Hadronic part of the muon anomalous magnetic moment: An improved evaluation

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(Received 15 June 1989; revised manuscript received 22 January 1990)

We have done a new evaluation of the lowest-order hadronic vacuum-polarization contribution $a_\mu^{\rm vac}$ to the anomalous magnetic moment of the muon. The result is $a_\mu^{\rm vac}=(7052\pm76)\times10^{-11}$ or $a_\mu^{\rm vac}=(7048\pm115)\times10^{-11}$ depending on the way in which the experimental systematic errors of the dominant two-pion contribution to $a_\mu^{\rm vac}$ are taken into account. The more pessimistic error, though numerically equal to the earlier result of Casas, López, and Ynduráin, is still an improvement over the latter since it includes also the model error omitted in the previous analysis. The increased accuracy of $a_\mu^{\rm vac}$ has been achieved through the use of global analytic models of the pion and kaon form factors for the two-pion and the two-kaon contributions as well as due to new experimental information mainly for the three-pion channel.

I. INTRODUCTION

Anomalous magnetic moments of leptons are traditional quantities for an extremely detailed confrontation of QED predictions with experimental results. Unlike the electron $g-2$ factor, which to the precision so far achieved is a pure leptonic effect, the consequence of a relatively large muon mass is that the interactions of a non-lepton-photon origin contribute to the total muon anomaly a_μ at the level of $6\times10^{-3}\%$. Because of precise QED calculations up to four loops,[1,2] yielding

$$a_\mu({\rm QED})=(116\,584\,800\pm30)\times10^{-11} \tag{1}$$

as well as highly accurate measurements[3]

$$\begin{aligned} a_{\mu^+}&=(116\,591\,000\pm1200)\times10^{-11}\,,\\ a_{\mu^-}&=(116\,593\,600\pm1200)\times10^{-11}\,, \end{aligned} \tag{2}$$

this number is by far not negligible. In fact it is about six times larger than the experimental uncertainty in the a_μ value. The non-QED part of the muon anomaly is dominated by the lowest-order hadronic vacuum-polarization contribution $a_\mu^{\rm vac}$ (Fig. 1). In spite of the gradual diminishing of the error of this component in recent years,[2,4] it is still known with an error four times larger than the error of the pure QED part. As has been stressed in Ref. 2, making the theoretical value of the hadronic part of a_μ more precise is crucial for the possibility of detecting in a measured anomoly the one-loop weak-interaction contribution evaluated as[2]

$$a_\mu({\rm weak})=(195\pm1)\times10^{-11}\,. \tag{3}$$

Since a new generation of $g-2$ experiments with considerably improved precision is under consideration,[5] it is desirable to come up with a value of $a_\mu^{\rm vac}$ as accurate as possible in relation to the accuracy level of the QED contribution, enabling one in this way to perform an important independent test of the Glashow-Weinberg-Salam (GWS) electroweak gauge theory.

In the present work we describe an attempt to diminish the error of the lowest-order hadronic vacuum-polarization contribution to a_μ. There are a few reasons one could hope to achieve this goal. First, we have developed global analytic models for pion and kaon form factors[10,16] in recent years. The models formulated in terms of physical parameters reproduce the data simultaneously in the spacelike and timelike regions. We use these parametrizations for the evaluation of the two-pion and two-kaon contributions to $a_\mu^{\rm vac}$ including in this way also the experimental information from the spacelike region. Another reason for a possible accuracy improvement of the theoretical value of $a_\mu^{\rm vac}$ is that, in addition to the new data on pion and kaon form factors, significantly better data on the three-pion e^+e^- annihilation have become available recently due to new measurements in Novosibirsk.[17] Last but not least, we believe that there are possibilities to perform the error analysis in the individual channels contributing to $a_\mu^{\rm vac}$ in a more quantitative and systematic way than has been done in the previous works.[2,4]

We describe our treatment of $a_\mu^{\rm vac}$ and the corresponding error analysis in Sec. III, while the final results with their discussion are given in Sec. IV. In the following

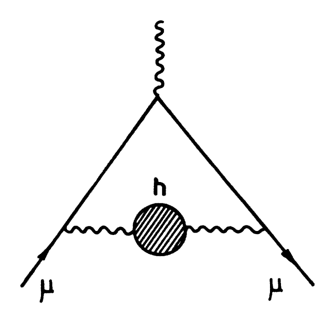

FIG. 1. The lowest-order hadronic vacuum-polarization contribution to the anomalous magnetic moment of the muon.

section the results of the earlier calculations[2,4] are briefly summarized.

II. PRESENT KNOWLEDGE OF HADRONIC CONTRIBUTIONS TO a_μ

The hadronic part of a_μ has been known with gradually higher precision in connection with the improvement of information on the cross section $\sigma(e^+e^-\to\text{hadrons})$ from the experiments on e^+e^- colliders in Frascati, Novosibirsk, and Orsay. The relevance of e^+e^- annihilation measurements to a_μ^{vac} is based on the fact that $\sigma(e^+e^-\to\text{hadrons})$ enters into the integral representation which serves as a basis for all calculations of a_μ^{vac} [see Eq. (5) below]. The last two evaluations of a_μ^{vac} have been done in 1985 and read

$$a_\mu^{\text{vac}}=(7070\pm60\pm170)\times10^{-11}\quad(\text{KNO})\ , \tag{4a}$$

$$a_\mu^{\text{vac}}=(7100\pm105\pm49)\times10^{-11}\quad(\text{CLY})\ , \tag{4b}$$

where the first error is statistical, the second is systematic and the abbreviations refer to Kinoshita *et al.*[2] and Casas *et al.*[4] In what follows we characterize the main features of both the analyses.

The first group of authors[2] has calculated the contributions from individual channels of the reaction $e^+e^-\to\text{hadrons}$ separately. The four-parameter modified Gounaris-Sakurai parametrization of the pion form factor has been used for the dominant two-pion part. While the statistical error has been evaluated by the covariance matrix of the fit ($\chi^2/N_{DF}=1.85$), the systematic error has been assessed from the deviation of the found mean value of a_μ and the mean value obtained by the trapezoidal-rule integration over the experimental points. The result 150×10^{-11} is the main contributor to the total error in (4a). The low-energy three-pion and two-kaon parts of a_μ^{vac} were treated by the Breit-Wigner formula for the ω and ϕ resonances. The statistical error was estimated from the statistical errors of the measured total and e^+e^- widths. The systematic error has been taken equal to the systematic error of both the widths. The same error estimates were done also for the contributions of the J/ψ and Υ resonances, treated in the narrow-width approximation. The contributions of other channels have been obtained by the trapezoidal-rule integration over the experimental data for

$$R=\sigma(e^+e^-\to\text{hadrons})/\sigma(e^+e^-\to\mu^+\mu^-)$$

whose errors were taken as an error estimate for this part of a_μ^{vac}.

The second group of authors[4] has reduced the essential part of the errors coming from the region $s>2\ \text{GeV}^2$ by employing the $O(\alpha_s^2)$ QCD expression for the quantity R. The error in this treatment comes from the uncertainty in the value of the QCD scale parameter Λ and from the neglected higher-order terms in R. The J/ψ and Υ resonances were evaluated in the narrow-width approximation and regions of $c\bar{c}$ and $b\bar{b}$ thresholds by the experimental data on R.

The integration over experimental points has been used also in the region $0.8\ \text{GeV}^2\le s\le2\ \text{GeV}^2$ for the 2π, 3π, 4π, 5π, 6π, K^+K^-, and $K_s^0K_L^0$ channels. The largest statistical error ($\sim17\%$) was found for the three-pion contribution and attributed to experimental uncertainties in the ϕ region. The total systematic error from this region was given implicitly in the overall systematic error of the a_μ^{vac} value (4b).

A great deal of Ref. 4 is devoted to the thorough numerical study of the dominant low-energy two-pion contribution to a_μ. It is performed in terms of a 15-parameter pion form-factor representation written as a product of the Omnes function and the inelastic part with correct analytic properties, normalization and the asymptotic behavior. The inelastic part is parametrized in terms of higher vector-meson contributions, a three-parameter background function and a function providing the asymptotic behavior of $F_\pi(s)$. The equality between the form-factor phase and the phase $\delta_1^1(s)$ of $I=J=1$ partial $\pi\pi$ scattering wave for $s\le0.8\ \text{GeV}^2$ is used in the integrand of the Omnes function. Two methods for the evaluation of the two-pion part $a_\mu^{2\pi}$ based on different parametrizations of $\delta_1^1(s)$ were applied to assess the systematic error of $a_\mu^{2\pi}$. The value 27×10^{-11} (compared to 150×10^{-11} of KNO) is an essential source of diminishing the total error of a_μ in Ref. 4. The mean value of $a_\mu^{2\pi}$ and its statistical error were obtained by the variational analysis of the experimental data on the form-factor inelastic part.

Closing this section we note that though KNO have found smaller statistical errors than CLY in all channels, the latter authors were able to diminish the total error of a_μ^{vac} for essentially two reasons: the use of QCD in the high-energy region and due to taking the deviation of the mean values of $a_\mu^{2\pi}$ in two methods as a measure of the systematic uncertainty of the dominant two-pion part of a_μ.

III. CALCULATION OF THE LOWEST-ORDER HADRONIC VACUUM CONTRIBUTION TO a_μ

All calculations of a_μ^{vac} are based on the integral representation[6]

$$a_\mu^{\text{vac}}=\frac{1}{4\pi^2\alpha}\int_{4m_\pi^2}^{\infty}\sigma^h(s)K_\mu(s)ds\ , \tag{5}$$

where α is the fine-structure constant, $\sigma^h(s)$ stands for the cross section $\sigma(e^+e^-\to\text{hadrons})$, and $K_\mu(s)$ is the function coming from the triangle Feynman diagram for a_μ corresponding to the exchange of a "particle" with the propagator $-ig_{\mu\nu}(q^2-s)^{-1}$:

$$K_\mu(s)=\frac{\alpha}{\pi}\int_0^1\frac{x^2(1-x)dx}{x^2+(1-x)s/m_\mu^2}\ . \tag{6}$$

A decomposition of the integrand to partial fractions leads to the explicit form

$$K_\mu(s)=\frac{\alpha}{\pi}\frac{1}{Sy}\left[\left[\frac{y}{2}(1+S)-y\right]\left[1-\frac{y}{2}(1+S)\right]\ln\left[1-\frac{2}{y(1+S)}\right]\right.$$
$$\left.-\left[\frac{y^2}{2}(1-S)-y\right]\left[1-\frac{y}{2}(1-S)\right]\ln\left[1-\frac{2}{y(1-S)}\right]+\frac{Sy}{2}-y^2S\right], \quad (7)$$

where $S=\sqrt{1-4/y}$, $y=s/m_\mu^2$.

As can be seen from Eq. (6), $K_\mu(s)$ behaves as $\alpha m_\mu^2/3\pi s$ for $s \gg m_\mu^2$, suppressing in this way the contributions from the higher-energy region.

Formula (5) can be derived by replacing the free photon propagator in the $O(\alpha)$ amplitude for a_μ by the exact photon propagator, defined in terms of the (hadronic) polarization operator $\Pi^h(s)$. Writing a dispersion integral for the latter and isolating the invariant function at the tensor structure $\sigma_{\mu\nu}k^\nu$, which at $k^2=0$ defines the anomalous magnetic moment (k is the four-momentum of the external photon), one obtains a_μ^{vac} as a superposition of the amplitudes $K_\mu(s)$ with the weight function $\text{Im}\Pi^h(s)/\pi s$. The usefulness of this representation for a_μ^{vac} follows from the well-known relation

$$\text{Im}\Pi^h(s)=\frac{s\sigma^h(s)}{16\pi^2\alpha^2}=\frac{1}{12\pi}R(s) , \quad (8)$$

providing the possibility to employ rich experimental information from the reaction $e^+e^-\to$hadrons for the calculation of a_μ^{vac} via relation (5). As a consequence, the accuracy of the result depends primarily on the precision of the measured cross section for individual annihilation channels. However, as we shall try to demonstrate, one can non-negligibly reduce the errors of a_μ^{vac} by choosing more realistic and adequate models for the cross sections $\sigma^h(s)$.

In our calculation of a_μ^{vac} we have divided the integral in (5) into the low-energy ($s<s_0=2$ GeV^2) and high-energy ($s>s_0$) parts. Following CLY we have used QCD in the latter, including, however, the $O(\alpha_s^3)$ term to the perturbative expansion of the ratio

$$R=\sigma(e^+e^-\to\text{hadrons})/\sigma(e^+e^-\to\mu^+\mu^-)$$

and confronting this calculation with the result obtained by integration over experimental data on R (Sec. III B). As to the chosen position of the point s_0, it is dictated from one side by the validity of perturbative QCD and from the other side by the fact that we are able to estimate the two-pion and the two-kaon contributions by means of the reliable form-factor models in the whole region $4m_\pi^2\le s\le s_0$ in which the corresponding integrals are saturated almost completely.

A. The low-energy region

We treat each channel in this region separately. In order to achieve realistic and quantitative error estimates, we include in our analysis the uncertainties coming from the experimental input as well as the ones induced by the models used for the cross section $\sigma^h(s)$. The first type of error consists of the statistical and systematic ones which we combine in quadrature (see a discussion in the Introduction of the Particle Data Group[20]). The total experimental error of a_μ is then computed from the covariance matrices of the corresponding fits and represents in fact an optimistic estimate. This point has negligible significance for all channels except the two-pion one since the latter contributes to a_μ^{vac} at the level of 70%. For this reason we have calculated the systematic uncertainty for the two-pion part also by another method which led to the more pessimistic estimate (see Sec. IV).

As already mentioned, we evaluate the model errors, too. Their actual value in each channel will be determined from the deviation of the a_μ value obtained by integration over experimental points using trapezoidal rule and by the integration based on the model parametrizations.

According to the remark after Eq. (7) it will be the contribution of the process $e^+e^-\to\pi^+\pi^-$ which will dominate in a_μ^{vac}. Its cross section is given by

$$\sigma^{2\pi}(s)=\frac{\pi\alpha^2\beta^3}{3s}\left|F_\pi(s)+\xi e^{i\phi}\frac{m_\omega^2}{m_\omega^2-s-im_\omega\Gamma_\omega}\right|^2 , \quad (9)$$

where $\beta=(1-4m_\pi^2/s)^{1/2}$ is the velocity of an outgoing pion in the c.m. system (c.m.s.) and the second term in (9) describes the part of the $\pi^+\pi^-$ final state due to the isospin-nonconserving ω-meson decay. Parameters ξ and ϕ are the modulus and the phase of the ρ-ω interference amplitude.[7,8]

It turns out that it is of crucial importance to find suitable and adequate parametrization of the complex function $F_\pi(s)$. For example, the modified Gounaris-Sakurai formula used by KNO which takes into account the inelastic ρ-ω channel by the effective factor with three parameters fixed by hand does not give a fully satisfactory description of the data. It manifests itself in a rather large deviation of the final result for the two-pion part $a_\mu^{2\pi}$ from the value obtained by direct integration over the data points of $\sigma^{2\pi}$. Problems with a simultaneous description of the spacelike and timelike pion-form-factor data (and data on δ_1^1)[9] indicate a possible inconsistency also in the model of the CLY caused probably by the choice of the parametrization of the inelastic part of $F_\pi(s)$. The nonadequate description of the data above 1 GeV^2 is likely the reason why the authors compute the contribution from the region $0.8\le s\le 2$ GeV^2 directly by means of the data instead of the model.

For our calculation of $a_\mu^{2\pi}$ we choose the analytic pion-form-factor model[10] which incorporates all well-established properties of $F_\pi(s)$, particularly the analytic ones, as precisely as possible. The model is formulated in the conformally mapped cut-free variable W:

$$W=i\frac{(q_1+q)^{1/2}-(q_1-q)^{1/2}}{(q_1+q)^{1/2}+(q_1-q)^{1/2}}\ ,$$
$$q=\tfrac{1}{2}(s-4m_\pi^2)^{1/2},\quad q_1\equiv q(s_1)\ , \tag{10}$$

where s_1 is the position of the square-root branch point which together with the elastic branch point at $s=4m_\pi^2$, corresponding cuts and complex-conjugated pairs of resonance poles define the pion-form-factor analytic structure in the complex s plane. As shown in Figs. 2(a)–2(c), the conformal mapping $s\to W$ transforms the first and the second sheet inside of the unit disc and the third and the fourth sheet outside of it with the simultaneous removal of both the elastic and inelastic cuts. The only remaining singularities are the four poles corresponding to every resonance and the pole at $W_P=0.23$ which, together with a zero at $W_Z=0.21$, simulates the left-hand cut from the second Riemann sheet.[11] Consequently, one can write a formula for $F_\pi(s)$ (Taylor series) reflecting its analytic structure in the W plane, as

$$F_\pi(W)=\frac{(W^2-1)^2(W-W_Z)\sum_0^4 A_n W^n}{(W-W_P)D_\rho(W)D_{\rho'}(W)D_{\rho''}(W)}\ , \tag{11}$$

where the unknown coefficients A_n are real due to the reality condition, the extracted factor $(W^2-1)^2$ ensures the asymptotic behavior $\sim 1/s$, and

$$D_\rho(W)=(W-W_\rho)(W-W_\rho^*)(W-W_\rho^{-1})(W-W_\rho^{*-1})\ ,$$
$$D_v(W)=(W-W_v)(W+W_v)(W-W_v^*)(W+W_b^*)\ ,$$
$$v=\rho',\rho''\ ,$$

with W_v $(v=\rho,\rho',\rho'')$ being the positions of the resonance poles. Five coefficients A_n can be expressed in terms of the resonance masses m_v, widths Γ_v, and coupling-constant ratios $g_v=f_{v\pi\pi}/f_v$ ($f_{v\pi\pi}$ and f_v correspond to the transitions $v\to\pi^+\pi^-$ and $v\to\gamma$, respectively) by requiring the correct normalization $F_\pi(0)=1$ and the threshold behavior $\delta_1^1\sim q^3$ for $q\to 0$ together with taking into account a connection of vector-meson-dominance (VMD) pion-form-factor representation with formula (11) in the limit $\Gamma_v\to 0$ separately for ρ, ρ', and ρ'' resonances (for a more complete treatment of the model see Refs. 10).

Formula (11) has been compared with 245 data on F_π (see Ref. 12, references therein, and Refs. 13 and 14) from the spacelike and timelike regions. The fitted parameters $A_i(i=1,\ldots,9)$ were $\mathrm{Re}W_\rho$, $\mathrm{Re}W_{\rho''}$, $\mathrm{Im}W_\rho$, $\mathrm{Im}W_{\rho''}$, the coupling-constant ratios g_ρ, $g_{\rho'}$, $g_{\rho''}$, the position of the effective inelastic threshold s_1, and the modulus of the ρ-ω interference amplitude ξ. The interference phase ϕ can be expressed by m_ρ, Γ_ρ, and m_ω.[15,8] The parameters $\mathrm{Re}W_{\rho'}$ and $\mathrm{Im}W_{\rho'}$ of the resonance $\rho'(1250)$ have been fixed at the values corresponding to $m_{\rho'}\sim 1310$ MeV, $\Gamma_{\rho'}\sim 400$ MeV which are typical for a few fits with small modifications of formula (11). The presence of the resonance ρ' is important for the quality of the fit; however, fixing its parameters is necessary due to the fact that data points are rather scattered in this region and making $m_{\rho'}$ and $\Gamma_{\rho'}$ free would introduce rather strong correlations to the covariance matrix.

The results of the best fit (transformed to the s plane) are shown in Table I(a). A good simultaneous description of the spacelike and timelike data has been achieved with $\chi^2/N_{\mathrm{DF}}=1.16$. Numerical evaluation of the integral (5) with K_μ, $\sigma^{2\pi}$, and F_π given in (7), (9), and (11) yields

$$a_\mu^{2\pi}=(4989\pm 41)\times 10^{-11}\ , \tag{12}$$

where the total experimental error 41×10^{-11} has been obtained by the formula

$$\sigma^2=\sum_{ij} C_{ij}D_iD_j,\quad D_i=\frac{\partial a_\mu}{\partial A_i}\ . \tag{13}$$

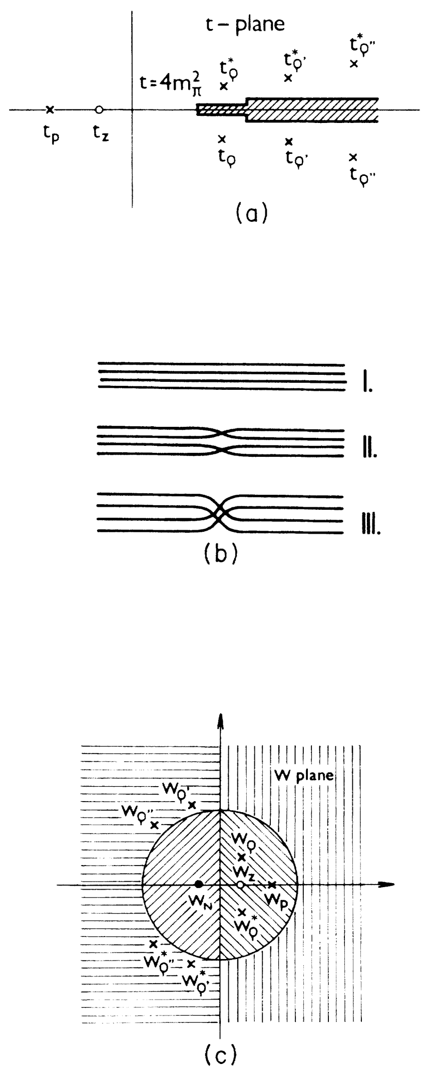

FIG. 2. (a) The analytic structure of the pion form factor in our model. In addition to the pairs of the ρ-meson poles also the pole t_P and the zero t_Z simulating the left-hand cut are shown. (b) The interconnection of individual sheets on crossing the real t axis: (I) below $4m_\pi^2$, (II) between the elastic threshold at $4m_\pi^2$ and the inelastic threshold, (III) above the inelastic threshold. (c) The W plane free of cuts with ρ,ρ',ρ'' poles and with W_P, W_Z simulating the left-hand cut.

TABLE I. (a) The results of the fit of the pion-form-factor model (11). (b) The results of the fit of the kaon-form-factor model (15). $s_{\rm inel}^{s}$ and $s_{\rm inel}^{v}$ are the effective inelastic thresholds. (c) The results of the fit of the cross section (20). Note: In this table only the statistical errors are shown.

	(a)	
$m_{\rho}=760\pm4$ MeV	$\Gamma_{\rho}=143\pm3$ MeV	$g_{\rho}=1.19\pm0.03$
$m_{\rho''}=1743\pm110$ MeV	$\Gamma_{\rho''}=280\pm96$ MeV	$g_{\rho''}=-0.06\pm0.02$
$s_1=1.42\pm0.06$ GeV^2	$\xi=0.0146\pm0.0006$	$g_{\rho'}=-0.40\pm0.06$
	(b)	
$m_{\Phi}=1019.4\pm0.7$ MeV	$\Gamma_{\Phi}=4.3\pm0.8$ MeV	$f_{\Phi K\bar{K}}/f_{\Phi}=0.33\pm0.01$
$m_{\Phi'}=1660\pm21$ MeV	$\Gamma_{\Phi'}=158\pm37$ MeV	$f_{\omega K\bar{K}}/f_{\omega}=0.20\pm0.01$
$m_{\rho'}=1315\pm183$ MeV	$\Gamma_{\rho'}=245\pm167$ MeV	$f_{\rho K\bar{K}}/f_{\rho}=0.57\pm0.01$
$m_{\rho''}=2114\pm140$ MeV	$\Gamma_{\rho''}=150\pm104$ MeV	$f_{\rho'' K\bar{K}}/f_{\rho''}=-0.04\pm0.01$
$s_{\rm inel}^{s}=1.68\pm0.03$ GeV^2	$s_{\rm inel}^{v}=1.72\pm0.04$ GeV^2	
	(c)	
$m_{\omega}=781.8\pm0.3$ MeV	$\Gamma_{\omega}=9.5\pm0.8$ MeV	$\sigma(\omega)=1519\pm120$ nb
$m_{\Phi}=1019.6\pm0.3$ MeV	$\Gamma_{\Phi}=4.3\pm0.7$ MeV	$\sigma(\Phi)=623\pm92$ nb

C_{ij} is the nine-by-nine external covariance matrix of the fit as given by the Hesse subroutine of the MINUIT program (with the parameter UP adjusted to nine parameters). The values of diagonal matrix elements have been checked by the MINOS subroutine. Evaluating the same integral by the trapezoidal rule (the CERN program TRAPER) we find

$$a_{\mu}^{2\pi}=(4906\pm24)\times10^{-11}\ . \tag{14}$$

From the difference between (12) and (14) we estimate the model error to be about 42×10^{-11}. We recall that the same method has been used by KNO to estimate the *systematic* experimental error with the result 150×10^{-11}. Adding our two errors in quadrature gives for the total error of the dominant two-pion part of $a_{\mu}^{\rm vac}$ a value of 59×10^{-11} (see a discussion in Sec. IV, however).

In principle the same procedure can be applied to the two-kaon contributions. A suitable generalized VMD model for the charged and neutral kaon form factors with correct analytic properties has been derived in Ref. 16. The final formulas for the isoscalar (s) and isovector (v) parts read

$$F_K^s(V)=\left[\frac{1-V^2}{1-V_N^2}\right]^2\sum_{s=\omega,\phi,\phi'}\frac{f_{sK\bar{K}}}{f_s}\frac{(V_N-V_s)(V_N-V_s^*)(V_N-\bar{V}_s)(V_N-\bar{V}_s^*)}{(V-V_s)(V-V_s^*)(V-\bar{V}_s)(V-\bar{V}_s^*)}\ , \tag{15a}$$

$$F_K^v(W)=\left[\frac{1-W^2}{1-W_N^2}\right]^2\sum_{v=\rho,\rho',\rho''}\frac{f_{vK\bar{K}}}{f_v}\frac{(W_N-W_v)(W_N-W_v^*)(W_N-\bar{W}_v)(W_N-\bar{W}_v^*)}{(W-W_v)(W-W_v^*)(W-\bar{W}_v)(W-\bar{W}_v^*)}\ . \tag{15b}$$

The variable W is the same as in (10); the variable V is defined in a similar way by means of the three-momentum $r=\frac{1}{3}(s-9m_{\pi}^2)^{1/2}$. An effective inelastic threshold in the r plane is assumed analogously to q_1 in the q plane. The points V_N and W_N correspond to the normalization point $s=0$. The factors in front of the sums in Eqs. (15a) and (15b) give the asymptotic behavior $\sim s^{-1}$ to the form factors. The ratios of the VMD coupling constants are restricted by the conditions

$$\sum_{s=\omega,\phi,\phi'}\frac{f_{sK\bar{K}}}{f_s}=\sum_{v=\rho,\rho',\rho''}\frac{f_{vK\bar{K}}}{f_v}=\frac{1}{2}\ , \tag{16}$$

which are the consequences of the normalization of F_K^s, F_K^v. As can be seen from Eqs. (15a) and (15b), each resonance is represented by four poles lying in the complex V, W planes (i.e., $\Gamma_v\neq0$, $\Gamma_s\neq0$) with

$$\bar{V}_s=V_s^{-1},\quad \bar{W}_v=W_v^{-1}\quad\text{or}\quad \bar{V}_s=-V_s,\quad \bar{W}_v=-W_v \tag{17}$$

depending on the relative position of the resonance and the effective threshold. Finally, the form factors of the charged and neutral kaons are given by linear combinations of F_K^s and F_K^v:

$$F_{K^\pm}=F_K^s+F_K^v,\quad F_{K^0}=F_K^s-F_K^v\ . \tag{18}$$

The number of free parameters of the model can be reduced to 14 by Eq. (16) and by fixing m_{ρ}, Γ_{ρ}, m_{ω}, Γ_{ω} at their table values as $\rho(770)$ and $\omega(783)$ lie in the unphysical region and one could hardly expect to be able to determine them with a sufficient accuracy from the fit. The optimal values of the fitted parameters (two inelastic thresholds, four ratios of coupling constants and positions of four resonances in complex V and W planes) from the analysis of all 138 available data (see Ref. 16) of the charged and neutral kaon form factor are given in Table I(b). The data are reproduced very well ($\chi^2/N_{\rm DF}=1.02$) for $s>0$ as well as for $s<0$. Since kaons are pseudoscalars, the cross section of the reactions

$e^+e^- \to K^+K^-$, $K^0_S K^0_L$ is completely analogous to (9) (without the second term, of course) and the rest of the analysis goes as for the pion contribution. The results are

$$a_\mu^{K^+K^-} = (223 \pm 21 \pm 3) \times 10^{-11} , \quad a_\mu^{2K^0} = (185 \pm 19 \pm 3) \times 10^{-11} . \tag{19}$$

Since the data on F_{K^0} do not cover the whole ϕ-meson region, the model error of the $K^0_S K^0_L$ contribution has been estimated by the corresponding error of the K^+K^- one. The model error is very small reflecting the reliable description of the data by the parametrization [(15a) and (15b)].

Further important contribution to a_μ^{vac} is the three-pion one. It is this portion which is determined in our work with substantially improved precision and contributes significantly to the reduction of the total error of a_μ^{vac}. The improvement comes from the two sources: new precise measurements[17] of the $e^+e^- \to \pi^+\pi^-\pi^0$ cross section in the ω region performed recently in Novosibirsk and, to some extent, from the use of Breit-Wigner formulas to *fit* the data on $\sigma^{3\pi}$. Though KNO have also used Breit-Wigner parametrizations for the ω and ϕ resonances, they have performed error estimates by means of the statistical errors of the measured total and e^+e^- widths of ω and ϕ and not by fitting the experimental cross sections. On the other hand, CLY integrate over experimental data with a large resultant statistical error of $\sim 17\%$.

For our calculation of the three-pion contribution we employ the Breit-Wigner parametrization of the form

$$\sigma^{3\pi}(s) = \left| \sqrt{\sigma(\omega)} \frac{m_\omega \Gamma_\omega}{m_\omega^2 - s - is^{1/2}\Gamma_\omega(s)} - \sqrt{\sigma(\phi)} \frac{m_\phi \Gamma_\phi}{m_\phi^2 - s - is^{1/2}\Gamma_\phi(s)} \right|^2 , \tag{20}$$

where $\sigma(\omega)$ and $\sigma(\phi)$ are the cross-section values in the ω and ϕ peaks and $\Gamma_i(s) = \Gamma_i s^{3/2} m_i^{-3}$, $i = \omega, \phi$. To take into account the ω-ϕ interference with negative relative sign is important for the correct description of the data in the region between the two resonances and above ϕ.[18] We have used essentially the same data above 0.81 GeV^2 as KNO and CLY with the addition of the 17 data points from Ref. 19. On the other hand, for $s < 0.81$ GeV^2 new high-quality data from the experiment with cryogenic magnetic detector in Novosibirsk have become available recently.[17] In the experiment a new method of resonance depolarization for the beam energy calibration has been applied for the first time. This procedure led to significant suppression of the systematic errors. Since the statistical errors of the measurement have also been reduced in comparison with earlier experiments and the results[17] are fully compatible with the world averages,[20] we take only these data for $s < 0.81$ GeV^2. The optimal values of the fitted parameters obtained by comparing formula (20) with 76 data points from the interval $9m_\pi^2 \le s \le 2$ GeV^2 are shown in Table I(c). Evaluation of the three-pion portion of the integral (5) by means of the CERN program RIWIAD using the Breit-Wigner cross section (20) with resonance parameters from Table I(c) yields

$$a_\mu^{3\pi} = (569 \pm 22 \pm 18) \times 10^{-11} . \tag{21}$$

The first error is obtained from the covariance matrix of the fit based on data with combined errors and is dominated by the statistical uncertainties. The model error is rather large because we have included in it the contribution coming from our ignorance of the experimental behavior of $\sigma^{3\pi}$ below 0.5 GeV^2. The value 16×10^{-11} was estimated from the difference of $a_\mu^{3\pi}$ values obtained by extrapolating the model curve (20) to the three-pion threshold and by the TRAPER integration over the experimental cross section starting at the point $s = (0.7502$ $\text{GeV})^2$.

The last contributions to a_μ^{vac} from the region below 2 GeV^2 come from the processes $e^+e^- \to 4\pi, 5\pi, 6\pi$. We perform TRAPER integration for these components of a_μ^{vac}. One could in principle try to fit the data by the Breit-Wigner functions in $2\pi^+2\pi^-$ and $\pi^0\pi^0\pi^+\pi^-$ channels, but the intermediate resonance states are not completely clear for these processes[19,21] and, moreover, for our purposes we need only a part of the corresponding cross sections below the peak. We use the same data as in Refs. 2 and 4, supplemented, however, by important new measurements[19] for both the four-pion channels. The results are displayed together with all other low-energy contributions in Table II where the combined experimental errors for the multipion channels are given. The systematic error of 10% has been added in quadrature to the statistical TRAPER error.

TABLE II. Contributions from the region $s < 2$ GeV^2 to $10^{11} a_\mu$.

Channel	Mean value	Experimental error	Model error
$\pi^+\pi^-$	4989	41	42
K^+K^-	223	21	3
$K^0_S K^0_L$	185	19	3
$\pi^0\pi^+\pi^-$	569	22	18
$\pi^0\pi^0\pi^+\pi^-$	140	14	
$\pi^+\pi^-\pi^+\pi^-$	55	6	
$5\pi, 6\pi$	7	2	
Total	6168	57	46

B. High-energy region

As noticed by CLY, one can considerably reduce the errors of the integral (5) coming from the region $s>2$ GeV^2 by considering the QCD expression for the quantity R instead of experimental data from individual channels. Really, KNO, who have used data, quote a rather large systematic error, for example, for the contribution of more than two hadrons (43×10^{-11}). On the other hand, there is no systematic error if one uses QCD. In our opinion, however, it is necessary in this case to check that the results obtained with the help of the QCD expression for the quantity R and by integrating over experimental data on R really coincide. In this section we describe our work in this direction.

The experimental information on $R(s)$ is rather rich. In the analysis we use data from 21 different experiments published during the last ten years as collected by Marshall.[22] The author has performed a simultaneous fit of these data sets to reliably determine the strong coupling constant α_s. One of his conclusions is that three data sets[23–25] should be renormalized modestly in order to be consistent with the remaining sets. We follow this prescription in the evaluation of the high-energy contribution to a_μ^{vac} by means of data on R. The result of TRAPER integration is

$$a_\mu^R=(817\pm13)\times10^{-11} \tag{22}$$

and the effect of the downward renormalization is to decrease the a_μ value by 55×10^{-11}. Of course, the above result concerns only the continuum. The contributions from the J/ψ and Υ resonance families should be added. In the narrow-width approximation they are expressed as

$$a_\mu^{\text{res}}=\frac{3\Gamma_{ee}}{\alpha m_{\text{res}}}K_\mu(m_{\text{res}}^2)\,, \tag{23}$$

where Γ_{ee} is the e^+e^- width of a given resonance, whose statistical and systematic errors induce the corresponding errors of a_μ^{res}. The total contribution from the J/ψ and Υ resonances is 71×10^{-11} (see Table III).

In the QCD calculation of the continuum contribution we have excluded the threshold regions 9.61–20.21 GeV^2 ($c\bar{c}$) and 81.0–196.0 GeV^2 ($b\bar{b}$) where the data have to be used. The systematic error of a_μ from these regions can be taken equal to the systematic uncertainties of the measurements, i.e., approximately 8%. The QCD expression for $R(s)$ calculated recently to $O(\alpha_s^3)$ is[26]

$$R(s)=3\sum_f Q_f^2\left[1+\frac{\alpha_s(s)}{\pi}+(1.986-0.115n_f)\left[\frac{\alpha_s(s)}{\pi}\right]^2+(70.985-1.200n_f-0.005n_f^2)\left[\frac{\alpha_s(s)}{\pi}\right]^3\right]-\left[\left[\sum_f Q_f\right]^2\times1.679\left[\frac{\alpha_s(s)}{\pi}\right]^3\right], \tag{24}$$

where Q_f is the electric charge of the quark of flavor f. It is interesting that the coefficient of the $O(\alpha_s^3)$ correction is unexpectedly large, affecting significantly the value of the extracted QCD scale parameter[26] $\Lambda_{\overline{\text{MS}}}$, where $\overline{\text{MS}}$ denotes the modified minimal-subtraction scheme. The effect of this next-next-to-leading term on the value of a_μ^R may therefore also be non-negligible. Indeed, we have found, for example, for the contribution from the region $2\le s\le9.61$ GeV^2,

$$\begin{aligned}O(\alpha_s^2)&:a_\mu^R=(562\pm8)\times10^{-11}\,,\\ O(\alpha_s^3)&:a_\mu^R=(586\pm17)\times10^{-11}\,.\end{aligned} \tag{25}$$

As can be seen the inclusion of the $O(\alpha_s^3)$ correction into R increases twice the error induced by the uncertainty of the parameter Λ (for the latter we took[27] $\Lambda=150\pm50$ MeV). Summing up all contributions from Table III, we find

TABLE III. Contributions from the region $s>2$ GeV^2 to $10^{11}a_\mu$.

Interval (GeV^2) and method	Mean value	Stat. error	Syst. error	Model error
$2\le s\le9.61$ QCD	586	17		2
$9.61\le s\le20.21$ $c\bar{c}$ thres., data	98	3	8	
$20.20\le s\le81.0$ QCD	90	1		0
$81.0\le s\le196.0$ $b\bar{b}$ thres., data	19	0	2	
$s>196$, QCD	20	0		0
ψ,Υ resonances	71	4	4	
Total	884	18	9	2

$$a_\mu^R=(884\pm 18\pm 9\pm 2)\times 10^{-11}\ , \quad (26)$$

where the first error is statistical and is dominated by the contribution induced by the uncertainty in the parameter Λ [see Eq. (25)]. The second error comes mainly from the $c\bar{c}$ and $b\bar{b}$ threshold regions. Its magnitude is estimated from the systematic errors of the data on R which are about 8%. The model error is negligible, since after the slight renormalization of three R data sets and the inclusion of the third-order term in (24) both the methods used yield the same value of a_μ^R.

We would like to finish this section by a few comments. First, notice that the statistical error in (26) comes from the QCD evaluation and is *larger* than the error of (22) obtained by means of data. The real advantage of using QCD rests in eliminating systematic error of the high-energy part of a_μ. As already mentioned, the coefficient of the $O(\alpha_s^3)$ term of $R(s)$ is unexpectedly large and it would be of crucial importance to know also the next term in the perturbative expansion to be sure that the QCD prediction for the quantity R is reliable. However, it has been argued in Refs. 26 and 22 that the coefficient is large not because the asymptotic nature of the QCD series has started to manifest itself but, rather because the second-order coefficient was accidentally small. This conjecture is supported by the fact that the fits with the inclusion of the $O(\alpha_s^3)$ term gave more consistent results for the parameter Λ than the second-order fits. This is in accord with our finding that the integration over the third-order QCD expression (24) and the integration over data yield almost identical results. A more prudent point of view is that the QCD calculation is unreliable. In this case one should forget the QCD result (24) and include also the systematic errors of the data which induce the systematic error of a_μ^R of about 60×10^{-11}.

IV. SUMMARY AND CONCLUSIONS

Our final result obtained by summing up all entries in Tables II and III is

$$a_\mu^{\rm vac}=(7052\pm 60\pm 46)\times 10^{-11}\ , \quad (27)$$

where the errors have been added quadratically. The first error in (27) is induced by the combined uncertainties of the data used in our analysis and the second one is our estimate for the total model error. Comparing (27) with the previous results (4a) and (4b) we see that while our central value is very close to them confirming thus the overall consistency of all three results, the real improvement over the last analysis[4] rests in diminishing the total error by 30% down to the value $76\times 10.^{-11}$ The increase in the accuracy of $a_\mu^{\rm vac}$ comes from the low-energy region. First, the statistical errors of the two-pion and two-kaon contributions have been reduced by a factor of 1.5 and 2, respectively, due to the use of rather accurate global analytic models of the pion and kaon electromagnetic form factors. Second, new precise experimental data on the cross section $\sigma(e^+e^-\to\pi^0\pi^+\pi^-)$ analyzed by means of the interfering ω and ϕ Breit-Wigner amplitudes led to a significant reduction of the errors of this channel.

As has been stressed several times, the main contribution to $a_\mu^{\rm vac}$ gives the two-pion part. Consequently, its error dominates in the total error. We have used a standard prescription of the Particle Data Group[20] to add the statistical and systematic errors of the data in quadrature before the fitting procedure. This has led to experimental uncertainty in the $a_\mu^{2\pi}$ value equal to 41×10^{-11}. A more prudent procedure of taking into account the systematic errors of the F_π measurements would be, for example, to shift the data upwards and downwards by their systematic error and to estimate $a_\mu^{2\pi}$ from the two fits to the shifted data. We have obtained in this way for a systematic part of the uncertainty of $a_\mu^{2\pi}$ a symmetric value 91×10^{-11}. The more pessimistic result is therefore

$$a_\mu^{\rm vac}=(7048\pm 105\pm 46)\times 10^{-11}\ . \quad (28)$$

We have done independent integrations directly over experimental data in the channels where models have been used. It was possible in this way to use the deviations of the two methods as a measure of possible model dependence of our results. The fact that the model errors are sufficiently small gives a certain credit to the final results on $a_\mu^{\rm vac}$. Further diminishing of the errors of the two-pion and three-pion components of $a_\mu^{\rm vac}$ will be possible when the recently proposed[28] precise measurement of $\sigma(e^+e^-\to\text{hadrons})$ at low energies will be realized.

Taking into account the QED and weak contributions as quoted in the Introduction together with the new value of higher hadronic (HH) contributions[2]

$$a_\mu(\text{HH})=(-41\pm 7)\times 10^{-11}\ , \quad (29)$$

the new value of the total anomalous magnetic moment of the muon will be

$$a_\mu=(116\,592\,006\pm 82)\times 10^{-11} \quad (30)$$

or

$$a_\mu=(116\,592\,002\pm 119)\times 10^{-11}\ , \quad (31)$$

where the latter corresponds to the more pessimistic estimate as described above. The total error is 42% (or 61%) of the one-loop effect of the weak interactions. This creates a real chance to detect this contribution (and also the possible one of the same order of magnitude predicted by some superstring-inspired models[29]) in the experimental value of a_μ after the improved $g-2$ measurements will be accomplished.

ACKNOWLEDGMENTS

The authors would like to thank Dr. S. B. Gerasimov, Dr. A. B. Govorkov, and Dr. J. Lánik for valuable comments and Professor V. A. Meshcheryakov for support. We are grateful also to Dr. R. Marshall for providing us the collected data sets on R.

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