configurations, which in some sense closely approximate Skyrmion configurations, may separate out as integer-winding-number Skyrmion configurations instead of decaying into pions. In the following we will attempt to classify such configurations. We find that the probability of production of such configurations will not be as drastically suppressed as in the above cases but it still is much less than the earlier estimates.[8]

B. Skyrmion production with relaxed boundary conditions

We will first consider the case of Skyrmions in two spatial dimensions as this case is easy to visualize. We will then generalize the arguments to the case of a Skyrmion in three spatial dimensions (here we will restrict our attention to the two-flavor case where the Skyrmion field is valued in the group SU(2) which as a manifold is S^3). In the case of two spatial dimensions the Skyrmion field is valued in a two-sphere S^2. As we have mentioned, the Skyrmion will arise when we impose the boundary conditions so that the Skyrmion field goes to some constant value at the spatial infinity of the two-dimensional space R^2 thereby compactifying R^2 to a two-sphere S^2. (In the following we use a construction employed in Ref. 12.)

Let us choose the coordinates for the (compactified) physical space to be the polar and azimuthal angles θ and ϕ whereas the coordinates on the target manifold S^2 (in which the Skyrmion field is valued) are denoted by polar and azimuthal angles Θ and Φ. Θ and Φ thus give a possible value of the Skyrmion field at some spatial point.

Figure 4(a) shows a winding-number-1 Skyrmion configuration. Dashed circles show contours of constant Θ and dashed lines (emanating from the center) show contours of constant Φ. $\Theta=0$ outside the outermost

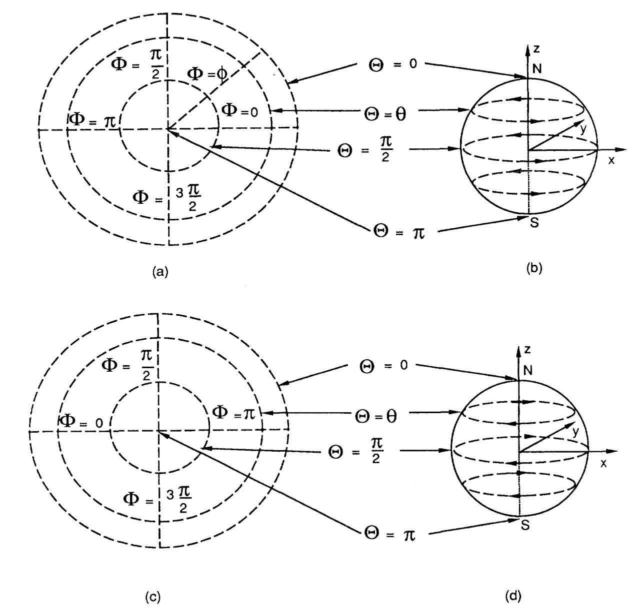

FIG. 4. (a) Shows a winding-number-1 Skyrmion configuration in R^2 while (b) shows the same on S^2 which is obtained by compactifying R^2. (c) and (d) show a winding-number-(-1) anti-Skyrmion configuration. θ and ϕ denote the polar and azimuthal angles on S^2 (the compactified R^2) while Θ and Φ give the value of the field at a given spatial point.

dashed circle (which is consistent with the compactification of R^2 to S^2). Figure 4(b) shows the same Skyrmion now on S^2, the compactified R^2. The whole region outside the outermost dashed circle in Fig. 4(a) corresponds to a neighborhood of the north pole N in Fig. 4(b) with the infinity of Fig. 4(a) corresponding to the north pole N. Figure 4(a) can be thought of as representing the stereographic projection of the S^2 in Fig. 4(b).

Figures 4(a) and 4(b) represent a winding-number-1 Skyrmion as can be seen from the fact that all the dashed circles in Fig. 4(a) [or equivalently the circles parallel to the equator in Fig. 4(b)] represent a winding-number-1 map of S^1 to S^1. We have here used a standard way of constructing the winding number n map of S^P to S^P wherein one starts with a winding number n map of S^{P-1} to S^{P-1}. By considering S^P as the suspension of S^{P-1}, this map is then extended to a map from S^P to S^P which can be shown to be a winding-number-n map.[13] (See also Ref. 12.)

In a similar way, we can construct a winding-number-(-1) anti-Skyrmion as shown in Figs. 4(c) and 4(d) (this will be useful to us when we discuss the baryon-antibaryon correlations). We start with a winding-number-(-1) map of the equatorial circle to S^1 and by considering S^2 as a family of circles parallel to the equatorial one (which shrink to zero size at the north and south poles); that is, by considering S^2 as the suspension of S^1, we extend this map to a winding-number-(-1) map of S^2 to S^2.

With this explicit construction of the Skyrmion configurations, let us now attempt to estimate their production following the Kibble mechanism. As we have mentioned, in the context of the Kibble mechanism (either for the interior of the jet in hadronic events or in the context of the Universe) it is not possible to force any boundary condition on the Skyrmion field. Let us make the standard assumption that the field between two points separated by the correlation length η varies along the shortest path in the vacuum manifold. We then see from Fig. 4(a) that, in order to have an integer-winding-number Skyrmion configuration, we need to specify field at least at the following points.

(a) At the center we must have $\Theta=\pi$. (Of course any point of the vacuum manifold could be chosen to be at the center. The value at this point governs the requirement for the field at other points.)

(b) On a circle enclosing this center, we must have some value of Θ between 0 and π and Φ must vary by winding number 1 around this circle (for a Skyrmion; for an anti-Skyrmion, Φ must vary by winding number -1 on this circle). This circle may be triangulated by using three points all of which are a correlation length (η) away from the center.

(c) We finally need an outer circle on which Θ must be zero. Now in order that we use the Kibble mechanism consistently, this outer circle must be about a correlation length away from the three points on the inner circle in item (b) above. This outer circle will have at least about six points (that is if we take it to have a diameter of about 2η) which are separated from each other by distance $\sim\eta$. The requirement the Θ be zero on these points gives a suppression factor of about $(\frac{1}{5})^6$. Note that we here need to specify five points in the vacuum manifold S^2 ($\Theta=\pi$, which is the south pole, $\Theta=0$ which is the north pole, and three points needed to triangulate a circle corresponding to some intermediate value of Θ) to get an integer winding instead of just four needed for the case of monopoles embedded in R^3.

The above discussion clearly shows the difference between the application of Kibble mechanism to the case of topological defects such as monopoles and the case of objects such as Skyrmions. For the latter class of objects one needs the information about the field at a much larger number of points than what one may naively expect. This is what gives rise to strong suppression in the production probability of such objects.

The generalization to the three-dimensional case is then immediate. We consider a winding-number-1 map of S^2 to S^2 such as the one shown in Fig. 4(b). We then consider a family of concentric two-spheres, S^2's, all of which correspond to winding-number-1 map of S^2 to S^2. These S^2's will be labeled by a third angle χ between 0 and π. The innermost sphere which corresponds to (say) $\chi=\pi$ is shrunk to a point at the origin (south pole of the S^3) whereas the outermost S^2 corresponds to $\chi=0$ (the north pole of the S^3) and leads to the compactification of R^3 to S^3. It can be easily seen from our discussion of the two-dimensional case (also see Ref. 13) that this leads to a winding-number-1 map of the (compactified) space S^3 (seen here as the suspension of S^2) to the target manifold S^3.

It is then clear in this three-dimensional case that in order to have an integer winding-number configuration, one must have (a) some fixed value at the center (say $\chi=\pi$), (b) there should be a winding-number-1 mapping on some S^2 (corresponding to some value of χ other than 0 and π), and finally (c) on a larger S^2 we should have $\chi=0$. As we have argued earlier, this final requirement leads to the drastic suppression in the production probability. We note here again that in order to have an integer-winding-number Skyrmion configuration, we need to consider six distinct points on the vacuum manifold S^3 ($\chi=\pi$, $\chi=0$, and four points needed to triangulate an S^2 corresponding to some intermediate value of χ). This is in contrast with the naive expectation of considering five points on S^3.

Now let us relax the requirement of an integer-winding-number configuration. Instead, we will attempt to find those (fractional winding number) field configurations which may separate out later as integer-winding-number configurations (given a suitable evolution of field at the boundary of the initial configuration). Consider again the two-dimensional case first. Suppose that we have some fixed value of the field (say $\Theta=\pi$) at the origin and winding number 1 for Φ variation on a circle enclosing the origin. Let this circle correspond to $\Theta=\theta$ where θ has some value other than 0 and π. Such a configuration does not wind around the S^2 completely and if we integrate a suitable winding-number density over this region, we will get a fractional number.

We first observe that given such a configuration (which represents the core of a Skyrmion) there is a large possibility that the field configuration in the neighboring region corresponds to the core of an anti-Skyrmion (as opposed to that of another Skyrmion). This is most clearly seen by comparing Figs. 4(a) and 4(c) which show that the variation of Φ in some segment of a circle in Fig. 4(a) actually corresponds to the Φ variation required on a segment of a circle in a nearby anti-Skyrmion configuration. (This is happening because, when viewed from a point inside the $\Theta=\theta$ circle in the Skyrmion configuration, Φ varies in an anticlockwise direction, whereas from a point outside this circle, the Φ variation is clockwise.) On the other hand if we want a Skyrmion configuration near the configuration in Fig. 4(a) then one will need to specify field at extra set of points to reverse the direction of Φ variation as one crosses the region of one Skyrmion into the region of the other Skyrmion. It is easy to see that the same argument holds for Skyrmions in three dimensions. Thus we are much more likely to find Skyrmion–anti-Skyrmion pairs as opposed to finding Skyrmion-Skyrmion pairs. Such a strong correlation between the production of Skyrmions and anti-Skyrmions was also obtained by Ellis *et al.*[8] though, as mentioned earlier, their treatment of the Skyrmion production differs from ours.

The question of the final evolution of the above configuration (representing the core of the Skyrmion) will crucially depend upon the field configuration in neighboring regions. We now assume that these neighboring configurations resemble portions of anti-Skyrmions. (as we have argued above it is quite likely to be so.) Then, it can be seen from Fig. 4(a) that, if $\theta<\pi/2$, then as the regions of partial Skyrmion and partial anti-Skyrmion separate, the field in between will tend to assume the value $\Theta=\pi$ (or some other constant value in the lower hemisphere to minimize the Φ variations). Combined with our assumption of the variation of the field along the shortest path in the vacuum manifold, we see that this does not lead to any nontrivial integer-winding-number configuration.

Now consider the case when $\theta>\pi/2$. Then above considerations show that the field in the intermediate regions will tend to assume a value $\Theta=0$ (or some other constant value in the upper hemisphere). Again the shortest path variation of the field shows that in this case we will get an isolated integer-winding-number Skyrmion. (We will not worry about the questions such as how many anti-Skyrmions can separate out this way since we are only attempting to find very rough criteria for which configurations have a good chance of separating as integer-winding-number Skyrmions.)

It is then easy to implement the above criteria for the Kibble mechanism. We triangulate the physical space (R^2, in this two-dimensional case) in terms of triangles such that the vertices are separated by η [see Fig. 5(a)]. Let us take $\Theta=\pi$ at some vertex. There are four neighboring vertices to this point and we require that, as we go full circle along these vertices, Φ changes by 2π and the corresponding value of Θ remain in the upper hemisphere. We see that this corresponds to the standard triangulation of the vacuum manifold S^2 using four points U_1, U_2, U_3, and U_4 as shown in Fig. 5(b). The thing which is new in this case is that to get even a fractional-

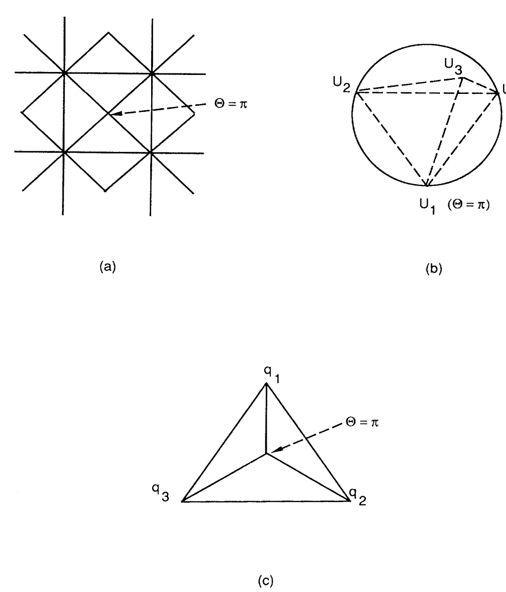

FIG. 5. (a) Triangulation of the two-dimensional space R^2. (b) Triangulation of the vacuum manifold S^2 by four points. With U_1 corresponding to $\Theta=\pi$, U_2, U_3, and U_4 correspond to $\Theta<\pi/2$. (c) Smallest region in R^2 needed to specify the two-dimensional (fractional winding number) Skyrmion configuration.

winding-number configuration (which may have some chance of later separating as an integer-winding-number one, of course depending on the neighboring configurations) we need to consider four elementary cells (triangles in this case) rather than just one. One may consider using only three adjacent triangles as shown in Fig. 5(c) since we are just trying to triangulate a circle enclosing the vertex with $\Theta=\pi$ and triangulating a circle requires only three points. Even though this is not consistent with the triangulation of full space R^2 using only one correlation length scale, we may still do this in order to get a sort of upper limit for the value of the production probability.

Using Figs. 5(b) and 5(c) we see that the probability of a fractional winding Skyrmion configuration in the region enclosed by points q_1, q_2, and q_3 with a specific value $\Theta=\pi$ at the center is (this type of situation will be relevant for the jet events where the Skyrmion field assumes some specific constant value, assumed to be $\Theta=0$ here, outside the jet; Skyrmion configurations then should have $\Theta=\pi$ at their centers)

$$P_1=(\tfrac{1}{4})(\tfrac{3}{4})^3(\tfrac{1}{4})=0.026 , \tag{4.1}$$

where the first $\frac{1}{4}$ factor comes from requiring that $\Theta=\pi$ at the center and $(\frac{3}{4})^3$ comes from requiring that field at none of q_i's is equal to $\Theta=\pi$. The last $\frac{1}{4}$ comes from calculating the probability that Φ changes by 2π around a circle on which points q_1, q_2, and q_3 lie (known from the string case, see Ref. 8), with the field restricted to the three points in the upper hemisphere of the vacuum manifold [see Fig. 5(b)].

Now suppose we do not require any specific value of the field at the center (which will be relevant for the case of the texture production in the Universe) then we get the probability to be

$$P_2=4P_1\simeq 0.1 . \tag{4.2}$$

We again emphasize here that these probabilities correspond to getting a (fractional winding number) Skyrmion configuration extended over at least three elementary (triangular) cells.

We can again generalize all this easily for the three-dimensional case. We consider the standard triangulation of the vacuum manifold S^3 using five points, one point being the south pole with $\chi=\pi$ (for the case when $\chi=0$ outside the jet; for the case of textures the choice of this point is arbitrary). The other four points triangulate the S^2 corresponding to some value of $\chi<\pi/2$. Consider a point in space with $\chi=\pi$ and enclose it with a tetrahedron (more appropriately with a cube which will be composed of elementary tetrahedron cells. We, however, follow our earlier approach used in the two-dimensional case and use only a tetrahedron to triangulate the S^2). The analog of P_1 (with $\chi=\pi$ at the center, appropriate for the jet events) is then

$$P_3=(\tfrac{1}{5})(\tfrac{4}{5})^4(\tfrac{1}{8})\simeq 0.01 , \tag{4.3}$$

where various factors here have similar interpretation as in Eq. (4.1) (e.g., the factor $\frac{1}{8}$ comes from requiring that the tetrahedron correspond to having a winding-number-1 map or S^2 to S^2; see the case of the monopole).[8]

The analog of P_2 (with no special value at the center, which is relevant for the case of texture formation in the Universe) is then

$$P_4=5P_3=0.05 . \tag{4.4}$$

We now see from Eq. (4.3) that even if we allow fractional-winding-number Skyrmion configurations, their formation probability (per four elementary tetrahedron cells) is quite small. Even if one says that there is no reason to have a specific value of the field (diametrically opposite to the value assumed for the outside region of the jet) at the center of the Skyrmion, we see from Eq. (4.4) that the formation probability (per four elementary tetrahedron cells) is still much less than the value of $4\times\frac{1}{16}$ obtained in Ref. 8. We again emphasize here that the probability of $\frac{1}{16}$ in Ref. 8 was calculated for a single four-simplex. When restricted to three physical dimensions, this corresponds to the probability of about $\frac{1}{16}$ per tetrahedron.

According to our considerations, the production of Skyrmions is mainly suppressed due to the requirement of constant boundary conditions. Thus the above conclusions may be greatly modified when we consider the regions near the boundary of the jet, due to naturally present constant boundary conditions outside the jet. For example, when enclosed by a surface on which the field assumes a constant boundary value almost everywhere, it may be enough to consider the winding (of S^2 to S^2) on a single tetrahedron. This will lead to enhancement in the probability of Skyrmion production. This however, suggests that the earlier suggestion,[8] that there should be less baryon production in a narrow jet event as compared to a more isotropic event, may be modified. For example, in the interior of a narrow jet, the surface of a Skyrmion may have more overlap with the surface of the jet. As argued above, this requires fewer points where one will need to specify the field in order to get a Skyrmion configuration. Baryon production in such a case may be thus enhanced as compared to a more isotropic case. The relevant thing to consider may be the surface-to-volume ratio of the region with a larger ratio leading to more baryon production.

V. CONCLUDING REMARKS

We have considered the question of boundary conditions for the case of texture formation in the early Universe[4] as well as for the topological model for the production of baryons.[8] Such considerations imply, for the case of textures in the Universe, that textures have no topological significance and are homotopic to trivial field configurations (when the horizon is much smaller than the Universe size). It is thus inappropriate to characterize them by a topologically invariant concept like winding number. One may occasionally find a configuration which at least initially may have integer winding number. But as this configuration evolves, there is no topological reason that this does not smoothly change into a fractional-winding-number configuration. Also, the probability of such configurations is highly suppressed and is at most 10^{-7}. (Note here that the probability of full knots was estimated to be about $\frac{1}{25}$ in Ref. 4.)

We point out (and as has been noted by others[11,4,6]) that the considerations of density fluctuations from textures as discussed in[4] are in fact valid for a much more general class of models and field configurations. In particular, any model with spontaneous breaking of a global symmetry leads to a scale-invariant distribution of density fluctuations as long as the connected component of the vacuum manifold is degenerate. It does not matter if the vacuum manifold has any nonvanishing homotopy group.

For the model of topological production of baryons in the interior of jets, our considerations lead to strong suppression (by a factor of about 10^{-7}) of the probability for the production of integer-winding-number Skyrmions (except for the possibility of enhancement near the surface of the jet). (We may note here that in this case, as opposed to the case of textures in the Universe, the final-state Skyrmions with integer-winding-number configurations are of clear topological significance due to the presence of an asymptotic region outside the jet where the field assumes some constant value, though inside the jet there still is no topologically invariant description of a Skyrmion.) We then consider certain fractional-winding-number Skyrmion configurations which (given an appropriate dynamical evolution of the field) may be expected to separate out as isolated, integer-winding-number Skyrmions. We find that such configurations necessarily extend over at least four elementary tetrahedron cells (used to triangulate the physical space R^3) and, in that region, their production probability is at most equal to 0.05, which amounts to a probability of about 0.013 per tetrahedron. (For jet events with fixed boundary conditions outside the jet, a more appropriate value is ~ 0.003 per tetrahedron.) Thus the production of Skyrmions is highly suppressed compared to earlier estimates,[8,9] where integer-winding-number Skyrmions were expected to form with the probability of $\frac{1}{16}$. We find that a strong correlation should exist in the production of baryons and antibaryons (similar correlation was also predicted by Ellis *et al.* in Ref. 8).

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