Large-N_c higher-order weak chiral Lagrangians coupled to external electromagnetic fields: Applications to radiative kaon decays

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Using factorization and bosonization of quark currents derived from the anomalous Wess-Zumino-Witten terms and from the electroweak perturbations to a QCD-motivated model for nonanomalous p^4 chiral Lagrangians, we obtain leading $1/N_c$ fourth-order $\Delta S=1$ weak chiral Lagrangians coupled to external electromagnetic fields, valid to the zeroth order of gluonic corrections. Applications to radiative K decays, e.g., $K\to\pi\gamma^*\to\pi l^+l^-$, $K\to\pi\gamma\gamma$, $K\to\pi\pi\gamma$, and $K\to\pi l\nu\gamma$, are studied. The twofold ambiguity for the predictions of $K\to\pi l^+l^-$ is suggestively resolved. The branching ratio of $K^\pm\to\pi^\pm\gamma\gamma$ is predicted to be 5.1×10^{-7}, while the current upper bound based on a pion phase-space spectrum is 10^{-6}. The direct $K^\pm\to\pi^\pm\pi^0\gamma$ decays offer an excellent test on both anomalous and nonanomalous weak chiral Lagrangians; the agreement between theory and experiment is marvelous. The branching ratio of the interference between inner-bremsstrahlung and direct-emission amplitudes in the process $K_L\to\pi e\nu\gamma$ is found to be of order 10^{-5}.

I. INTRODUCTION

In a recent paper[1] we have derived higher-order effective chiral Lagrangians for $\Delta S=1$ nonleptonic weak interactions within the framework of the $1/N_c$ approach and applied them successfully to $K\to\pi\pi\pi$ decays. We extend in the present paper the previous work to derive electromagnetically induced effective weak chiral Lagrangians and study the applications to various radiative kaon decays: $K\to\pi\gamma^*\to\pi l^+l^-$, $K\to\pi\gamma\gamma$, $K\to\pi\pi\gamma$, and $K\to\pi l\nu\gamma$.

It is well known that the infrared properties of QCD can be elaborated on by chiral symmetry and PCAC (partial conservation of axial-vector current). The description of the interactions of pseudo-Goldstone bosons at low energies is model independent (e.g., chiral perturbation theory, current algebra, linear or nonlinear σ model) as long as chiral symmetry is respected. However, beyond the low-energy limit, the dynamics of meson interactions is no longer fixed by the requirement of chiral invariance alone: Higher-order chiral Lagrangians and unitarity corrections arising from chiral loops become important at moderate energies, say, 200–500 MeV.

Chiral symmetry is realized nonlinearly in chiral perturbation theory which is confined to describe only pseudoscalar mesons. The feature of nonlinear realization brings two effects: First, the chiral-Lagrangian description of strong and electroweak interactions at low energies is given in terms of perturbative expansion in powers of particle four-momenta and masses. Second, higher-order chiral Lagrangians depend on the choice of the renormalization scale μ as divergences of chiral loops are absorbed by the counterterms which have the same structure as that of higher-derivative Lagrangian terms. As a consequence, the couplings of higher-order chiral perturbation theory are running parameters and hence can be determined only *empirically* from various low-energy hadronic processes. (In general, only certain combinations of the running couplings are empirically extracted from experiment.) This means that no first-principles predictions can be made in the standard framework of chiral perturbation theory.

The aforementioned drawbacks with the chiral-Lagrangian approach can be circumvented in the large-N_c limit. A QCD-inspired model for p^4 nonanomalous effective action for strong interactions can be derived from the integration of nontopological chiral anomalies,[2,3] just as the well-known Wess-Zumino-Witten action is derived from the integration of topological Bardeen anomalies. Moreover, the coupling constants are renormalization scale independent as chiral loops are suppressed in the leading $1/N_c$ expansion. This approach is phenomenologically successful when applied to various low-energy physical processes.

In the previous paper,[1] we derived a large-N_c effective Lagrangian for nonleptonic $\Delta S=1$ weak interactions at order p^4 based on the following three ingredients: a rather simple structure of the effective weak Hamiltonian in the leading $1/N_c$ expansion, bosonization up to the subleading order, and factorization valid in the limit of large N_c. Confrontation with experiment for $K\to3\pi$ decays reveals a good agreement for two of the measured parameters in the Dalitz expansion of $K\to3\pi$ amplitudes. Based on the same approach, we derive in the present paper nonanomalous and anomalous fourth-order chiral Lagrangians (valid to the zeroth order in α_s) responsible for $\Delta S=1$ radiative weak transitions.

The radiative kaon decay is an ideal place to test the above-mentioned effective weak Lagrangians coupled to external photon fields: It cannot be generated from the lowest-order chiral Lagrangian since Lorentz and gauge invariance requires at least two powers of momenta in the radiative transition amplitude. Since to the zeroth order of α_s the coupling constants are fixed in the $1/N_c$ approach, predictions thus can be made at least in the limit of large N_c and in the absence of gluonic corrections. Especially, the study of the structure-dependent component of $K\to\pi\pi\gamma$, a formidable task before, now becomes manageable.

This paper is organized as follows. We present in Sec. II a brief overview of the formulism of chiral perturbation theory. A detailed derivation of the $\Delta S=1$ electromagnetically induced weak Lagrangians is given in Sec. III and applications to various radiative kaon transitions are discussed in Sec. IV. Section V is devoted to summary and conclusions.

II. OVERVIEW OF CHIRAL PERTURBATION THEORY

In this section we give a brief overview of the formulism of chiral perturbation theory for strong and electroweak interactions. The lowest-order chiral Lagrangian including explicit chiral-symmetry breaking for low-energy QCD is given by

$$\mathcal{L}_S^2=\frac{f_\pi^2}{8}\mathrm{Tr}(\partial_\mu U\,\partial^\mu U^\dagger)+\frac{f_\pi^2}{8}\mathrm{Tr}(MU^\dagger+UM^\dagger)\ , \tag{2.1}$$

where

$$U=\exp\left[2i\frac{\phi}{f_\pi}\right],\quad \phi=\frac{1}{\sqrt{2}}\phi^a\lambda^a\ ,$$
$$\mathrm{Tr}(\lambda^a\lambda^b)=2\delta^{ab},\quad f_\pi=132\ \mathrm{MeV} \tag{2.2}$$

and M is a meson mass matrix with the nonvanishing matrix elements (isospin invariance being assumed):

$$M_{11}=M_{22}=m_\pi^2,\quad M_{33}=2m_K^2-m_\pi^2\ . \tag{2.3}$$

It was established by Gasser and Leutwyler[4] that the most general expressions for the p^4 effective chiral Lagrangians including external vector V_μ and axial-vector A_μ gauge fields are (in the chiral limit)

$$\begin{aligned}\mathcal{L}_S^4=&L_1[\mathrm{Tr}(D^\mu U^\dagger D_\mu U)]^2+L_2[\mathrm{Tr}(D_\mu U^\dagger D_\nu U)]^2\\&+L_3\mathrm{Tr}(D^\mu U^\dagger D_\mu U)^2\\&+L_9\mathrm{Tr}(F_{\mu\nu}^R D^\mu U^\dagger D^\nu U+F_{\mu\nu}^L D^\mu U\,D^\nu U^\dagger)\\&+L_{10}\mathrm{Tr}(U^\dagger F_{\mu\nu}^R U F^{\mu\nu L})\ ,\end{aligned} \tag{2.4}$$

with

$$\begin{aligned}&D_\mu U=\partial_\mu U+A_\mu^L U-UA_\mu^R\ ,\\&F_{\mu\nu}^{L,R}=\partial_\mu A_\nu^{L,R}-\partial_\nu A_\mu^{L,R}+[A_\mu^{L,R},A_\nu^{L,R}]\ ,\\&A_\mu^{L,R}=V_\mu\pm A_\mu\ .\end{aligned} \tag{2.5}$$

For the physical applications discussed in this paper the external gauge fields are identified with the photon A_μ and the left-handed $W_\mu^\pm$ boson fields

$$\begin{aligned}&A_\mu^L=-ieA_\mu Q-2i\Lambda W_\mu\ ,\\&A_\mu^R=-ieA_\mu Q\ ,\end{aligned} \tag{2.6}$$

where

$$Q=\begin{bmatrix}\frac{2}{3}&0&0\\0&-\frac{1}{3}&0\\0&0&-\frac{1}{3}\end{bmatrix}$$

and $\Lambda_{ij}=1$ and vanishes otherwise for the quark current $\bar{q}_i\gamma_\mu(1-\gamma_5)q_j$. For convenience, we have factored out the factor $(g/\sqrt{2})\sin\theta_C$ (or $\cos\theta_C$) in Eq. (2.6).

Before proceeding it should be stressed that the coupling constants L_i of $\mathcal{L}_S^4$ are in general renormalization scale dependent: Counterterms necessary to renormalize one-loop meson graphs have the same structure as that of $\mathcal{L}_S^4$.[5] That is, divergences of the one-loop chiral graphs generated from $\mathcal{L}_S^2$ are of order p^4 and hence can be absorbed by the counterterms which have the same form as $\mathcal{L}_S^4$. Let the bare coupling constant of $\mathcal{L}_S^4$ be denoted by L_i^0; the renormalized couplings are then given by

$$L_i(\mu)=L_i^0+\frac{b_i}{32\pi^2}\ln\frac{\Lambda^2}{\mu^2}\ , \tag{2.7}$$

where Λ is a cutoff, μ is an arbitrary renormalization scale, and b_i are coefficients to be determined from chiral loops. As a consequence, although the lowest-order chiral Lagrangian $\mathcal{L}_S^2$ is scale independent, higher-order effective Lagrangians do depend on the choice of μ, reflected by the nonrenormalizability of the nonlinear chiral-Lagrangian approach. Of course, physical quantities should be independent of the renormalization scale. For example, the μ dependence of the one-loop diagram generated by $\mathcal{L}_S^2$ must be canceled by the tree amplitude induced by $\mathcal{L}_S^4$.

It is well known that the lowest-order chiral Lagrangian responsible for $\Delta S=1$ and $\Delta I=\frac{1}{2}$ nonleptonic weak interactions reads[6]

$$\mathcal{L}_W^2=-g_8\mathrm{Tr}(\lambda_6 L_\mu L^\mu)\ , \tag{2.8}$$

where $L_\mu\equiv(\partial_\mu U)U^\dagger$ is an $SU(3)_R$ singlet and $L_\mu^\dagger=-L_\mu$. The parameter g_8 of the octet weak interaction is determined from the measured $K\to\pi\pi$ rates. In the chiral limit and in the absence of external gauge fields there are seven independent CP-even quartic-derivative weak Lagrangian terms[7,8] which transforms as $(8_L,1_R)$ under chiral rotations:

$$\begin{aligned}\mathcal{L}_W^4=\frac{g_8}{f_\pi^2}\{&h_1\mathrm{Tr}(\lambda_6L_\mu L^\mu L_\nu L^\nu)+h_2\mathrm{Tr}(\lambda_6L_\mu L_\nu L^\mu L^\nu)+h_3\mathrm{Tr}(\lambda_6L_\mu L_\nu L^\nu L^\mu)+h_4\mathrm{Tr}(\lambda_6L_\mu L_\nu)\mathrm{Tr}(L^\mu L^\nu)\\&+h_5\mathrm{Tr}(\lambda_6\tilde{Y}\tilde{Y})+h_6\mathrm{Tr}([\lambda_6,\tilde{Y}]L_\mu L^\mu)+h_7\mathrm{Tr}([\lambda_6,\tilde{Y}_{\mu\nu}]L^\mu L^\nu)\}\ ,\end{aligned} \tag{2.9}$$

where $Y_{\mu\nu}=(\partial_\mu\partial_\nu U)U^\dagger$, $\tilde{Y}_{\mu\nu}=Y_{\mu\nu}-Y^\dagger_{\mu\nu}$, and $\tilde{Y}=g^{\mu\nu}\tilde{Y}_{\mu\nu}$. Under the CP transformation, $\tilde{Y}_{\mu\nu}\rightarrow -\tilde{Y}^T_{\mu\nu}$ (Ref. 7).

The coupling constants of the higher-order chiral Lagrangians $\mathcal{L}^4_S$ and $\mathcal{L}^4_W$ are *a priori* unknown parameters: They are not fixed by the requirement of chiral symmetry alone. In fact, these couplings are not really fundamental coupling constants as they depend on the choice of the renormalization scale μ. For strong interactions, the coefficients b_i and hence the μ dependence of the couplings $L_i(\mu)$ in $\mathcal{L}^4_S$ have been calculated by Gasser and Leutwyler[4(b)] with the results

$$b_1=\tfrac{3}{32},\quad b_2=\tfrac{3}{16},\quad b_3=0,\quad b_9=\tfrac{1}{4},\quad b_{10}=-\tfrac{1}{4}\ . \tag{2.10}$$

Furthermore, they have empirically determined those parameters at the mass scale $\mu=m_\eta$ from various low-energy hadronic processes in conjunction with the Zweig-rule argument. For nonleptonic $\Delta S=1$ weak decays, there is only one process, namely, $K\rightarrow\pi\pi\pi$, relevant for the determination of the unknown parameters $h_i(\mu)$ in $\mathcal{L}^4_W$. Consequently, only combinations of h_i can be extracted from experiment.[8]

In the limit of large N_c (N_c being the number of colors), all aforementioned coupling constants are fixed at least to the zeroth order in α_s within the framework of a QCD-motivated model. First of all, the chiral-loop contribution is suppressed by at least a factor of $1/N_c$ relative to the quark loop at the same order of p^n in the leading $1/N_c$ expansion. Subsequently, the higher-order couplings in the large N_c chiral perturbation theory are renormalization scale independent. Second, consider QCD coupled to external gauge fields. The integration of both quark and gluonic degrees of freedom yields two categories of global chiral anomalies: proper (Bardeen) anomalies which contain the totally antisymmetric tensor $\epsilon_{\mu\nu\alpha\beta}$ and spurious anomalies which do not. The variation of the fermion determinant under a local chiral transformation is governed by chiral anomalies. The resulting effective action is thus the difference of the generating function before and after chiral rotations. It is well known that the integration of topological (Bardeen) anomalies gives rise to the Wess-Zumino-Witten effective action.[9] Likewise, in the absence of gluonic corrections, the integration of nontopological (spurious) chiral anomalies yields an action for the quartic-derivative nonanomalous chiral Lagrangians for strong interactions.

The large-N_c limit is a well-defined approximation to QCD. A consistent leading $1/N_c$ expansion for the nontopological chiral anomalies requires one to include not only the contributions from the quark loops but also the gluon effects arising from all planar diagrams without the internal quark loops. The gluonic corrections which have been ignored in all previous publications[3,10] were discussed recently by Espriu, de Rafael, and Taron[11] (ERT). Technically, the flaws with previous work come from stopping at the a_2 coefficient in the heat-kernel expansion of the fermion determinant. To the first order in α_s, ERT found that only the couplings L_3 and L_{10} receive gluonic modifications. Because of the theoretical difficulty in estimating the nonperturbative part of gluonic corrections, we shall consider in the present paper the strong chiral Lagrangian to the zeroth order of α_s. In the absence of gluonic modifications, the p^4 strong chiral Lagrangian derived from the integration of spurious anomalies has the couplings

$$8L_1=4L_2=-2L_3=L_9=-2L_{10}=\frac{N_c}{48\pi^2}\ . \tag{2.11}$$

It should be stressed that the strong Lagrangian given by Eqs. (2.4) and (2.11) should be considered as a QCD-motivated model rather than a formal chiral Lagrangian derived from large-N_c QCD. First, the effective Lagrangian is derived by coupling QCD to "external" meson and gauge fields. Second, gluonic contributions in the large-N_c limit are not included in the naive QCD-inspired model.

Since the parameters L_i under the large-N_c approximation are scale-independent constants, one should in principle *not* compare L_i directly with the running renormalized couplings $L_i^r(\mu)$ determined empirically by Gasser and Leutwyler,[4] though numerically they are quite close at the mass scale between 0.5 and 1 GeV.[12] However, it is evident from Eq. (2.10) that the combinations, e.g.,

$$\begin{aligned}&2L_1^r(\mu)-L_2^r(\mu)=2L_1-L_2\ ,\\&L_3^r(\mu)=L_3\ ,\\&L_9^r(\mu)+L_{10}^r(\mu)=L_9+L_{10}\ ,\end{aligned} \tag{2.12}$$

are independent of the renormalization point μ, and hence can be utilized to test the validity of the quark-loop approach. Empirically, L_1^r, L_2^r, and L_3^r extracted from the D-wave $\pi\pi$ scattering length together with the Zweig rule[4(b),13] are consistent with Eq. (2.12). The value of $L_9^r+L_{10}^r$ may be extracted from the pion polarizability α_π measured in the pion Compton effect in the reaction $\pi^-A\rightarrow A\pi^-\gamma$:[14]

$$\alpha_\pi=\frac{8\alpha}{m_\pi f_\pi^2}(L_9^r+L_{10}^r)\ . \tag{2.13}$$

The experimental result[15] $\alpha_\pi=(6.8\pm1.4)\times10^{-43}$ cm^3 translates into

$$L_9^r+L_{10}^r=(3.69\pm0.76)\times10^{-3}\ , \tag{2.14}$$

which is in good agreement with the "zeroth-order" theoretical prediction

$$L_9+L_{10}=\frac{1}{32\pi^2}=3.2\times10^{-3}\ . \tag{2.15}$$

Another bit of information on the combination $L_9^r+L_{10}^r$ comes from the radiative pion decay $\pi\rightarrow e\nu\gamma$ [Ref. 3(b)] or the related process $\pi\rightarrow e\nu e^+e^-$:

$$\gamma\equiv\frac{f_A}{f_V}=32\pi^2(L_9^r+L_{10}^r)\ , \tag{2.16}$$

where f_A and f_V are structure-dependent axial-vector and vector form factors, respectively. A very recent measurement at the Swiss Institute for Nuclear Research[16] (SIN) of these form factors from the process

$\pi^+ \to e^+ \nu e^+ e^-$ yields

$$f_A = 0.021^{+0.011}_{-0.013}, \quad f_V = 0.023^{+0.015}_{-0.013} \tag{2.17}$$

in good accord with the "lowest-order" prediction $\gamma = 1$.[17]

Several remarks are in order. (i) The fact that the higher-order effective chiral Lagrangians in the leading $1/N_c$ expansion have been applied successfully to various low-energy hadronic processes[1,3(b),18–23] indicates that chiral-loop effects induced by $\mathcal{L}_S^2$ are in general small compared to the tree-level contributions from $\mathcal{L}_S^2$ and $\mathcal{L}_S^4$, as expected from the large-N_c argument. Of course, this does not mean that meson-loop contributions are always trifling or negligible. Examples which show the importance of the one-loop correction are the rare decays $K_S \to \gamma\gamma$ and $K_L \to \pi\gamma\gamma$ which, to the order p^4, receive contributions only from the loop diagrams. Nevertheless, it should be stressed from the outset that only in the limit of large N_c can one use the scale-independent coupling constants, Eq. (2.11), derived from the integration of spurious chiral anomalies. (ii) It is evident from Eqs. (2.1), (2.4), and (2.11) that contributions from $\mathcal{L}^4$ are suppressed by factors of p^2/Λ_χ^2, where $\Lambda_\chi = 2\pi f_\pi = 830$ MeV is the scale of chiral-symmetry breaking.[24] This explains why the kaon system is an ideal place to test chiral perturbation theory. (iii) As far as the color factor is concerned, the $\mathcal{L}_S^4$ contribution to the amplitude relative to that of $\mathcal{L}_S^2$ is of order N_c/f_π^2. Since f_π scales with N_c as $N_c^{1/2}$, this factor is independent of N_c when N_c is large. In practice we can thus put $N_c = 3$ as f_π is taken to be the physical value 132 MeV in our $N_c = 3$ world.

As for the weak couplings h_i of $\mathcal{L}_W^4$, they can also be theoretically calculated in the large-N_c approximation based on factorization, bosonization, and a simple structure of the effective weak Hamiltonian, which will be elucidated on in the next section. The predictions[1] are (in the absence of gluonic corrections)

$$h_1 = -h_2/3 = h_4/3 = h_6 = -h_7 = \frac{N_c}{24\pi^2}, \quad h_3 = h_5 = 0 . \tag{2.18}$$

The effective weak chiral Lagrangians $\mathcal{L}_W^2 + \mathcal{L}_W^4(1/N_c)$ have been tested successfully in the study of the nonleptonic $K \to \pi\pi\pi$ decay.[1]

III. ELECTROMAGNETICALLY INDUCED ANOMALOUS AND NONANOMALOUS WEAK CHIRAL LAGRANGIANS

It was pointed out recently by Ecker, Pich, and de Rafael[25] (EPR) that the p^4 electromagnetically induced $\Delta S = 1$ nonanomalous weak Lagrangians which satisfy the constraints of chiral and CPS symmetry[26] should have the form

$$\mathcal{L}_{\text{nonanom}}^{\Delta S=1} = i\left[\frac{2}{f_\pi^2}\right] g_8 e F^{\mu\nu}[\omega_1 \text{Tr}(\lambda_6 L_\mu L_\nu Q) + \omega_2 \text{Tr}(\lambda_6 L_\nu Q L_\mu)] + \omega_4 \left[\frac{2}{f_\pi^2}\right] g_8 e^2 F^{\mu\nu} F_{\mu\nu} \text{Tr}(\lambda_6 Q U Q U^\dagger) , \tag{3.1}$$

while a possible anomalous term is

$$\mathcal{L}_{\text{anom}}^{\Delta S=1} = i\omega_3 \left[\frac{2}{f_\pi^2}\right] g_8 e \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} \text{Tr}(Q L_\rho) \text{Tr}(\lambda_6 L_\sigma) \tag{3.2}$$

in which the ordinary derivative in L_μ is replaced by the covariant derivative in the presence of external gauge fields. It was argued by EPR that Eqs. (3.1) and (3.2) complete the structure of the effective weak chiral Lagrangians at the p^4 level necessary for a consistent one-loop calculation of nonleptonic radiative $\Delta I = \frac{1}{2}$, $\Delta S = 1$ transitions with at most two external photon fields. However, as we shall see later in this section, there are additional contributions to the anomalous weak interaction.

For radiative $\Delta S = 1$ transitions, EPR found that the relation

$$\omega_2 = 4L_9 \tag{3.3}$$

must hold at least for the divergent parts of the counterterm coupling constants because they must render the divergent loop amplitudes finite.[25(a)] From the experimental $K^+ \to \pi^+ e^+ e^-$ decay rate, EPR obtained two solutions for the renormalized constant ω_1^r:

$$\omega_1^r(\mu = m_\eta) = (1.67 \pm 0.19) \times 10^{-2} \quad \text{or } (4.95 \pm 0.19) \times 10^{-2} . \tag{3.4}$$

As a result, predictions for $K^+ \to \pi^+ \mu^+ \mu^-$ and $K_S \to \pi^0 l^+ l^-$ $(l = e, \mu)$ are subject to twofold ambiguity.[25(a)]

The main task of this section is to determine the parameters ω_i within the framework of $1/N_c$ chiral perturbation theory. This requires three ingredients: the $\Delta S = 1$ effective weak Hamiltonian at the quark level, bosonization, and factorization, as we are going to elaborate on. The $\Delta S = 1$ effective nonleptonic Hamiltonian in the limit of large N_c has a rather simple structure:[27,28]

$$\mathcal{H}_{\text{eff}}^{\Delta S=1} = \frac{G_F}{\sqrt{2}} \sin\theta_C \cos\theta_C [c_8(Q_2 - Q_1) + c_{27}(Q_2 + 2Q_1)] , \tag{3.5}$$

$$Q_1 = (\bar{s}d)(\bar{u}u), \quad Q_2 = (\bar{s}u)(\bar{u}d) ,$$

where $(\bar{q}_i q_j) \equiv \bar{q}_i \gamma_\mu (1 - \gamma_5) q_j$. The combination $(Q_2 - Q_1)$ is a $\Delta I = \frac{1}{2}$ four-quark operator which transforms as $(8_L, 1_R)$ under chiral rotations. Using the fact the matrix elements of $\sum_q (\bar{s}d)(\bar{q}q)$ vanish in the leading $1/N_c$ expansion and that $(\bar{s}d)(\bar{s}s)$ does not contribute to nonleptonic kaon decays, it is easily seen that $(Q_2 + 2Q_1)$ is equivalent to the 27-plet $\Delta I = \frac{3}{2}$ operator

$$O_4 = (\bar{s}u)(\bar{u}d) + (\bar{s}d)(\bar{u}u) - (\bar{s}d)(\bar{d}d) . \tag{3.6}$$

The short-distance contributions to the Wilson coefficient functions c_8 and c_{27} are perturbatively calculable from M_W down to the renormalization scale $\mu \sim 1$ GeV. The long-distance contribution below 1 GeV is, however, beyond the task of perturbative QCD.

From Eq. (2.1) it is easily seen that the bosonization of the quark current $J_\mu^{ij} \equiv (\bar{q}^i q^j)$ to the leading order in chiral expansion reads

$$(J_\mu)_{ji}=i\frac{f_\pi^2}{2}(L_\mu)_{ij} . \tag{3.7}$$

Since factorization is valid in the large-N_c limit, we may substitute Eq. (3.7) into (3.5) to obtain the octet weak chiral Lagrangian Eq. (2.8) with

$$g_8(1/N_c)=-\frac{G_F}{\sqrt{2}}\frac{f_\pi^4}{4}\sin\theta_C\cos\theta_C c_8 \tag{3.8}$$

and the 27-plet $\Delta I=\frac{3}{2}$ weak Lagrangian

$$\mathcal{L}_{27}^{\Delta S=1}=g_{27}(L_{\mu 13}L_{21}^{\mu}+L_{\mu 23}L_{11}^{\mu}-L_{\mu 23}L_{22}^{\mu}) \tag{3.9}$$

with

$$g_{27}(1/N_c)=-\frac{G_F}{\sqrt{2}}\frac{f_\pi^4}{4}\sin\theta_C\cos\theta_C c_{27} . \tag{3.10}$$

Therefore, though the sign of g_8 and g_{27} cannot be determined from the measured $K\to\pi\pi$ rates, it is fixed to be negative via Eqs. (3.8) and (3.10) as the Wilson coefficients c_8 and c_{27} are positive.[29] The experimental values of the weak couplings are given by[29]

$$g_8=-0.26\times10^{-8}m_K^2 ,$$
$$g_{27}=-0.86\times10^{-10}m_K^2 . \tag{3.11}$$

It is worth stressing that chiral-Lagrangian coupling constants receive both short- and long-distance contributions.

We next proceed to incorporate the external photon fields into the bosonization of the quark current. An easy way of deriving this current is to recast the effective Lagrangian in the form $\mathrm{Tr}(W_\mu J^\mu)$, where W_μ is the left-handed vector-boson field. Substituting Eq. (2.6) into $\mathcal{L}_S^4(1/N_c)$ [i.e., Eq. (2.4)+Eq. (2.11)] and expanding to first order in W_μ and second order in A_μ, we find the relevant Lagrangian terms to be[30]

$$\frac{N_c}{24\pi^2}eF^{\mu\nu}W_\mu\mathrm{Tr}([Q,\Lambda]L_\nu)+i\frac{N_c}{6\pi^2}e^2F^{\mu\nu}W_\mu A_\nu\mathrm{Tr}(QU^\dagger Q\Lambda U-QU^\dagger\Lambda QU) . \tag{3.12}$$

Hence, the bosonization of the quark current reads

$$(J^\mu)_{ji}=i\frac{f_\pi^2}{2}(L^\mu)_{ij}-\frac{N_c}{24\pi^2}eF^{\mu\nu}(QL_\nu-L_\nu Q)_{ij}-i\frac{N_c}{12\pi^2}e^2F^{\mu\nu}A_\nu(UQU^\dagger Q-QUQU^\dagger)_{ij} . \tag{3.13}$$

Writing $J_\mu=(if_\pi^2/2)(L_\mu+\hat{L}_\mu)$ and substituting into (Q_2-Q_1) we find

$$Q_2-Q_1\to\frac{f_\pi^2}{4}[\mathrm{Tr}(\lambda_6L_\mu L^\mu)+\mathrm{Tr}(\lambda_6L_\mu\hat{L}^\mu)+\mathrm{Tr}(\lambda_6\hat{L}_\mu L^\mu)-\mathrm{Tr}(\lambda_6L_\mu)\mathrm{Tr}(\hat{L}^\mu)] , \tag{3.14}$$

where we have neglected the term quadratic in $\hat{L}_\mu$. This, together with Eqs. (3.5) and (3.8), yields the nonanomalous $\Delta S=1$ Lagrangian $\mathcal{L}_{\mathrm{nonanom}}^{\Delta S=1}$ [Eq. (3.1)] with

$$\omega_1=\omega_2=\frac{N_c}{12\pi^2},\quad \omega_4=0 , \tag{3.15}$$

where uses of $[Q,\lambda_6]=0$ and integration by parts have been made. The previous observation of $\omega_2=4L_9$ made by EPR is numerically reproduced here. The implications of this large-N_c chiral Lagrangian to the radiative kaon decay will be discussed in the next section.

To derive the anomalous weak chiral Lagrangian coupled to the external photon fields, we first write down the relevant Wess-Zumino-Witten terms[9]

$$\begin{aligned}\mathcal{L}_{\mathrm{WZW}}=-\frac{N_c}{48\pi^2}\epsilon^{\mu\nu\rho\sigma}\mathrm{Tr}[&-(A_\mu^RR_\nu R_\rho R_\sigma+A_\mu^LL_\nu L_\rho L_\sigma)-\tfrac{1}{2}A_\mu^LL_\nu A_\rho^LL_\sigma-A_\mu^RU^\dagger A_\nu^LUR_\rho R_\sigma+A_\mu^LUA_\nu^RU^\dagger L_\rho L_\sigma\\&+\partial_\mu A_\nu^RU^\dagger A_\rho^LUR_\sigma+\partial_\mu A_\nu^LUA_\rho^RU^\dagger L_\sigma+(A_\mu^L\partial_\nu A_\rho^L+\partial_\mu A_\nu^LA_\rho^L)L_\sigma]+\cdots ,\end{aligned} \tag{3.16}$$

where $R_\mu\equiv U^\dagger\partial_\mu U$. After a lengthy evaluation we find[31]

$$\begin{aligned}\mathcal{L}_{\mathrm{WZW}}=&\frac{iN_c}{12\pi^2f_\pi^2}e\epsilon^{\mu\nu\rho\sigma}\partial_\mu A_\nu W_\rho\mathrm{Tr}(3f_\pi\{\Lambda,Q\}\partial_\sigma\phi+3i\{\Lambda,Q\}[\phi,\partial_\sigma\phi]-2i\Lambda\{[Q,\phi],\partial_\sigma\phi\})\\&-\frac{N_c}{3\pi^2f_\pi^3}e\epsilon^{\mu\nu\rho\sigma}A_\mu\mathrm{Tr}(Q\partial_\nu\phi\partial_\rho\phi\partial_\sigma\phi)+\cdots .\end{aligned} \tag{3.17}$$

We have proven explicitly that there are not any gauge-noninvariant terms in the expansion of the Wess-Zumino-Witten action.

Since

$$L_\mu=\frac{2i}{f_\pi}\partial_\mu\phi-\frac{2}{f_\pi^2}[\phi,\partial_\mu\phi]+\cdots , \tag{3.18}$$

the bosonization in the anomalous case has the form

$$(J^\mu)_{ji}=\frac{if_\pi}{2}(L^\mu)_{ij}+\frac{N_c}{12\pi^2 f_\pi^2}e\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}\{[Q,\phi],\partial_\nu\phi\}_{ij}+\frac{N_c}{16\pi^2}e\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}\{Q,L_\nu\}_{ij} \ . \tag{3.19}$$

Plugging this into Eq. (3.14) leads to

$$\mathcal{L}^{\Delta S=1}_{\text{anom}}=i\left[\frac{2}{f_\pi^2}\right]g_8 e\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}\left[\frac{N_c}{12\pi^2}\text{Tr}(QL_\rho)+\frac{N_c}{24\pi^2}\text{Tr}(QR_\rho)\right]\text{Tr}(\lambda_6 L_\sigma)-\frac{N_c}{3\pi^2 f_\pi^5}g_8 e\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}\text{Tr}(\lambda_6[\partial_\rho\phi\partial_\sigma\phi,[Q,\phi]]) \ , \tag{3.20}$$

where we have used

$$L_\mu-R_\mu=-\frac{4}{f_\pi^2}[\phi,\partial_\mu\phi]+O(\phi^4) \ . \tag{3.21}$$

Comparing (3.20) with (3.2) we find not only

$$\omega_3=\frac{N_c}{12\pi^2} \tag{3.22}$$

but also two additional contributions to the electromagnetically induced $\Delta S=1$ anomalous interactions. The term $\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}\text{Tr}(QR_\rho)\text{Tr}(\lambda_6 L_\sigma)$ does survive all symmetry constraints including *CPS* invariance. Since the last term in $\mathcal{L}^{\Delta S=1}_{\text{anom}}$ is not written in terms of L_μ, R_μ, or U fields, the fact that it satisfies all symmetry constraints is thus not manifest. Equations (3.1), (3.15), and (3.20) are the main results in this section. It should be stressed again that gluonic modifications, arising from all planar diagrams without the internal quark loops, to the coupling constants of higher-order chiral Lagrangians are not considered in the present paper.

IV. APPLICATIONS TO RADIATIVE KAON DECAY

In the preceding section we have derived p^4 anomalous and nonanomalous weak chiral Lagrangians responsible for $\Delta S=1$ radiative transitions within the framework of the $1/N_c$ approach. The radiative K decay is an ideal place to test higher-order effective Lagrangians since it cannot be generated by lowest-order tree Lagrangians: Lorentz and gauge invariance requires at least *two* powers of momenta in the radiative amplitude whereas the amplitude induced by $\mathcal{L}^2$ is only linear in the momenta in the case of one-photon emission.

In this section we apply the electromagnetically induced weak Lagrangians to the following radiative K decays: $K\to\pi\gamma^*\to\pi l^+l^-$, $K\to\pi\gamma\gamma$, $K\to\pi\pi\gamma$, and $K\to\pi l\nu\gamma$. The structure-dependent $K\to\pi\gamma\gamma\gamma$, $\pi\pi\pi\gamma$ transitions will not be investigated here as the experimental feasibility for them is still remote. As we shall see, the direct decay $K\to\pi\pi\gamma$, which has not yet been studied in chiral perturbation theory (except for pole contributions), offers the most excellent test on both anomalous and nonanomalous $\Delta S=1$ effective Lagrangians. Effects of *CP* violation will not be addressed in the present paper.

A. $K\to\pi l^+l^-$ transitions

We first consider the transition $K\to\pi\gamma^*$ with γ^* being a virtual photon. ($K\to\pi\gamma$ is known to be prohibited by either gauge invariance or conservation of angular momentum.) A direct calculation of the $K(k)\to\pi(p)+\gamma^*(q)$ amplitudes induced by $\mathcal{L}^{\Delta S=1}_{\text{nonanom}}$ yields

$$A(K^+\to\pi^+\gamma^*)=i\frac{4}{3}\frac{eg_8}{f_\pi^4}(\omega_1+2\omega_2)q^2(p+k)\cdot\epsilon \ ,$$
$$A(K^0\to\pi^0\gamma^*)=-i\frac{4}{3\sqrt{2}}\frac{eg_8}{f_\pi^4}(\omega_1-\omega_2)q^2(p+k)\cdot\epsilon \ . \tag{4.1}$$

However, the $K^+\to\pi^+\gamma^*$ amplitude receives an additional pole contribution from Fig. 1 induced by $\mathcal{L}^2_W$ and the L_9 term of $\mathcal{L}^4_S$. The total $K^+\to\pi^+\gamma^*$ amplitude becomes

$$A(K^+\to\pi^+\gamma^*)_{\text{total}}=i\frac{4}{3}\frac{eg_8}{f_\pi^4}(\omega_1+2\omega_2-12L_9)q^2(p+k)\cdot\epsilon \ . \tag{4.2}$$

Our results are in agreement with Ref. 25(a).

Based on the observation that the one-loop amplitudes of $K\to\pi\gamma^*$ satisfy the relation

$$A(K^+\to\pi^+\gamma^*)_{\text{loop}}=-\sqrt{2}A(K^0\to\pi^0\gamma^*)_{\text{loop}} \tag{4.3}$$

in the SU(3) limit, EPR conclude[25(a)] that the correlation $\omega_2=4L_9$ must hold for the divergent parts of the counterterm coupling constants in order for the tree amplitudes (4.1) and (4.2) to satisfy the same relation (4.3) as the loop amplitudes in the SU(3) limit. Moreover, EPR conjectured that (3.3) holds also for the finite renormalization

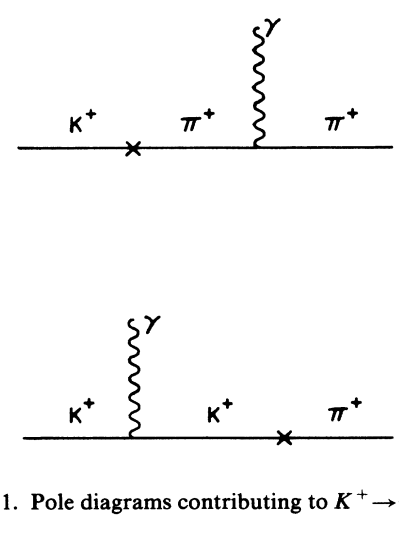

FIG. 1. Pole diagrams contributing to $K^+\to\pi^+\gamma^*$.

parts. We have already seen in the preceding section that the relation $\omega_2=4L_9$ is a natural consequence in the large-N_c chiral-Lagrangian approach.

Unfortunately, it is easily seen from Eq. (3.15) that the $K\to\pi\gamma^*$ amplitudes vanish in the limit of large N_c. This means that the $K\to\pi\gamma^*$ transitions receive contributions first from the $1/N_c$ corrections in chiral perturbation theory. At the next-to-leading level, chiral loops contribute to the decay amplitude and couplings ω_i get renormalization. Using the experimental measurement of $K^+\to\pi^+e^+e^-$ as an input, EPR obtained two solutions for ω_1^r at the renormalization point $\mu=m_\eta$ as shown in Eq. (3.4). Consequently, predictions for $K^+\to\pi^+\mu^+\mu^-$ and $K_S\to\pi^0 l^+l^-$ are twofold ambiguous. Nevertheless, the empirical fact[12] that the predicted parameters L_i for strong interactions in the $1/N_c$ approach are very close to those renormalized couplings L_i^r determined from various low-energy hadronic processes at the mass scale 0.5 GeV $<\mu<1$ GeV suggests that one of the solutions, namely, $\omega_1^r(m_\eta)=(1.67\pm0.19)\times10^{-2}$, is more favored. (The value of ω_1^r will be sharpened by a new high-statistics experiment at BNL, E777, and a near-future experiment at BNL, E851.) This in turn implies that the predictions done in Ref. 25(a),

$$\begin{aligned} &B(K^+\to\pi^+\mu^+\mu^-)=(6.1^{+1.1}_{-1.0})\times10^{-8}\ ,\\ &B(K_S\to\pi^0e^+e^-)=(4.8^{+2.0}_{-1.7})\times10^{-10}\ ,\\ &B(K_S\to\pi^0\mu^+\mu^-)=(1.0\pm0.4)\times10^{-10}\ , \end{aligned}\tag{4.4}$$

are preferred to the other set of predictions,

$$\begin{aligned} &B(K^+\to\pi^+\mu^+\mu^-)=(4.5^{+1.0}_{-0.8})\times10^{-8}\ ,\\ &B(K_S\to\pi^0e^+e^-)=(4.9\pm0.6)\times10^{-9}\ ,\\ &B(K_S\to\pi^0\mu^+\mu^-)=(1.0\pm0.1)\times10^{-9}\ , \end{aligned}\tag{4.5}$$

based on the solution $\omega_1^r=(4.95\pm0.19)\times10^{-2}$. Three candidate events for the rare decay $K^+\to\pi^+\mu^+\mu^-$ were recently reported by the Brookhaven E787 Collaboration,[32] Assuming an estimate background of 0.3 events, those three events result in a measured branching ratio of $(9\pm6)\times10^{-8}$. Since a significant signal for this decay mode has not yet been established, an upper limit of 2.3×10^{-7} is set at the 90% confidence level.[32] However, analysis of new data with many more candidate events is still ongoing.

Before ending this subsection, we remark that in the short-distance effective Hamiltonian approach, the $K^+\to\pi^+l^+l^-$ decay is dominated by the so-called electromagnetic penguin diagram, which is realized in chiral perturbation theory as the ω_1 term of $\mathcal{L}^{\Delta S=1}_{\rm nonanom}$. It is easily seen from the analysis of Ref. 25(a) [cf. Eq. (3.40)] that the electromagnetic penguin contribution alone already saturates the observed $K^+\to\pi^+e^+e^-$ rates if the long-distance effect to the coefficient of the electromagnetic penguin operator and to ω_1^r is the same.

B. The $K^+\to\pi^+\gamma\gamma$ decay

As first pointed out by EPR,[25(b)] the loop amplitudes of $K_{L,S}\to\pi\gamma\gamma$ and $K^+\to\pi^+\gamma\gamma$ are finite. From the point of view of large-N_c chiral-Lagrangian approach, the mode $K^+\to\pi^+\gamma\gamma$ is more interesting since it also receives contributions from the tree Lagrangians $\mathcal{L}^{\Delta S=1}_{\rm nonanom}$ and $\mathcal{L}_S^4$ via pole diagrams (except for the ω_4 term, which contributes via the direct-emission diagram).

The total decay rate of $K^+\to\pi^+\gamma\gamma$ was calculated in Ref. 25(c) to be

$$\Gamma(K^+\to\pi^+\gamma\gamma)=\Gamma_{\rm loop}+\Gamma_{\rm tree}+\Gamma_{\rm int}+\Gamma_{\rm WZW}\ ,\tag{4.6}$$

with

$$\begin{aligned} &\Gamma_{\rm loop}=2.80\times10^{-23}\ {\rm GeV},\quad \Gamma_{\rm tree}=0.17\hat{c}^{\,2}\times10^{-23}\ {\rm GeV}\ ,\\ &\Gamma_{\rm int}=0.87\hat{c}\times10^{-23}\ {\rm GeV},\quad \Gamma_{\rm WZW}=0.26\times10^{-23}\ {\rm GeV}\ , \end{aligned}\tag{4.7}$$

and

$$\hat{c}=32\pi^2[4(L_9+L_{10})-\tfrac{1}{3}(\omega_1+2\omega_2+2\omega_4)]\ .\tag{4.8}$$

Since the loop amplitude is finite, the combination $\omega_1+2\omega_2+2\omega_4$ is thus scale independent. (It was already shown in Sec. II that $L_9^r+L_{10}^r$ is independent of μ.) From Eqs. (2.11) and (3.15) we obtain $\hat{c}=-4$ and the prediction

$$B(K^+\to\pi^+\gamma\gamma)=5.1\times10^{-7}\ .\tag{4.9}$$

Notice that this decay is dominated by chiral-loop effects.

The present best upper limit[33] for $K^+\to\pi^+\gamma\gamma$ is 8.4×10^{-6}. This bound was recently pushed to the level of 10^{-6} by the Brookhaven E-787 experiment.[34] At first glance, the prediction (4.9) seems to indicate that this radiative decay should be in the vicinity of being observed. However, it should be stressed that all previous experimental upper bounds were obtained by assuming a phase-space spectrum. If the theoretical spectrum is used, then the upper limit will be pulled back to the level of 10^{-4}.[35] Figure 2 depicts the expected two-photon

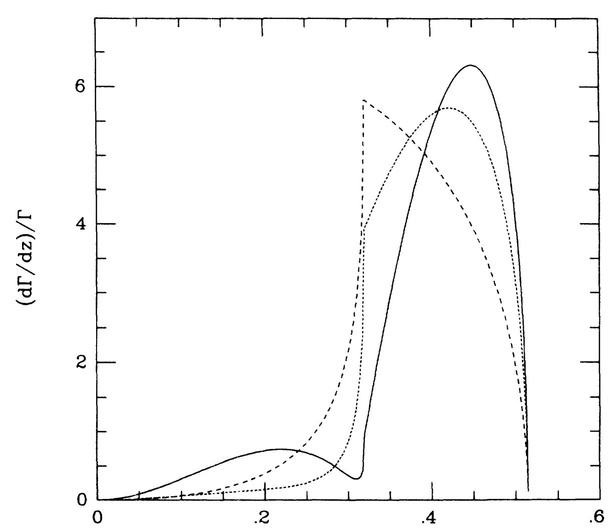

FIG. 2. The normalized differential decay rate of $K^+\to\pi^+\gamma\gamma$ vs $z=m^2_{\gamma\gamma}/m_K^2$, for $\hat{c}=-4$ (solid curve), where $m^2_{\gamma\gamma}$ is the invariant-mass squared of the two-photon pair. Also shown for comparison is the two-photon spectrum for $\hat{c}=-2$ (dotted curve) and $\hat{c}=0$ (dashed curve).

spectrum as a function of the invariant-mass squared z of the photon pair for $\hat{c}=-4$. For the purpose of comparison, the spectra for $\hat{c}=0$ and $\hat{c}=-2$ are also shown in Fig. 2.

C. Direct $K\to\pi\pi\gamma$ transitions

The radiative decay $K\to\pi\pi\gamma$ receives two contributions: direct emission (DE) and inner bremsstrahlung. The process of direct photon emission has not yet been studied within the framework of chiral perturbation theory except for the long-distance pole effects. Previous calculations using the vector-meson-dominance model, the short-distance effective weak Hamiltonian, current algebra, etc., are crude, unreliable, and erroneous in some respects.[36] For example, it was pointed out in Ref. 21 that the soft-pion technique cannot be applied to the magnetic transition amplitude as in the case of $\pi^0\to\gamma\gamma$. Armed with the Lagrangians $\mathcal{L}^{\Delta S=1}_{\rm nonanom}$ and $\mathcal{L}^{\Delta S=1}_{\rm anom}$ derived in the $1/N_c$ approach, this once formidable task now becomes manageable. It turns out that the direct radiative decay, especially $K^+\to\pi^+\pi^0\gamma$, offers an excellent test on the electromagnetically induced effective weak Lagrangians. First, unlike the previous two transitions $K\to\pi l^+l^-$ and $K\to\pi\gamma\gamma$, the decay $K\to\pi\pi\gamma$ consists of the anomalous $\Delta S=1$ direct emission. Second, it is dominated by tree contributions in contrast with the previous two examples, which are dominated by loop effects.

Under Lorentz and gauge invariance, the general expression for the invariant DE amplitude of the decay $K(k)\to\pi(p_1)\pi(p_2)\gamma(q)$ reads

$$\begin{aligned} A_{\rm DE}&=\tilde{\beta}M+\tilde{\gamma}E\ ,\\ M&\equiv e\epsilon_{\mu\nu\rho\sigma}p_1^{\mu}p_2^{\nu}q^{\rho}\epsilon^{\sigma}\ ,\\ E&\equiv e[(p_1\cdot\epsilon)(p_2\cdot q)-(p_2\cdot\epsilon)(p_1\cdot q)]\ , \end{aligned}\tag{4.10}$$

where ϵ_μ is the polarization vector of the photon. The first term of $A_{\rm DE}$ corresponds to magnetic transitions whereas the second term is caused by electric transitions. Taking into account the experimental cutoff on the photon energy, we have the branching ratios[21]

$$\begin{aligned} B(K^+\to\pi^+\pi^0\gamma)_{\rm DE}&=1.32\times10^5\ {\rm GeV}^6\ (|\tilde{\beta}|^2+|\tilde{\gamma}|^2)\ ,\\ B(K_L\to\pi^+\pi^-\gamma)_{\rm DE}&=1.33\times10^6\ {\rm GeV}^6\ (|\tilde{\beta}|^2+|\tilde{\gamma}|^2)\ ,\\ B(K_S\to\pi^+\pi^-\gamma)_{\rm DE}&=2.28\times10^3\ {\rm GeV}^6\ (|\tilde{\beta}|^2+|\tilde{\gamma}|^2)\ . \end{aligned}\tag{4.11}$$

1. Long-distance pole contributions

We will first concentrate on the long-distance pole contributions induced by the pole diagrams, Fig. 3. The anomalous $\phi\phi\phi\gamma$ vertex is governed by the last term of Eq. (3.17). The results are[21]

$$\begin{aligned} A(K^\pm\to\pi^+\pi^0\gamma)_{\rm pole}&=\mp\frac{4}{\sqrt{2}}\frac{g_8}{\pi^2f_\pi^5}M\ ,\\ A(K_L\to\pi^+\pi^-\gamma)_{\rm pole}&=-\frac{4}{\sqrt{2}}\frac{g_8}{\pi^2f_\pi^5}\frac{m_K^2}{m_K^2-m_\pi^2}\chi M\ , \end{aligned}\tag{4.12}$$

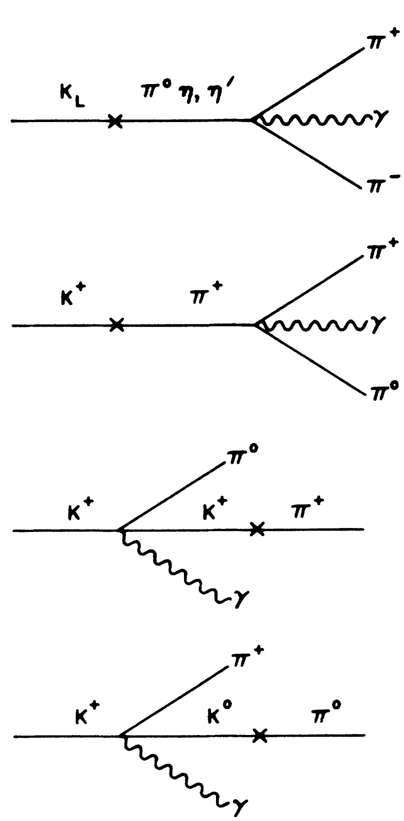

FIG. 3. Long-distance pole contributions to the direct photon emission of $K_L\to\pi^+\pi^-\gamma$ and $K^+\to\pi^+\pi^0\gamma$.

where[37]

$$\begin{aligned} \chi=1&+\frac{m_K^2-m_\pi^2}{m_K^2-m_\eta^2}[(\tfrac{1}{3})^{1/2}(1+\xi)\cos\theta+2(\tfrac{2}{3})^{1/2}\rho\sin\theta]\\ &\times[(\tfrac{1}{3})^{1/2}(f_\pi/f_8)^3\cos\theta-(\tfrac{2}{3})^{1/2}(f_\pi/f_0)^3\sin\theta]\\ &+\frac{m_K^2-m_\pi^2}{m_K^2-m_{\eta'}^2}[(\tfrac{1}{3})^{1/2}(1+\xi)\sin\theta-2(\tfrac{2}{3})^{1/2}\rho\cos\theta]\\ &\times[(\tfrac{1}{3})^{1/2}(f_\pi/f_8)^3\sin\theta+(\tfrac{2}{3})^{1/2}(f_\pi/f_0)^3\cos\theta]\ , \end{aligned}\tag{4.13}$$

and uses have been made of

$$\begin{aligned} \langle\pi^+(k)|\mathcal{L}_W|K^+(k)\rangle&=4\frac{g_8}{f_\pi^2}k^2\ ,\\ \langle\pi^0(k)|\mathcal{L}_W|K_L(k)\rangle&=-4\frac{g_8}{f_\pi^2}k^2\ , \end{aligned}\tag{4.14}$$

with the 27-plet contributions being neglected. In Eq. (4.13), ξ measures SU(3) breaking in relating the matrix element $\langle\eta_8|\mathcal{L}_W|K_L\rangle$ to $\langle\pi^0|\mathcal{L}_W|K_L\rangle$

$$\langle\eta_8|\mathcal{L}_W|K_L\rangle=(\tfrac{1}{3})^{1/2}(1+\xi)\langle\pi^0|\mathcal{L}_W|K_L\rangle\ ,\tag{4.15}$$

the decay constants f_8 and f_0 are not identical to f_π if SU(3) breaking in η_8, $\eta_0\to\pi\pi\gamma$ is included, θ is the mixing angle of the SU(3) octet η_8 and singlet η_0,

$$\eta=\eta_8\cos\theta-\eta_0\sin\theta,\quad \eta'=\eta_8\sin\theta+\eta_0\cos\theta\ , \tag{4.16}$$

and ρ is the complex parameter introduced via

$$\langle \eta_0|\mathcal{L}_W|K_L\rangle=-2(\tfrac{2}{3})^{1/2}\rho\langle \pi^0|\mathcal{L}_W|K_L\rangle \tag{4.17}$$

as the matrix element $\langle \eta_0|\mathcal{L}_W|K_L\rangle$ is not related to the K_L-π^0 transition by SU(3) symmetry.

Unlike the radiative charged kaon decay, there exists a large theoretical uncertainty in the estimate of pole contributions to $K_L\to\pi^+\pi^-\gamma$ owing to the presence of the η' pole. The SU(3) singlet η_0 is outside of the framework of SU(3) chiral perturbation theory, and hence *a priori* the parameter ρ cannot be fixed theoretically. The value of χ is fairly sensitive to the unknown parameter ρ, the η-η' mixing angle θ, and SU(3) breaking in f_π/f_8, f_π/f_0 and in the matrix element $\langle \eta_8|\mathcal{L}_W|K_L\rangle$ measured by the parameter ξ. Nevertheless, the mixing angle θ can be taken reliably as $-20°$, as implied by the $1/N_c$ approach[4(b),38] and confirmed by the recent measurements of $\eta,\eta'\to\gamma\gamma$ rates. As for the parameter ρ, U(3) chiral perturbation theory suggests $\rho=1$, a consequence of nonet symmetry. However, assuming that the $K_L\to\gamma\gamma$ is dominated by the same pole contributions, a fit to experiment yields[39]

$$\rho=(0.22\pm0.05)+0.74\xi \quad \text{or} \quad (0.63\pm0.05)+0.74\xi\ . \tag{4.18}$$

To see the sensitivity of χ with ρ, we find $\chi=1.44$, 0.63, and -0.27, respectively, for $\rho=1$, 0.63, and 0.22 in the absence of SU(3) breaking. Therefore, there is a large theoretical uncertainty in the magnitude and even in the sign of χ. The value of $\rho=0.22$ was employed in Ref. 21 for the consideration of long-distance effects.

As we are going to show shortly, since the contact-term contributions can be calculated reliably in the $1/N_c$ approach, information on the parameter ρ may be extracted from the experimental measurement of $K_L\to\pi^+\pi^-\gamma$. It turns out that a large value of $\rho\approx1.1$ is suggested by this work.

2. *Contact-term contributions*

To compute contact-term contributions (i.e., direct weak transitions) we expand Eqs. (3.1) and (3.20) and retain terms relevant for our purposes:

$$\begin{aligned}\mathcal{L}^{\Delta S=1}_{\rm nonanom}=\frac{2g_8}{\pi^2 f_\pi^5}eF^{\mu\nu}\Bigg[&\frac{2}{\sqrt2}\pi^-\partial_\mu\pi^0\partial_\nu K^+-\frac{1}{\sqrt2}\partial_\mu\pi^-\partial_\nu\pi^0K^+\\&-\frac{1}{\sqrt2}\partial_\mu\pi^-\pi^0\partial_\nu K^++\frac{2}{\sqrt2}\pi^+\partial_\mu\pi^0\partial_\nu K^--\frac{1}{\sqrt2}\partial_\mu\pi^+\partial_\nu\pi^0K^-\\&-\frac{1}{\sqrt2}\partial_\mu\pi^+\pi^0\partial_\nu K^--\partial_\mu\pi^-\partial_\nu\pi^+K^0+\partial_\mu\pi^-\pi^+\partial_\nu K^0-\partial_\mu\pi^+\partial_\nu\pi^-\bar K^0+\partial_\mu\pi^+\pi^-\partial_\nu\bar K^0\Bigg]+\cdots\end{aligned} \tag{4.19}$$

and

$$\begin{aligned}\mathcal{L}^{\Delta S=1}_{\rm anom}=\frac{g_8}{\pi^2 f_\pi^5}e\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}\Bigg[&-\frac{1}{\sqrt2}\partial_\rho\pi^0\partial_\sigma\pi^-K^++\frac{2}{\sqrt2}\partial_\rho\pi^0\pi^-\partial_\sigma K^+\\&+\frac{1}{\sqrt2}\partial_\rho\pi^0\partial_\sigma\pi^+K^--\frac{2}{\sqrt2}\partial_\rho\pi^0\pi^+\partial_\sigma K^-+\pi^+\partial_\rho\pi^-\partial_\sigma K^0\\&-2\partial_\rho\pi^+\pi^-\partial_\sigma K^0-\partial_\rho\pi^+\pi^-\partial_\sigma\bar K^0+2\pi^+\partial_\rho\pi^-\partial_\sigma\bar K^0\Bigg]+\cdots\ .\end{aligned} \tag{4.20}$$

From Eqs. (4.19) and (4.20) it is straightforward to obtain[40]

$$\begin{aligned}A(K^\pm\to\pi^\pm\pi^0\gamma)_{\rm contact}&=\frac{g_8}{\sqrt2\pi^2f_\pi^5}(\mp6M+4E)\ ,\\A(K_L\to\pi^+\pi^-\gamma)_{\rm contact}&=\frac{g_8}{\sqrt2\pi^2f_\pi^5}(12M)\ ,\\A(K_S\to\pi^+\pi^-\gamma)_{\rm contact}&=\frac{g_8}{\sqrt2\pi^2f_\pi^5}(-8E)\ ,\end{aligned} \tag{4.21}$$

where $K_{L,S}=(1/\sqrt2)(K^0\pm\bar K^0)$ within the convention of chiral perturbation theory. We see that if *CP* violation is neglected, the decay mode $K_L\to\pi^+\pi^-\gamma$ proceeds only via the magnetic transition, whereas $K_S\to\pi^+\pi^-\gamma$ is caused by electric transitions.

Thus far, only $\Delta I=\frac12$ contributions to the radiative weak decay have been considered. The $\Delta I=\frac32$, $\Delta S=1$ electromagnetically induced weak chiral Lagrangians can be obtained by substituting Eqs. (3.13) and (3.19) into the operator Q_2+2Q_1 in (3.5), analogous to the $\Delta I=\frac12$ ones.

sults for the total direct weak contributions including $\Delta I=\frac{3}{2}$ effects:

$$A(K^{\pm}\to\pi^{\pm}\pi^0\gamma)_{\rm contact}=\frac{g_8}{\sqrt{2}\pi^2 f_\pi^5}[\mp 6(1-\delta)M+4(1+\delta)E]\ ,$$

$$A(K_L\to\pi^+\pi^-\gamma)_{\rm contact}=\frac{g_8}{\sqrt{2}\pi^2 f_\pi^5}12(1-\tfrac{7}{3}\delta)M\ , \quad (4.22)$$

$$A(K_S\to\pi^+\pi^-\gamma)_{\rm contact}=-\frac{g_8}{\sqrt{2}\pi^2 f_\pi^5}8(1+\delta)E\ ,$$

with $\delta=g_{27}/g_8$. Since $\delta\approx\frac{1}{30}$ from Eq. (3.11), it is clear that $\Delta I=\frac{3}{2}$ contributions are very small.

Numerical values for the predictions (4.12) and (4.22) are shown in Table I with the experimental results taken from Refs. 41–43. It is evident from Table I that the agreement between theory and experiment for the direct emission of $K^+\to\pi^+\pi^-\gamma$ is striking. This indicates that very little room is left for chiral-loop corrections. For $K_S\to\pi^+\pi^-\gamma$, the branching ratio of the structure-dependent component is predicted to be 2×10^{-7}, which is beyond the present upper limit 6×10^{-5}.[43]

We cannot make a definite prediction for the direct emission of $K_L\to\pi^+\pi^-\gamma$ owing to a large theoretical uncertainty in the estimate of the long-distance effect. The direct weak contribution alone will yield a branching ratio of 2×10^{-4} from Eqs. (4.11) and (4.22), which is too large by an order of magnitude. This means that a large *destructive* interference between pole and direct-transition amplitudes of $K_L\to\pi^+\pi^-\gamma$ is required in order to explain data. Fitting to the experimental branching fraction of $(2.89\pm0.28)\times10^{-3}$ (Ref. 42) and neglecting SU(3) breaking, we find surprisingly $\rho\approx1.1$, recalling that $\rho=1$ is the naive prediction of nonet symmetry or U(3) chiral perturbation theory.[44] However, it is still not clear to us how to explain the decay $K_L\to\gamma\gamma$ satisfactorily within the framework of the effective-Lagrangian approach with this value of ρ.

Three remarks are in order. (i) Pole and contact-term contributions are equally important for the direct radiative transition of K^+ and K_L, whereas only the latter one contributes to $K_S\to\pi^+\pi^-\gamma$ in the limit of *CP* symmetry. (ii) Unlike inner bremsstrahlung, the direct-emission amplitudes of $K^{\pm}\to\pi^{\pm}\pi^0\gamma$ and $K_L\to\pi^+\pi^-\gamma$ are no longer subject to the $\Delta I=\frac{1}{2}$ rule and *CP* violation, respectively. This explains why the branching ratio of K^+ and K_L is larger than that of K_S by two orders of magnitude and why structure-dependent effects can be seen in those two decay modes. (iii) The two additional terms in Eq. (3.20), which were not considered in the original EPR work, reduce the ω_3-term contribution to magnetic transitions by a factor of 2. In other words, the ω_3 term alone will lead to a $\tilde{\beta}_{\rm contact}$ which is too large by a factor of 2.

D. The $K\to\pi l\nu\gamma$ decay

The radiative K_{l3} decay was systematically studied by Fearing, Fischbach, and Smith two decades ago.[45] They wrote down the most general Lorentz- and gauge-invariant direct-emission amplitude for the radiative decay $K(k)\to\pi(p_\pi)l(p_l)\nu(p_\nu)\gamma(q)$:

$$T=i\frac{eG_F\sin\theta_C}{\sqrt{2}m_K^2}[A(\epsilon\cdot lk\cdot q-\epsilon\cdot kl\cdot q)+B\epsilon^{\mu\nu\rho\sigma}\epsilon_\mu l_\nu k_\rho q_\sigma+C(\epsilon\cdot lp_\pi\cdot q-\epsilon\cdot p_\pi l\cdot q)+D\epsilon^{\mu\nu\rho\sigma}\epsilon_\mu l_\nu p^\pi_\rho q_\sigma]\ , \quad (4.23)$$

where $l_\mu=\bar{u}(p_l)\gamma_\mu(1-\gamma_5)v(p_\nu)$ is the leptonic current, and A, B, C, and D are four unknown parameters.

In the $1/N_c$ approach advocated in Sec. III, the coefficients of the structure-dependent $K_{l3\gamma}$ amplitude are completely determined. First of all, we expand Eqs. (3.12) and (3.17) and keep those terms relevant for $K_{l3\gamma}$ decays:

$$\frac{eg\sin\theta_C}{4\sqrt{2}\pi^2 f_\pi^2}\left[\epsilon^{\mu\nu\rho\sigma}\partial_\mu A_\nu W_\rho\left[\frac{3}{\sqrt{2}}K^+\partial_\sigma\pi^0-\frac{1}{\sqrt{2}}\pi^0\partial_\sigma K^+-\frac{3}{\sqrt{2}}K^-\partial_\sigma\pi^0+\frac{1}{\sqrt{2}}\pi^0\partial_\sigma K^-+\partial_\sigma(\pi^+K^0)-\partial_\sigma(\pi^-\bar{K}^0)\right]\right.$$
$$\left.+F^{\mu\nu}W_\mu\left[\frac{1}{\sqrt{2}}\pi^0\overleftrightarrow{\partial}_\nu K^++\frac{1}{\sqrt{2}}K^-\overleftrightarrow{\partial}_\nu\pi^0+\pi^+\overleftrightarrow{\partial}_\nu K^0+\bar{K}^0\overleftrightarrow{\partial}_\nu\pi^-\right]\right]\ , \quad (4.24)$$

TABLE I. Theoretical predictions for the direct-emission contributions to the radiative decay $K\to\pi\pi\gamma$. Form factors $\tilde{\beta}$ and $\tilde{\gamma}$ are defined in Eq. (4.10) and are in units of 10^{-6} GeV^{-3}. As explained in the text, because of the large theoretical uncertainties in the long-distance effects (i.e., the form factor $\tilde{\beta}_{\rm pole}$), no definite prediction for the branching ratio of $K_L\to\pi^+\pi^-\gamma$ can be made. Experimental values are taken from Refs. 41–43.

Decay mode	$\tilde{\beta}_{\rm pole}$	$\tilde{\beta}_{\rm contact}$	$\tilde{\gamma}_{\rm contact}$	$(B)_{\rm theor}$	$(B)_{\rm expt}$
$K^{\pm}\to\pi^{\pm}\pi^0\gamma$	∓4.56	∓6.62	4.70	1.94×10^{-5}	$(2.05\pm0.46^{+0.39}_{-0.23})\times10^{-5}$
$K_L\to\pi^+\pi^-\gamma$	?	-12.23		?	$(2.89\pm0.28)\times10^{-5}$
$K_S\to\pi^+\pi^-\gamma$			-9.40	2.02×10^{-7}	$<6\times10^{-5}$

where the factor $g\sin\theta_C/\sqrt{2}$ is now displayed explicitly. From Eq. (4.24) we find

$$K^+_{l3\gamma}: A=-B=C=-D/3=\frac{m_K^2}{2\sqrt{2}\pi^2 f_\pi^2}\ ,$$
$$K^0_{l3\gamma}: A=B=C=-D=\frac{m_K^2}{2\pi^2 f_\pi^2}\ , \qquad (4.25)$$

where the form factors of $K^-_{l3\gamma}$ ($\bar K^0_{l3\gamma}$) are opposite to that of $K^+_{l3\gamma}$ ($K^0_{l3\gamma}$), and use of the relation $g^2/M_W^2=4\sqrt{2}G_F$ has been made.

It is known that this radiative decay is dominated by inner bremsstrahlung[46] and hence experimentally it is fairly difficult to isolate the direct-emission component from the "structureless" bremsstrahlung contributions. To have a theoretical estimate of the structure-dependent contribution, we apply the formula given in Ref. 45(b) for the decay rate of interference between inner bremsstrahlung and direct emission:

$$\Gamma(\bar K^0\to\pi^+e^-\bar\nu\gamma, E_\gamma>30\ \text{MeV})_{\text{int}} = \frac{G_F^2\sin^2\theta_C m_K^5}{64\pi^3} f_+(0)\times 10^{-6} \times(3.7A+1.2B+2.8C+1.2D)\ . \qquad (4.26)$$

This, together with Eq. (4.25), leads immediately to the prediction [with $f_+(0)=1$]

$$B(K_L\to\pi^+e^-\bar\nu\gamma, E_\gamma>30\ \text{MeV})_{\text{int}}=2\times10^{-5}\ . \qquad (4.27)$$

As a consequence, the branching fraction of the direct emission should be $\sim 10^{-7}$–10^{-8}. For $K^+\to\pi^0e^+\nu\gamma$, a rough estimate using the phase-space integral given in Ref. 45(a) yields

$$B(K^+\to\pi^0e^+\nu\gamma, E_\gamma>30\ \text{MeV})_{\text{int}}\sim 1\times10^{-6}\ , \qquad (4.28)$$

which in turn implies a branching ratio of order 10^{-8} for the structure-dependent component. At present, the best upper limit for the direct-emission rate is[41]

$$B(K^+\to\pi^0e^+\nu\gamma, E_\gamma>10\ \text{MeV})_{\text{DE}}<5.3\times10^{-5}\ . \qquad (4.29)$$

V. SUMMARY AND CONCLUSIONS

In the standard framework of chiral perturbation theory, the couplings of higher-order chiral Lagrangians are running parameters; that is, they depend on the choice of the renormalization scale. *A priori*, those couplings are unknown and can only be empirically determined from low-energy hadronic experiments. However, predictions can be made in the QCD-inspired model for higher-order chiral Lagrangians. The main purpose of this paper is to continue this approach to study $\Delta S=1$ radiative weak transitions.

In the QCD-motivated model, the nonanomalous higher-order strong chiral Lagrangians $\mathcal{L}_S^4$ arise from the integration of spurious (nontopological) chiral anomalies, in analog to the anomalous Wess-Zumino-Witten terms $\mathcal{L}_{\text{WZW}}$ determined by the integration of topological Bardeen anomalies. The couplings of these effective $1/N_c$ Lagrangians are scale-independent constants since chiral-loop effects are suppressed in the large-N_c limit. The higher-derivative large-N_c Lagrangian can be tested on the ground that certain combinations of the running renormalized coupling constants are independent of the renormalization point, for example, the combination L_9+L_{10}. Experiment on the pion polarizability reveals a good agreement with the "lowest-order" theoretical prediction on L_9+L_{10}, which is also in accord with the very recent measurement of form factors in the decay amplitude of $\pi^+\to e^+\nu e^+e^-$.

The higher-order $\Delta S=1$ effective weak chiral Lagrangians in the $1/N_c$ approach are derived based on the following three ingredients: an extremely simple structure of the effective weak Hamiltonian at the quark level in the leading $1/N_c$ expansion, bosonization determined from the electroweak perturbations to $\mathcal{L}_S^2+\mathcal{L}_S^4+\mathcal{L}_{\text{WZW}}$, and factorization valid on the large-N_c approximation. Equations (3.1), (3.15), and (3.20) are the main results for the weak chiral Lagrangians $\mathcal{L}^{\Delta S=1}_{\text{nonanom}}$ and $\mathcal{L}^{\Delta S=1}_{\text{anom}}$ coupled to external electromagnetic fields. For anomalous $\Delta S=1$ weak interactions, we find two additional contributions not considered before.

The radiative kaon transition cannot be generated from the lowest-order chiral Lagrangian since Lorentz and gauge invariance requires at least two powers of momenta in the amplitude. Therefore, radiative $\Delta S=1$ weak decays offer a nice test on the Lagrangians $\mathcal{L}^{\Delta S=1}_{\text{nonanom}}$ and $\mathcal{L}^{\Delta S=1}_{\text{anom}}$. Unfortunately, we find vanishing $K\to\pi\gamma^*\to\pi l^+l^-$ transitions in the large-N_c approximation. This means that the $K\to\pi l^+l^-$ ($l=e,\mu$) decay receives contributions first from the $1/N_c$ corrections due to chiral loops and hence weak couplings ω_i get renormalization. Nevertheless, the relation $\omega_2=4L_9$ found previously from the study of meson loop contributions to $K\to\pi\gamma^*$ is numerically reproduced in the large-N_c approach advocated in the present paper. Moreover, the fact that the $1/N_c$ predicted couplings L_i are consistent with the empirically determined renormalized constants L_i^r at the mass scale between 0.5 and 1 GeV enables us to pick up one of the solutions for the weak renormalized constant ω_1^r determined from $\Gamma(K^+\to\pi^+e^+e^-)$. The twofold ambiguity for the predictions of the decay rates of $K^+\to\pi^+\mu^+\mu^-$, $K_S\to\pi^0l^+l^-$ ($l=e,\mu$) given by Ecker, Pich, and de Rafael is thus suggestively resolved. On the experimental side, three candidate events were recently reported by the Brookhaven E-787 Collaboration. However, a significant signal for this decay mode has not yet been established.

As for the decay $K^+\to\pi^+\gamma\gamma$, the loop amplitude is finite but it also receives contributions from tree Lagrangians. The branching ratio is predicted to be 5.1×10^{-7}, which is close to the present best upper limit 10^{-6} set by the Brookhaven E-787 experiment derived by assuming a pion phase-space spectrum.

The transitions $K\to\pi\pi\gamma$ provide an excellent test on the Lagrangians $\mathcal{L}^{\Delta S=1}_{\text{nonanom}}$ and $\mathcal{L}^{\Delta S=1}_{\text{anom}}$. Unlike the previous two decay modes $K\to\pi l^+l^-$ and $K\to\pi\gamma\gamma$, the structure-dependent photon emission of $K\to\pi\pi\gamma$ receives contributions from $\mathcal{L}^{\Delta S=1}_{\text{anom}}$ and is dominated by the tree amplitude. Previous considerations in the framework of, e.g., current algebra, the vector-meson-dominance model, and the effective weak Hamiltonian are unreliable, crude, and erroneous in some respects. The computation for the direct decays $K^+ \to \pi^+\pi^0\gamma$ and $K_S \to \pi^+\pi^-\gamma$ is now amenable to the large-N_c chiral perturbation theory. We find a striking agreement for the direct emission of $K^\pm \to \pi^\pm\pi^0\gamma$, implying that very little room is left for loop corrections. The branching ratio of the direct $K_S \to \pi^+\pi^-\gamma$ decay is predicted to be 2×10^{-7}.

Owing to a large theoretical uncertainty in the estimate of the pole contribution to $K_L \to \pi^+\pi^-\gamma$, no definite prediction on the branching fraction can be made. The pole effect on this mode is very sensitive to SU(3) breaking and especially to the parameter ρ which relates the K-η_0 matrix element to the K-π transition. After subtracting the contact-term part, which is calculable in the large-N_c chiral perturbation theory, we find from the experimental result for the structure-dependent component of $K_L \to \pi^+\pi^-\gamma$ that $\rho \approx 1.1$, recalling that $\rho = 1$ is predicted by nonet symmetry.

A low-energy test of the higher-derivative chiral Lagrangians is also devoted to the radiative K_{l3} decay. The structure-dependent $K_{l3\gamma}$ amplitude is completely determined in the framework of $1/N_c$ chiral perturbation theory. The branching fractions of the direct emission of $K_L \to \pi^+ e^- \bar{\nu}\gamma$ and its interference with inner bremsstrahlung are estimated to be of order 10^{-7}–10^{-8} and 10^{-5}, respectively.

To conclude, we have extended the QCD-motivated approach for higher-order chiral Lagrangians to radiative kaon decays. Decay rates and spectra are unambiguously predictable (at least) to the leading $1/N_c$ expansion and to the zeroth order in gluonic modifications. Future high-statistics experiments with great sensitivity will be able to test those predictions.

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