Multifractal structure of multiparticle production in branching models

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A procedure is described for the multifractal analysis of data on multiparticle production obtained at high energy either in experiment or in Monte Carlo simulation. It is shown how the spectrum $f(\alpha)$ of the rapidity-density index α can be determined from the multiplicity fluctuation of the rapidity distribution, as the resolution is changed. The branching model is used to illustrate the procedure. It is found that the ϕ^3 model has a narrower $f(\alpha)$ than the gluon model, suggesting that multifractality is a useful arena for confrontation between theory and experiment.

I. INTRODUCTION

The study of multiplicity fluctuation in particle production at high energy has revealed self-similar properties[1] conjectured by Bialas and Peschanski,[2] who called the phenomenon intermittency. This has, on the one hand, opened up the possibility of more stringent tests on the dynamical models describing multiparticle production, while on the other provided a strong hint of the possible existence of fractal properties in such production processes. The latter presents a new field of exploration in particle physics. We report here the results of an initial quantitative probing into that field and show in the framework of the branching model that the fluctuation in rapidity distribution indeed has nontrivial multifractal structure.

The mathematical procedure in which we carry out this investigation is by means of Monte Carlo (MC) simulation. The branching model used serves mainly as a concrete dynamical scheme that is simple yet sufficiently close to reality to generate rapidity distributions whose fractal properties can be analyzed in quantitative detail. Apart from the intrinsically interesting multifractal structure thus revealed, an important aspect of this study is also the charting of a procedure for analysis that can be adopted by experiments in their attempt to extract fractal information from their data. In our opinion the richness of that information has not come close to being fully revealed in the study of intermittency thus far carried out.[3]

Consider the rapidity distribution $\mathcal{N}(y)$ of one event, whether obtained in an actual experiment or by MC simulation according to a particular theoretical model. Assume that the energy is high enough so that $\mathcal{N}(y)$ is well distributed over a certain rapidity interval, though not necessarily smooth. Indeed, it is the fluctuation of $\mathcal{N}(y)$ that is the focus of fractal analysis. While the fluctuations themselves change from event to event, what we want to extract are the properties of those fluctuations that are universal. The degree of fluctuation, of course, depends on the resolution in which $\mathcal{N}(y)$ is examined. Let Y_0 be an interval in which the multifractal analysis is to be carried out. The result may depend upon the location and range of Y_0. If $\mathcal{N}(y)$ has a pronounced peak, it is sensible to choose Y_0 to straddle the peak but not to include the fringes. Subdivide Y_0 into M bins, each having rapidity widths $\delta = Y_0/M$. As with intermittency,[2] we look for power-law dependences on δ of appropriate quantities so that the exponents can deliver information about the self-similarity. Unlike intermittency, we abandon factorial moments F_l, which are defined only for positive integers l, but work instead with the moments G_q, which are defined for all real q, positive or negative, and not necessarily integral.[4] Furthermore, a crucial point to be stressed here is that the power-law behavior G_q is to be analyzed event by event, and not for the event-averaged $\langle \mathcal{N}(y) \rangle$, which suppresses the importance of fluctuations in low-multiplicity events.

II. MULTIFRACTAL ANALYSIS

For a given partition of Y_0 into M bins, let k_i be the multiplicity of particles detected in the ith bin and n be the total multiplicity within the Y_0 interval so that $n = \sum_{i=1}^{M} k_i$. When M is large, some bins may have no particles. Let $\mathcal{M}$ be the number of nonempty bins, which constitute a set of bins that have fractal properties, when M becomes large. This set contains many subsets, each of which is characterized by an index α defined as follows. Let p_i be defined by

$$p_j = k_j / n \tag{1}$$

for nonempty bins only, so that $\sum_{j=1}^{\mathcal{M}} p_j = 1$. Each bin has its own δ dependence, as δ is decreased; let it be described by a power-law behavior[4]

$$p_j \propto \delta^{\alpha} . \tag{2}$$

Into every interval $(\alpha, \alpha + d\alpha)$ can be mapped many bins all having the same behavior as (2); the collection of all such bins constitutes a fractal subset, which is therefore labeled by the index α. Let the number of bins of such a subset be M_α, whose δ dependence may be written as

$$M_\alpha \propto \delta^{-f(\alpha)} . \tag{3}$$

We aim to determine this spectrum $f(\alpha)$ of all α indices, which exhaust all bins in Y_0 through the mapping (2). The collection of all subsets M_α of M constitutes the multifractal structure of the rapidity distribution. Thus by decreasing the bin width δ, we map an erratically behaving $\mathcal{N}(y)$, rich in peaks and valleys, into a smooth function $f(\alpha)$. This line of analysis is useful only if upon event averaging stable, average $f(\alpha)$ emerges, characterizing the nature of multiplicity fluctuations for all events. For convenience, we shall refer to the determination of $f(\alpha)$ using an increasing number of bins in the fixed interval Y_0 for each event as the horizontal analysis, and to event averaging as vertical averaging.[4]

The execution of the horizontal analysis can be readily carried out by using the G_q moments. Define

$$G_q = \sum_{j=1}^{\mathcal{M}} p_j^q , \tag{4}$$

where the sum is over all nonempty bins, and q is any real value. For experimental convenience we can define (for a given event) Q_k as the number of bins with k particles, satisfying $\sum_{k=1}^{\infty} Q_k = \mathcal{M}$. Then (4) may be rewritten as

$$G_q = \sum_{k=1}^{\infty} k^q Q_k \Bigg/ \left[\sum_{k=1}^{\infty} k Q_k \right]^q . \tag{5}$$

Note that because $k \geq 1$, q can be negative. G_q is different from the normalized C_l moments for horizontal analysis:

$$C_l = \sum_{k=1}^{\infty} k^l P_k \Bigg/ \left[\sum_{k=1}^{\infty} k P_k \right]^l , \tag{6}$$

where $P_k = Q_k / \sum_{k=0}^{\infty} Q_k$, a normalization that includes the $k=0$ empty bins. Usually, C_l is defined in (6) with $k=0$ included, since $P_0 \neq 0$ in general; however, for l being a positive integer, it is equivalent to the sum with the $k=0$ term excluded. Equation (6) permits the integral value of l to be extended to all real values of q. Let the corresponding moments be denoted by C_q. Since $\sum_{k=0}^{\infty} Q_k = M$, G_q and C_q can be related by

$$G_q = C_q M^{1-q} \tag{7}$$

for all real q.

For every event in an experiment or MC calculation, G_q can be determined as a function of δ. The particle production process exhibits self-similar behavior, when a region of δ can be found such that

$$G_q \propto \delta^{\tau(q)} . \tag{8}$$

This behavior does not occur in the limit $\delta \to 0$. In fact, because $\sum_{j=1}^{M} k_j = n$, where n is a fixed number for a given event, the lower limit of k_j for a nonempty bin is 1, as $\delta \to 0$; consequently, by (1) and (4), G_q approaches n^{1-q}. Thus the power-law behavior in (8) cannot be achieved if $\delta \to 0$. However, we have found in our model calculation to be discussed below that (8) can be used to describe the behavior of G_q for a range of δ not too small.

Once $\tau(q)$ is determined, we apply the theory of multifractals[5,6] to calculate[4] $f(\alpha)$ by Legendre transform

$$f(\alpha_q) = q\alpha_q - \tau(q) , \tag{9}$$

$$\alpha_q = d\tau(q)/dq . \tag{10}$$

Since a derivative is involved, it is necessary to determine $\tau(q)$ for small incremental changes of q, especially in the neighborhood of $q=0$, where $f(\alpha)$ has its maximum. It follows from the general theory that $f(\alpha)$ is a concave function and that $f(\alpha_0) = D_0$, the fractal dimension. In our model calculation our results for $f(\alpha)$ satisfy the general properties, even though the self-similarity in our problem is not always in the limit of vanishing δ, as is assumed in the theory of multifractals.[7]

To determine the average $\langle f(\alpha) \rangle$ by vertical averaging, it is essential that the procedure guarantees the determination of the average $\langle \tau(q) \rangle$. That means that the averaging should not be performed on G_q, but on $\ln G_q$. In practice, we let $M = 2^\nu$, and calculate

$$\begin{aligned} \langle \tau(q) \rangle &= -\langle \Delta \ln G_q / \Delta \ln \delta \rangle \\ &= -(\ln 2)^{-1} \Delta \langle \ln G_q \rangle / \Delta\nu , \end{aligned} \tag{11}$$

where angular brackets denote event averaging, and $\Delta\nu$ is the range of ν in which self-similarity exists. $\langle \alpha_q \rangle$ is determined from $d\langle \tau(q) \rangle / dq$, and then $\langle f(\alpha_q) \rangle$ is obtained from $q\langle \alpha_q \rangle - \langle \tau(q) \rangle$. When no confusion can arise, we shall, for brevity, use $f(\alpha)$ to denote the vertically averaged $\langle f(\alpha_q) \rangle$ thus obtained.

It should be remarked that the type of horizontal analysis and vertical averaging described above has not been carried out so far in the study of intermittency,[1,3] let alone the study of multifractals. The usual factorial moments considered are[2]

$$F_l = \left\langle M^{-1} \sum_{j=1}^{M} k_j(k_j-1) \cdots (k_j-l+1) \right\rangle \Bigg/ \left\langle M^{-1} \sum_{j=1}^{M} k_j \right\rangle^l \tag{12}$$

whose power-law behavior in small δ,

$$F_l \propto \delta^{-a_l} , \tag{13}$$

is examined for the determination of a_l. We suggest instead that F_l without vertical averaging in (12) should first be determined for each event and that the average $\langle a_l \rangle$ then be calculated according to

$$\langle a_l \rangle = -\Delta \langle \ln F_l \rangle / \Delta(\ln \delta) , \tag{14}$$

in analogy with (11). The significance of this procedure is that the unexhibited multiplicative factor in (13) denoted by γ_l, say, becomes immaterial in the averaging; otherwise, the γ_l, for events with smaller a_l, are given less weight. This may be important for hadronic and nuclear collisions, where fluctuations due to impact-parameter variations from event to event should be minimized, if the effects of fluctuations due to dynamical process of hadronization are to be identified. It is possible that the weak intermittency observed in the hadronic and nuclear collisions thus far[1] is due to the prevailing procedure in which the intermittency indices have been determined.

III. BRANCHING MODELS

Before we describe the mulitfractal properties of the branching model, let us consider two simple examples for illustrative purpose. First, suppose that $\mathcal{N}(y)$=constant for every event even for arbitrarily small δ. Then $M = \mathcal{M}$ and $k = n/M$, so $p_j = 1/M$, and $G_q = M^{1-q}$. From (8) we get $\tau(q) = q - 1$. It then follows from (9) and (10) that $f(\alpha) = \delta_{\alpha,1}$ for every event, and therefore so also after vertical averaging. This is the trivial result when there are no fluctuations. It may also be remarked parenthetically that the uniform triadic Cantor set has $f(\alpha)=\alpha\delta_{\alpha,\alpha_0}$ where $\alpha_0=\ln 2/\ln 3$.[5] The absence of a width for $f(\alpha)$ is an indication of the lack of randomness in the problem, despite the presence of self-similarity. Consider next an example in which p_j has a peak rising above a uniform background. Take, for instance, $p_j=N\{1+2\exp[-(y_j-y_0)^2/\sigma^2]\}$, where N is fixed by $\sum_j p_j=1$, y_j is the rapidity at the center of the jth bin, and y_0 fluctuates randomly from event to event so that the single-particle rapidity distribution after event averaging is uniform. The Gaussian width σ provides a scale in the problem; its exact relationship to the pair-correlation length is not immediately clear, although the existence of some correlation is obvious. This mathematical example illustrates the existence of self-similarity in a finite range of M (i.e., δ does not $\to 0$), and that the corresponding $f(\alpha)$, which can readily be calculated, has a narrow peak situated at $\alpha_0=1$ with $f(\alpha_0)=1$, having a width that decreases with increasing σ. The full width $\Delta\alpha$ at half-maximum is about 0.2, if $\sigma=2$ and $Y_0=6$. Here we see that although p_j is a smooth function of y_j, the existence of correlation (and therefore also multiplicity fluctuation) results in a nontrivial $f(\alpha)$ for an appropriate range of resolution. It should further be noted that, because p_j is an analytic function, the spectrum is trivial $[f(\alpha)=\delta_{\alpha,1}]$, if the limit $\delta\to 0$ is taken.

We now consider the branching model to explore fully the multifractal structure of multiparticle production. The model has been described in detail in Ref. 8, where intermittency is investigated. Briefly stated, the model specifies the successive branchings of partons, as their virtualities degrade from the initial value Q^2 to the final values $q^2\le q_0^2$, at which point a parton is identified as a final-state particle. At each vertex a mother parton splits into two daughter partons with momentum fractions z and $1-z$ according to the probability function $P(z)$. We consider here two forms of $P(z)$: (a) ϕ^3 model $P(z)=6z(1-z)$; and (b) gluon model $P(z)=c[(1-z)/z+z/(1-z)+z(1-z)]$. The rapidity of a final particle with momentum fraction x is determined by $y=\operatorname{arcsinh}(xQ/m_T)$, where $m_T=q_0/2$. The kinematic range of y is therefore between 0 and $Y=\operatorname{arcsinh}(Q/m_T)$. The reader is referred to Ref. 8 for details of how the model is applied to the MC simulation of particle production.

For the ϕ^3 model the peak of $\mathcal{N}(y)$ is around $y=3$ at $Q=2$ TeV, so we choose $Y_0=2$ straddling the peak. For the gluon model $\mathcal{N}(y)$ is sharply peaked at $y=0$; thus the range between 0 and $Y_0=0.5$ has been chosen so that a substantial fraction of the final partons are included. As Q is varied, the overall Y range changes; thus we change Y_0 accordingly in order to keep Y_0/Y fixed. A discussion of the dependence on Y_0/Y is deferred to a longer paper later, since it is not central to our problem at hand.

In Fig. 1 we show for the gluon model the results on the event-averaged $\ln G_q$ vs ν for some typical values of q. For each q, a range of ν can be identified in which there is not only approximate linearity but also universality, i.e., independence on Q. It is in those ranges that we determine $\langle\tau(q)\rangle$ by use of (11). In practice we imagine the asymptotes for each q to be the straight lines drawn between $\nu=1$ and 3 for $Q=20$ TeV and determine $\langle\tau(q)\rangle$ accordingly. The result after averaging over 500 events, which include more than 50 000 final particles, is shown by the solid line in Fig. 2(a), together with that for the ϕ^3 model shown by the dashed line. $\langle\tau(q)\rangle$ contains all the information that is conveyed by intermittency. It follows from (7) that, for positive q,[9,10]

$$\langle\tau(q)\rangle+\langle a(q)\rangle=q-1,\quad q=l=1,2,\ldots, \tag{15}$$

assuming that F_l and C_l have the same intermittency indices.[2] We have calculated $\langle a_l\rangle$ from (14) in the two branching models and verified the validity of (15). It should be pointed out that the $\langle a_l\rangle$ indices here are not identical to the ones determined in Ref. 8, because it is the horizontal analysis that is studied here, not the vertical analysis considered there. Figure 2(a), however, gives also $\langle\tau(q)\rangle$ for $q<1$; consequently, it goes beyond intermittency.

From $\langle\tau(q)\rangle$ we have calculated $\langle\alpha_q\rangle$, which, when used in conjunction with $\langle\tau(q)\rangle$ in (11), yields the spectrum function $f(\alpha)$. In Fig. 2(b) is shown $f(\alpha)$ for both models. Evidently, the fractal properties of the two models are quite different. Both curves are obtained by MC calculation at various values of q, which are densely spaced near the peaks. The curves satisfy the general properties that the peaks occur at α_0 and that they are tangent to the 45° line at α_1.[4] In the case of the gluon model the peak is not exactly for $q=0$, an artifact due to our procedure for selecting a tangent that exhibits self-similarity.[7] The portion to the left of a curve is for $q>0$, that to the right being for $q<0$. The points corresponding to $q=0$, 1, and 2 are denoted by special symbols in the figure. The corresponding dimensions $D_q\equiv\tau(q)/(q-1)$ can be determined accordingly;[11] in particular, we have the fractal dimension $D_0=f(\alpha_0)$, the information dimension $D_1=f(\alpha_1)$, the correlation di-

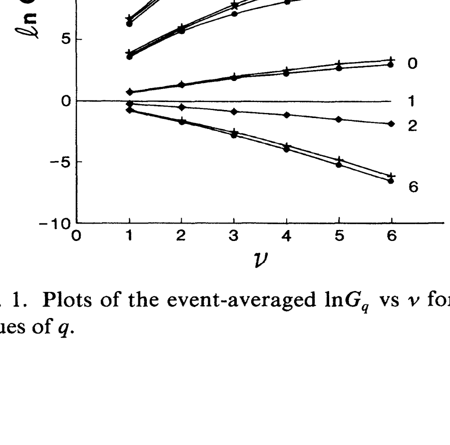

FIG. 1. Plots of the event-averaged $\ln G_q$ vs ν for some typical values of q.

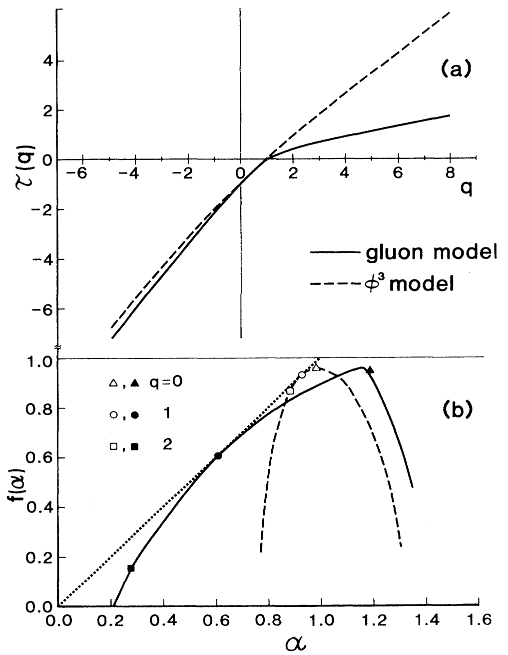

FIG. 2. (a) Event-averaged $\tau(q)$ at various q values determined at small Δq apart, especially for $0<q<2$. (b) The spectrum $f(\alpha)$ of α for two branching models. The dotted line represents a 45° line, tangent to $f(\alpha)$ at $\alpha=\alpha_1$.

mension $D_2=2\alpha_2-f(\alpha_2)$. We note that the information conveyed by the intermittency analysis for $q\geq 2$ represents, for the gluon model say, only the very small region of $f(\alpha)$ near $\alpha\simeq 0.2$, since $\langle\tau(q)\rangle$ is very nearly a straight line for $q>2$. Clearly, the fractal analysis reveals much more.

IV. COMMENTS

The fact that $f(\alpha_0)$ in Fig. 2(b) is very close to 1 means that for the range of resolution in which there is universal self-similarity, there are not too many empty bins, since the number of nonempty bins $\mathcal{M}$ behaves as $\delta^{-f(\alpha_0)}$. The wider spectrum for the gluon model implies that its rapidity distribution deviates more from homogeneity, which is hardly surprising, since we already know that it is sharply peaked at $y=0$. More importantly, $f(\alpha)$ gives a quantitative description of the multiplicity fluctuation in both the dense and sparse regions in rapidity space, corresponding to the $\alpha<\alpha_0$ and $\alpha>\alpha_0$ regions of $f(\alpha)$, respectively. The difference in the shapes of the two curves is due not only to the dissimilarity in the two models, but also to the different values of Y_0 chosen for the two cases. Thus in a comparison between theory and experiment it is necessary to have the same Y_0 intervals. Since for the calculation that we have done the values of Y_0 have been chosen somewhat arbitrarily, the absolute shapes of the $f(\alpha)$ curves are not to be taken seriously without specific data in mind to be compared with. Phenomenology is not the purpose of this paper. Our aim has been to describe a method of multifractal analysis in multiparticle production and to demonstrate it by use of two models in MC simulation. Our result shows that the proposed multifractal analysis can be a fertile ground for the study of multiparticle production. Although other investigations of fractal structures have also been suggested,[12,13] the quantitative study by MC simulation here renders the approach concrete as well as directly accessible to phenomenology.

The issue of statistical fluctuation has been addressed in Ref. 2 and can be raised here with regard to the G moments and the spectrum $f(\alpha)$. The reader interested in the subject is referred to Ref. 14 for an extensive discussion. It is shown there how the statistical contribution to the multiplicity fluctuation can be filtered out.

To summarize, we have demonstrated that the multifractal analysis discussed here can extract the crucial properties of multiplicity fluctuation and present the result in the form of a smooth function $f(\alpha)$. The method should apply equally well to both experimental data and MC simulation in the framework of a model. Thus this field of study has the potential of becoming an arena for effective confrontation between theory and experiment on the physics of multiparticle production.

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