Field-theory calculations of the pion mass to one-loop order

T. Hakioḡlu and M. D. Scadron
Department of Physics, University of Arizona, Tucson, Arizona 85721
(Received 27 November 1989; revised manuscript received 12 February 1990)

We compute the pion mass to one-loop order in the SU(2) linear σ and four-fermion model field theories both in terms of the nonstrange current-quark mass $\hat{m}_{\rm cur}$. The answers in these two chiral-symmetry schemes are the same and predict $\hat{m}_{\rm cur}\approx 34$ MeV for two quark flavors.

Thirty years ago our first understanding of the resulting Nambu-Goldstone theorem for (almost) massless pions[1] began with apparently independent chiral field theories: The SU(2) linear σ model[2] (LSM) and the four-fermion Nambu–Jona-Lasinio (NJL) model.[3] These chiral-invariant field theories demonstrate in different ways how chiral symmetry may be broken spontaneously or dynamically while still keeping the pion massless in the chiral limit (CL). Such CL mechanisms were initially worked out in the tree order for the LSM and to all orders for the NJL model.

In this paper we study the SU(2) LSM and NJL models to one-loop order, first demonstrating that the pion remains massless in the CL as expected. By requiring the otherwise general σ mass in the LSM to be fixed at the NJL value[3] $m_\sigma=2m_{\rm qk}$ (where $m_{\rm qk}$ is the quark mass), we in fact show that the models become essentially equivalent through one-loop order. Away from the CL each model then gives the same value for the average nonstrange current-quark mass in terms of the pion mass, $\hat{m}_{\rm cur}\approx\frac{9}{16}m_\pi^2/m_{\rm qk}\approx 34$ MeV for two quark flavors $N_f=2$.

We begin with the SU(2) LSM, but first shift the σ field vacuum expectation value from the spontaneously broken value $\langle\sigma_{\rm old}\rangle=-f_\pi\neq 0$ to $\sigma=\sigma_{\rm old}+f_\pi$. Then $\langle\sigma\rangle=0$ signals the true vacuum and the new CL Lagrangian density has the interacting part[2,4]

$$\mathcal{L}_{\rm int}=g'\sigma(\sigma^2+\boldsymbol{\pi}^2)-(g'/4f_\pi)(\sigma^2+\boldsymbol{\pi}^2)^2+g\bar{\psi}(\sigma+i\gamma_5\boldsymbol{\tau}\cdot\boldsymbol{\pi})\psi-gf_\pi\bar{\psi}\psi\ . \tag{1}$$

In (1) we take the fermion fields as quark operators. At this quark level the CL couplings in (1) are

$$g=m_{\rm qk}/f_\pi\approx 3.5,\quad g'=m_\sigma^2/2f_\pi\approx 2.2\ \text{GeV}\ , \tag{2}$$

where $f_\pi\approx 90$ MeV, $m_{\rm qk}\approx M_N/3\approx 313$ MeV, and we invoke the NJL value[3,5] $m_\sigma=2m_{\rm qk}\approx 626$ MeV.

In the CL one requires $m_\pi=0$, not only to tree order in the Lagrangian, but in higher-loop orders as well. This null result in fact holds for the quark loop graphs in one-loop order. For the quark "vacuum polarization" (VP) and tadpole (qktad) pion self-energy graphs of Fig. 1, the corresponding CL amplitudes M^0 as $q\to 0$ are

$$M^0_{\rm VP}=-i4N_cN_fg^2\int\frac{d\!\!{}^-{}^4p}{p^2-m_{\rm qk}^2}\ , \tag{3a}$$

$$M^0_{\rm qktad}=\frac{i4N_cN_f2g'g}{m_\sigma^2}\int\frac{d\!\!{}^-{}^4p\, m_{\rm qk}}{p^2-m_{\rm qk}^2}\ , \tag{3b}$$

where $d\!\!{}^-{}^4p=d^4p/(2\pi)^4$. With the Lagrangian couplings (2), it is indeed clear from (3) that the pion remains massless in the CL: $M^0_{\rm VP}+M^0_{\rm qktad}=0$.

Likewise the meson-loop graphs of Fig. 2 "conspire" to keep the pion massless in the CL. More specifically, these one-loop order graphs, respectively, give the CL amplitudes

$$M^0_{\text{meson loops}}=4g'^2i\int\frac{d\!\!{}^-{}^4p}{p^2(p^2-m_\sigma^2)}+\left[\frac{5g'}{f_\pi}-\frac{6g'^2}{m_\sigma^2}\right]i\int\frac{d\!\!{}^-{}^4p}{p^2}+\left[\frac{g'}{f_\pi}-\frac{6g'^2}{m_\sigma^2}\right]i\int\frac{d\!\!{}^-{}^4p}{p^2-m_\sigma^2}\ . \tag{4}$$

Using (2) and the partial fraction identity

$$m_\sigma^2p^{-2}(p^2-m_\sigma^2)^{-2}=(p^2-m_\sigma^2)^{-2}-p^{-2}\ ,$$

the first integral in (4) contributes equally (but with opposite sign) to the second and third integrals in (4). Then again the coefficients of these latter two integrals $(-4+10-6, 4+2-6)$ both identically vanish and the pion remains massless in the CL: $M^0_{\text{meson loops}}=0$.

The vanishing of (3a) plus (3b) and (4) in the CL emphasizes the importance of the quark (and meson) σ tadpole graphs in Figs. 1 and 2. They ensure chiral symmetry of the theory, keeping $m_\pi=0$ to one-loop order. Some texts[4] on the LSM stress that σ tadpoles signal a clumsy expansion around the wrong (false) vacuum and that these tadpoles can be eliminated by shifting to the true vacuum with $\langle\sigma\rangle=0$. This statement refers[4] only to *tree*-level tadpoles generating an $\mathcal{L}\sim a\sigma$ term, which in our CL case automatically vanishes when $m_\pi=0$. These latter tree-level tadpoles must not be confused with the one-loop tadpoles of Figs. 1 and 2, which exist even after the shift to the true vacuum.

For future reference one sets the scale of the formally

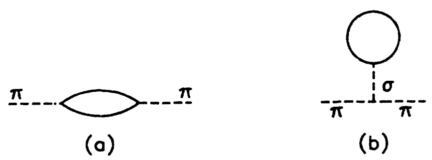

FIG. 1. LSM quark loop contributions to the pion mass.

divergent loop graphs in (3) and (4) by introducing an ultraviolet cutoff Λ in the LSM determined by the CL spontaneous chiral-symmetry-breakdown scale $f_\pi \approx 90$ MeV. The latter is found from the quark loop of Fig. 3, which leads to the decay constant $f_\pi = m_{qk}/g$ and the exact gap equation

$$1 = -i4N_c g^2 \int \frac{d^4p}{(p^2 - m_{qk}^2)^2} . \tag{5}$$

With $g \approx 3.5$ given by (2), the cutoff Λ in (5) is found to be $\Lambda \approx 2.45 m_{qk} \approx 767$ MeV for $N_c = 3$, reasonably close to $\Lambda \sim m_\sigma = 2m_{qk}$. Also for the latter cutoff, an approximate gap-type equation numerically holds for the squared σ mass with $N_f = 2$:

$$m_\sigma^2 \approx i4N_c g^2 2 \int \frac{d^4p}{p^2 - m_{qk}^2} . \tag{6}$$

We will make use of the gap equations (5) and (6) shortly.

Away from the CL for general N_c, N_f, the LSM self-energy amplitudes (3) representing Fig. 1, but for $q^2 = m_\pi^2 \neq 0$, are

$$M_{VP} = -i4N_c N_f g^2 \times \int \frac{d^4p (p^2 - \hat{m}^2 - m_\pi^2/4)}{[(p + \frac{1}{2}q)^2 - \hat{m}^2][(p - \frac{1}{2}q)^2 - \hat{m}^2]} , \tag{7a}$$

$$M_{qktad} = \frac{i4N_c N_f}{m_\sigma^2} 2g'g \int \frac{d^4p\, \hat{m}}{p^2 - \hat{m}^2} , \tag{7b}$$

where now[2,4] $2f_\pi g' = m_\sigma^2 - m_\pi^2$ guarantees that there is no linear σ field term in the chiral broken Lagrangian (1). Also the chiral-broken nonstrange quark mass is $\hat{m} = m_{qk} + \hat{m}_{cur}$ with $\hat{m}_{cur}$ being the nonstrange current-quark mass. Then subtracting Eqs. (3) from Eqs. (7), and using the gap equations, it is straightforward to show that the incremental self-energy shifts for massive pions are, for quark loops with $N_f = 2$,

$$\delta M_{VP} = M_{VP} - M_{VP}^0 \approx -\tfrac{5}{4} m_\pi^2 + 4\hat{m}_{cur} m_{qk} , \tag{8a}$$

$$\begin{aligned} \delta M_{qktad} &= (M - M^0)_{qktad} \\ &\approx -4\hat{m}_{cur} m_{qk} + i4N_c N_f g^2 \int \frac{d^4p}{p^2 - m_{qk}^2} \left[\frac{\hat{m}_{cur}}{m_{qk}} - \frac{m_\pi^2}{m_\sigma^2} \right] \\ &\approx -m_\pi^2 . \end{aligned} \tag{8b}$$

The expression (8a) is worked out in the Appendix. Note that the $\hat{m}_{cur}$ coefficient in (8b) vanishes due to the approximate gap equation (6). In a similar manner one can compute the meson-loop amplitudes of Fig. 2 for $q^2 = m_\pi^2$. The incremental shift of (4) again using $2f_\pi g' = m_\sigma^2 - m_\pi^2$ and cutoff $\Lambda \sim 700$ MeV is

$$\delta M_{\text{meson loops}} = (M - M^0)_{\text{meson loops}} \approx m_\pi^2 . \tag{9}$$

The sign change of the m_π^2 contributions in (8) relative to (9) is due to the Feynman rule for fermion versus boson loops.

Adding the amplitudes (8) to (9), we then obtain the net pion self-energy shift in the LSM away from the CL to one-loop order with $m_\sigma = 2m_{qk}$ and $N_f = 2$,

$$\begin{aligned} \delta M_{LSM} &= \delta M_{VP} + \delta M_{qktad} + \delta M_{\text{meson loops}} \\ &\approx -\tfrac{5}{4} m_\pi^2 + 4\hat{m}_{cur} m_{qk} . \end{aligned} \tag{10}$$

Identifying this net chiral-breaking mass shift with the entire squared pion mass (the mass shift is the mass) $\delta M_{LSM} = m_\pi^2$, Eq. (10) then implies for $N_f = 2$,

$$\begin{aligned} &(1 + \tfrac{5}{4}) m_\pi^2 \approx 4\hat{m}_{cur} m_{qk} , \\ &\hat{m}_{cur} \approx \tfrac{9}{16} m_\pi^2 / m_{qk} \approx 34 \text{ MeV} . \end{aligned} \tag{11}$$

Note that the $\pm m_\pi^2$ factors in (8b) and (9) essentially cancel [even if we do not employ the approximate gap equation (6)], leaving the $-\frac{5}{4} m_\pi^2$ term due to the VP quark loop in (8a) as controlling the nonstrange current-quark mass scale in (11).

In order to support the above LSM results, we now turn to the four-fermion NJL chiral model with the Lagrangian density[3]

$$\mathcal{L} = i\bar{\psi}\partial\!\!\!/\psi + \tilde{g}[(\bar{\psi}\psi)^2 + (\bar{\psi} i\gamma_5 \tau \psi)^2] . \tag{12}$$

Unlike the LSM with elementary quark, pion, and σ meson fields coupled in a chirally invariant manner, the NJL theory treats only quarks as elementary. The π and σ mesons are then bound states with $m_\pi = 0$ and $m_\sigma = 2m_{qk}$ in the CL following from the NJL nonperturbative gap equation[3] depicted in Fig. 4:

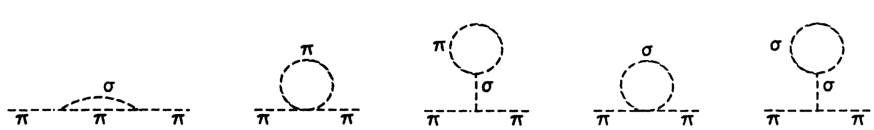

FIG. 2. LSM meson loop contributions to the pion mass.

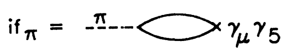

FIG. 3. Quark loop representation of f_π.

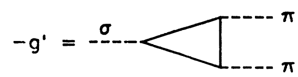

FIG. 5. NJL representation of the LSM coupling g' in one-loop order.

$$m_{\rm qk}=i2\tilde{g}N_cN_f4m_{\rm qk}\int\frac{d^4p}{p^2-m_{\rm qk}^2}\ , \tag{13}$$

quite similar to Eq. (6) in the LSM, but with g^2/m_σ^2 in (6) replaced by $2\tilde{g}$ in (13). However, the LSM gap equation for f_π, Eq. (5), remains unchanged in the NJL model. The two fit NJL parameters $\tilde{g}$ and the invariant cutoff[6] $\Lambda\sim700$ MeV are fixed to yield the chiral-limiting quark mass $m_{\rm qk}\approx313$ MeV and pion decay constant $f_\pi\approx90$ MeV as supported by the gap equation (5).

In order to link this NJL theory to the LSM, we first point out that one-loop-order NJL graphs should be identified with tree-order LSM couplings g and g'. More specifically, Fig. 3 is a one-loop-order NJL graph representing (5) which for the cutoff of $\Lambda\approx2.2m_{\rm qk}\approx700$ MeV requires

$$g\approx2\pi/\sqrt{N_c}\approx3.6\ . \tag{14a}$$

Likewise g' in the NJL picture is the loop graph of Fig. 5, which in the $q\to0$ chiral limit has the value

$$\begin{aligned}g'&=-i4N_c2g^3m_{\rm qk}\int\frac{d^4p}{(p^2-m_{\rm qk}^2)^2}\\&=2g^2f_\pi\approx2.2\ \text{GeV}\ ,\end{aligned} \tag{14b}$$

where we have used the gap equation (5) with (2) to arrive at (14b). We see that the NJL one-loop level g and g' in (14) are quite close to the LSM tree-level couplings (2). In fact, setting (14b) equal to $m_\sigma^2/2f_\pi$ from (2) gives $m_\sigma=2m_{\rm qk}$, so indeed the NJL model and the LSM are intimately related.

Next we examine the pion mass in the NJL model. Since now pions and σ mesons are bound states in the NJL model one "sums bubbles" to all orders. The pion propagator in the NJL picture is then[3]

$$-g^2D_\pi(q^2)=\frac{2\tilde{g}}{1+2\tilde{g}J(q^2)}\ , \tag{15}$$

where the quark loop representing $J(q^2)$ is depicted in Fig. 6. In the CL with $q\to0$ one obtains, for general N_c and N_f,

$$J(0)=-i4N_cN_f\int\frac{d^4p}{p^2-m_{\rm qk}^2}\ . \tag{16}$$

Comparing (16) with the gap equation (13), one finds $-2\tilde{g}J(0)=1$. Then the pion propagator (15) has a pole as $q\to0$, signaling $m_\pi=0$ in this (chiral) limit.

This Goldstone equivalence of $m_\pi=0$ between the LSM and the NJL model also extends away from the CL. Then the NJL bubble of Fig. 6 corresponds to the amplitude

$$\begin{aligned}J(q^2)=-iN_cN_f\int d^4p\,\mathrm{Tr}[&i\gamma_5(\not p+\not q/2-\hat{m})^{-1}\\&\times i\gamma_5(\not p-\not q/2-\hat{m})^{-1}]\end{aligned} \tag{17}$$

for $q^2=m_\pi^2$. Carrying out the trace in (17) and comparing with the LSM VP amplitude (7a), one sees that

$$M_{\rm VP}=g^2J(m_\pi^2)\ . \tag{18}$$

The inverse of the pion propagator (15) is then

$$\begin{aligned}-D_\pi^{-1}(q^2)&=(g^2/2\tilde{g})[1+2\tilde{g}J(q^2)]\\&=(g^2/2\tilde{g})+g^2J(q^2)\ .\end{aligned} \tag{19}$$

To extract the self-energy part away from the CL, we associate (19) with the Schwinger-Dyson inverse propagator $D_\pi^{-1}(q^2)=D_\pi^{0-1}(q^2)-\Pi(q^2)$. Hence, in this chiral theory, the pion mass shift is the entire pion mass

$$\begin{aligned}\Pi(q^2=m_\pi^2)&=-\delta D_\pi^{-1}=D_\pi^{-1}(q^2=0)-D_\pi^{-1}(q^2=m_\pi^2)\\&=m_\pi^2\ .\end{aligned} \tag{20}$$

Since both g and $\tilde{g}$ are constant in the NJL model, $g^2/2\tilde{g}$ subtracts out when we apply the incremental shift (20) to (19), giving, by virtue of (18) and the Appendix,

$$m_\pi^2=g^2\delta J(m_\pi^2)=\delta M_{\rm VP}\approx-\tfrac{5}{4}m_\pi^2+4\hat{m}_{\rm cur}m_{\rm qk}\ . \tag{21a}$$

Solving (21a) for the nonstrange current-quark mass, we are again led back to the LSM value (11):

$$\hat{m}_{\rm cur}\approx\tfrac{9}{16}m_\pi^2/m_{\rm qk}\approx34\ \text{MeV}\ . \tag{21b}$$

Our analysis is along the same lines as Ref. 6, which ties the pion mass to a quark condensate that is somewhat elusive in the NJL model. Instead we have linked the NJL model closely to the LSM via Eq. (20), which is consistent with LSM shift $\delta M=m_\pi^2$.

The important observation for the calculation of the pion mass in these two models is that the σ meson quark tadpole negative contribution to the self-energy $-m_\pi^2$ in (8b) almost exactly cancels the positive meson loop contribution $+m_\pi^2$ in (9). This leaves the *quark* vacuum-

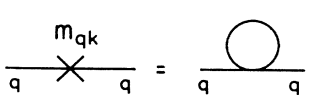

FIG. 4. NJL representation of quark mass.

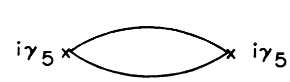

FIG. 6. NJL quark bubble J with pseudoscalar coupling.

polarization (VP) contribution (8a) to control the *net* pion self-energy in the LSM. But this VP amplitude $M_{\rm VP}$ is directly related to the NJL quark bubble J in (18). Thus the squared pion mass in the four-fermion NJL model is also proportional to the same current-quark mass in (21) as the LSM gives in (11).

Second, we comment on the physical pion mass now converted into the nonstrange current-quark mass scale $\hat{m}_{\rm cur}\approx 34$ MeV in (11) and in (21). It has long been appreciated that current-quark mass scales are model dependent. In the combined LSM-NJL model, the quark masses do not "run" with momenta as they do in QCD. Nonetheless we can make contact with the "physical" constituent nonstrange quark mass which is typically[7] taken as 350 MeV. In our picture we can express this constituent quark mass as the sum of the dynamical ($M_N/3$) part and the (nonrunning) SU(2)×SU(2) chiral-breaking current mass just obtained:

$$\hat{m}_{\rm con}=m_{\rm qk}+\hat{m}_{\rm cur}\approx 313\ \text{MeV}+34\ \text{MeV}\approx 347\ \text{MeV}\ . \tag{22}$$

In our opinion, all proposed models for current-quark masses should pass a constituent quark mass test analogous to (22).

A third point in favor of this combined LSM-NJL chiral-symmetry theory is its compatibility with experimental data. In particular, $g\sim 3.5$ in (2) and (14a) is what one expects based on πN scattering with $g_{\pi NN}\approx 3g_A g\sim 13.2$, near the experimental value 13.4 ± 0.08. Also $g'\approx 2.2$ GeV from (2) and (14b) is close to the magnitude extracted from $\delta\to\eta\pi$ decay with the measured width $\Gamma\approx 57$ MeV suggesting $|g'|\sim 2.7$ GeV for a singlet-octet mixing angle $\theta\sim -15°$. Finally, there have been many measurements[8] over the past decade of the σ meson mass and width, finding $m_\sigma\sim 600-700$ MeV, $\Gamma\sim 400-800$ MeV. These measurements must not be ignored even though $\Gamma_\sigma\sim m_\sigma$. Indeed, we have shown that such a scalar $\sigma(600)$ particle is necessary to satisfy the requirements of chiral symmetry.

The authors are grateful for partial support from the U.S. Department of Energy. M.D.S. acknowledges stimulating discussions with V. Elias, N. Fuchs, and T. Hatsuda. T.H. is grateful for discussions with P. Carruthers. Both of us appreciate a critical reading of the manuscript by A. Kocić.

APPENDIX

Given the vacuum-polarization (VP) amplitude away from the CL, (7a), and its chiral-limiting value (3a), the self-energy shift is

$$\begin{aligned}\delta M_{\rm VP}&=M_{\rm VP}-M^0_{\rm VP}\\&=-i4N_cN_fg^2\\&\quad\times\int d^4p\left[\frac{(p^2-\hat{m}^2-m_\pi^2/4)}{(p^2-\hat{m}^2+m_\pi^2/4)^2-(p\cdot q)^2}\right.\\&\qquad\left.-\frac{1}{p^2-m_{\rm qk}^2}\right]\ .\end{aligned} \tag{A1}$$

Since $\hat{m}^2=(m_{\rm qk}+\hat{m}_{\rm cur})^2\approx m_{\rm qk}^2+2\hat{m}_{\rm cur}m_{\rm qk}$, the partial fraction difference in (A1) has one power of $p^2-m_{\rm qk}^2$ canceling in the resulting numerator, with the denominator in (A1) then reduced to $(p^2-m_{\rm qk}^2)^2$ in the leading order. The incremental VP self-energy can thus be expressed as

$$\begin{aligned}\delta M_{\rm VP}&=-i4N_cN_fg^2\\&\quad\times\int\frac{d^4p}{(p^2-m_{\rm qk}^2)^2}\left[2\hat{m}_{\rm cur}m_{\rm qk}-\frac{m_\pi^2}{2}\right]\\&\quad-im_\pi^2N_cN_fg^2\int\frac{d^4p\ m_{\rm qk}^2}{(p^2-m_{\rm qk}^2)^3}\ .\end{aligned} \tag{A2}$$

The first integral in (A2) is evaluated from the exact gap equation (5), while the second integral in (A2) coming from the $(p\cdot q)^2$ term in (A1) is $(-i32\pi^2)$. The latter can be further simplified using $g^2\approx 4\pi^2/N_c$ from (14a). Then (A2) becomes, for $N_f=2$,

$$\delta M_{\rm VP}\approx 2\left[2\hat{m}_{\rm cur}m_{\rm qk}-\frac{m_\pi^2}{2}\right]-\tfrac{1}{4}m_\pi^2\ . \tag{A3}$$

which is equivalent to Eq. (8a).

---

[1] Y. Nambu, Phys. Rev. Lett. **4**, 380 (1960); J. Goldstone, Nuovo Cimento **19**, 154 (1961).
[2] M. Gell-Mann and M. Lévy, Nuovo Cimento **16**, 705 (1960).
[3] Y. Nambu and G. Jona-Lasinio, Phys. Rev. **122**, 345 (1961).
[4] V. de Alfaro, S. Fubini, G. Furlan, and C. Rossetti, *Currents in Hadron Physics* (North-Holland, Amsterdam, 1973), Chap. 5; B. W. Lee, *Chiral Dynamics* (Gordon and Breach, New York, 1972); C. Itzykson and J.-B. Zuber, *Quantum Field Theory* (McGraw-Hill, New York, 1980), Chap. 11-4.
[5] R. Delbourgo and M. D. Scadron, Phys. Rev. Lett. **48**, 379 (1982); V. Elias and M. D. Scadron, *ibid.* **53**, 1129 (1984).
[6] T. Hatsuda and T. Kunihiro, Prog. Theor. Phys. **74**, 765 (1985).
[7] See, e.g., A. De Rújula, H. Georgi, and S. Glashow, Phys. Rev. D **12**, 147 (1975); N. Isgur and G. Karl, *ibid.* **20**, 119 (1979).
[8] See, e.g., P. Estabrooks, Phys. Rev. D **19**, 2678 (1979); J. Donoghue and Y. Leroyer, Nucl. Phys. **B158**, 123 (1979); N. Biswas *et al.*, Phys. Rev. Lett. **47**, 1378 (1981); T. Akesson *et al.*, Phys. Lett. **133B**, 241 (1983); A. Courau *et al.*, Nucl. Phys. **B271**, 1 (1986); J. Augustin *et al.*, *ibid.* **B320**, 1 (1989).