Bound states for a massive spin-one particle and a magnetic monopole

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(Received 18 December 1989)

We analyze the interaction of a magnetic monopole with a massive spin-one particle that has an extra permanent magnetic moment, as well as an extra gauge-invariant interaction, quadratic in the monopole magnetic field, an induced magnetization. This interaction can lead to bound states. The corresponding bound-state energies depend critically on the strength of the induced magnetization.

I. INTRODUCTION

In an accompanying paper[1] (referred to as I), we show that a massive spin-one particle that has a suitable permanent magnetic moment can be attracted to a magnetic monopole, but that this system is unstable at short distances. More specifically, the radial equations that describe the would-be bound states do not permit a wave function that vanishes at the origin as it must because of the way the eigenfunctions of angular momentum behave.[2,3]

If, however, the spin-one particle has some additional, intrinsic property that makes it repel the monopole at short distances, without having any significant effect at large distances, then bound states could exist. One possibility which we consider in this paper is to endow the spin-one particle with an extra gauge-invariant interaction, quadratic in the monopole magnetic field, an induced magnetization. It turns out that if this term is repulsive at short distances it is sufficiently singular to make the wave function vanish at the origin.

The paper is organized as follows. In Sec. II we find the eigenvalue equation for the vector field $\boldsymbol{\phi}$. In Sec. III we determine the corresponding coupled (scalar) radial equations. Sections IV and V are devoted to a study of these equations at short distances, for type-A and -B states, respectively. In Secs. VI and VII we discuss the type-C eigenvalue equation in some more detail, and determine the bound-state spectrum.

II. PROCA EQUATION FOR THE CASE OF INDUCED MAGNETIZATION

Let $\mathcal{L}_0$ be the Lagrangian of Ref. 1:

$$\mathcal{L}_0=-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}-\tfrac{1}{2}G^{\dagger}_{\mu\nu}G^{\mu\nu}+m^2\phi^{\dagger\mu}\phi_\mu+ie\kappa F_{\mu\nu}\phi^{\dagger\mu}\phi^\nu \ . \tag{2.1}$$

We have seen[1] that while the extra magnetic-moment interaction $ie\kappa F_{\mu\nu}\phi^{\dagger\mu}\phi^\nu$ can lead to attraction at large distances (i.e., wave functions that are oscillatory in some region, and then fall off exponentially as $r\to\infty$), it cannot at the same time provide an acceptable solution at short distances. In this paper we discuss, as an example, how the introduction of a term that is bilinear in the magnetic field,

$$\mathcal{L}=\mathcal{L}_0+\lambda(e/m)^2F_{\mu\nu}F_\rho{}^\nu\phi^{\dagger\mu}\phi^\rho \ , \tag{2.2}$$

can lead to wave functions that are well behaved also at short distances.

Written out in terms of the monopole magnetic field $\mathbf{B}=g\mathbf{r}/r^3$, the extra interaction term is given by

$$-\lambda\frac{q^2}{m^2r^4}(|\boldsymbol{\phi}|^2-|\boldsymbol{\phi}\cdot\mathbf{r}/r|^2) \ . \tag{2.3}$$

The eigenvalue equation corresponding to the present Lagrangian is derived in the same way as in Sec. II of paper I. The equation of motion takes the form [compare Eq. (I.2.4)]

$$D_\mu G^{\mu\nu}+m^2\phi^\nu-ie\kappa\phi_\mu F^{\mu\nu}+\lambda(e/m)^2F^{\mu\nu}F_{\mu\rho}\phi^\rho=0 \ , \tag{2.4}$$

or

$$D_\mu D^\mu\phi^\nu-D_\mu D^\nu\phi^\mu+m^2\phi^\nu-ie\kappa\phi_\mu F^{\mu\nu}+\lambda(e/m)^2F^{\mu\nu}F_{\mu\rho}\phi^\rho=0 \ . \tag{2.5}$$

We next consider the covariant divergence of this expression:

$$D_\nu D_\mu D^\mu\phi^\nu-D_\nu D_\mu D^\nu\phi^\mu+m^2D_\nu\phi^\nu-ie\kappa D_\nu\phi_\mu F^{\mu\nu}+\lambda(e/m)^2D_\nu F^{\mu\nu}F_{\mu\rho}\phi^\rho=0 \ . \tag{2.6}$$

Invoking Eqs. (I.2.7) and (I.2.9), we find

$$m^2D_\nu\phi^\nu-ie(1-\kappa)F_{\mu\nu}D^\mu\phi^\nu+\lambda(e/m)^2D_\nu F^{\mu\nu}F_{\mu\rho}\phi^\rho=0 \ . \tag{2.7}$$

We next rewrite Eq. (2.5) as [cf. Eq. (I.2.11)]

$$(D_\mu D^\mu+m^2)\phi_\nu-D_\nu D_\mu\phi^\mu-ie(1+\kappa)\phi^\mu F_{\mu\nu}+\lambda(e/m)^2F_{\mu\nu}F^{\mu\rho}\phi_\rho=0 \ , \tag{2.8}$$

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which, by virtue of Eq. (2.7) takes the form

$$(D_\mu D^\mu+m^2)\phi_\nu-ie(1+\kappa)\phi^\mu F_{\mu\nu}+\lambda(e/m)^2F_{\mu\nu}F^{\mu\rho}\phi_\rho - \frac{ie}{m^2}(1-\kappa)D_\nu F_{\alpha\beta}D^\alpha\phi^\beta+\frac{\lambda e^2}{m^4}D_\nu D_\beta F^{\alpha\beta}F_{\alpha\rho}\phi^\rho=0 . \tag{2.9}$$

For the spatial components of ϕ, this equation takes the form

$$(D_\mu D^\mu+m^2)\phi^k-ie(1+\kappa)\phi_i F^{ik}+\lambda(e/m)^2F^{ik}F_{ij}\phi^j - \frac{ie}{m^2}(1-\kappa)D^kF_{ij}D^i\phi^j+\frac{\lambda e^2}{m^4}D^kD_jF^{ij}F_{il}\phi^l=0 . \tag{2.10}$$

By Eq. (I.2.14),

$$F^{ij}F_i{}^k=g^{jk}\mathbf{B}^2+B^jB^k , \tag{2.11}$$

with $g^{\mu\nu}=\mathrm{diag}(1,-1,-1,-1)$ the metric tensor, and thus Eq. (I.2.15) gets replaced by

$$\left[\frac{\partial^2}{\partial t^2}-(\nabla-ie\,\mathbf{A})^2+m^2\right]\phi^k-ie(1+\kappa)\epsilon_{ikl}B^l\phi^i+\lambda(e/m)^2(g^{kj}\mathbf{B}^2+B^kB^j)\phi_j + \frac{ie}{m^2}(1-\kappa)\left[\frac{\partial}{\partial x^k}-ieA^k\right]\epsilon_{ijl}B^l\left[\frac{\partial}{\partial x^i}-ieA^i\right]\phi^j-\frac{\lambda e^2}{m^4}\left[\frac{\partial}{\partial x^k}-ieA^k\right]\left[\frac{\partial}{\partial x^j}-ieA^j\right](g^{jl}\mathbf{B}^2+B^jB^l)\phi_l=0 . \tag{2.12}$$

The time dependence is given by $\phi^k\sim e^{-iEt}$. Invoking also Eq. (I.2.18), we find the equation of motion of the spin-one particle in the field of the magnetic monopole:

$$\left[-E^2+m^2-\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\frac{\partial}{\partial r}\right]+\frac{1}{r^2}(\mathbf{L}^2-q^2)\right]\boldsymbol{\phi}-ie(1+\kappa)\mathbf{B}\times\boldsymbol{\phi}+\lambda(e/m)^2[\mathbf{B}^2\boldsymbol{\phi}-\mathbf{B}(\mathbf{B}\cdot\boldsymbol{\phi})] + \frac{ie}{m^2}(1-\kappa)(\nabla-ie\,\mathbf{A})\{\mathbf{B}\cdot[(\nabla-ie\,\mathbf{A})\times\boldsymbol{\phi}]\}-\frac{\lambda e^2}{m^4}(\nabla-ie\,\mathbf{A})\{(\nabla-ie\,\mathbf{A})\cdot[\mathbf{B}^2\boldsymbol{\phi}-\mathbf{B}(\mathbf{B}\cdot\boldsymbol{\phi})]\}=0 . \tag{2.13}$$

III. RADIAL EQUATIONS

We write the wave function for a type-A state of angular momentum (j,j_z) as

$$\boldsymbol{\phi}_{jj_z}=\sum_{l=j-1}^{j+1}f_{jl}(r)\mathbf{Y}^{(q)}_{jlj_z}(\hat{\mathbf{r}}) , \tag{3.1}$$

where, in the notation of Sec. III of paper I,

$$f_{j,j-1}(r)=f(r),\quad f_{j,j}(r)=g(r),\quad f_{j,j+1}(r)=h(r) , \tag{3.2}$$

and $\mathbf{Y}^{(q)}_{jlj_z}(\hat{\mathbf{r}})$ are the monopole vector harmonics.

A. Terms multiplying λ

As compared with Eq. (I.2.19), Eq. (2.13) here has two more terms, namely, those multiplying λ. We start out by expressing the first of those in terms of $\mathbf{Y}^{(q)}_{jLj_z}(\hat{\mathbf{r}})$. Since $\mathbf{B}=g\hat{\mathbf{r}}/r^2$, we obtain

$$\mathbf{B}^2\boldsymbol{\phi}_{jj_z}-\mathbf{B}(\mathbf{B}\cdot\boldsymbol{\phi}_{jj_z})=\frac{g^2}{r^4}\sum_l f_{jl}(r)\{\mathbf{Y}^{(q)}_{jlj_z}(\hat{\mathbf{r}})-\hat{\mathbf{r}}[\hat{\mathbf{r}}\cdot\mathbf{Y}^{(q)}_{jlj_z}(\hat{\mathbf{r}})]\} . \tag{3.3}$$

Here, we can simplify the last term, using Eqs. (I.4.1) and (I.4.13),

$$\hat{\mathbf{r}}[\hat{\mathbf{r}}\cdot\mathbf{Y}^{(q)}_{jlj_z}(\hat{\mathbf{r}})]=\sum_L C^{(q)}_{jl}C^{(q)}_{jL}\mathbf{Y}^{(q)}_{jLj_z}(\hat{\mathbf{r}}) , \tag{3.4}$$

and therefore

$$\mathbf{B}^2\boldsymbol{\phi}_{jj_z}-\mathbf{B}(\mathbf{B}\cdot\boldsymbol{\phi}_{jj_z})=\frac{g^2}{r^4}\sum_{lL}h_{jlL}f_{jl}(r)\mathbf{Y}^{(q)}_{jLj_z}(\hat{\mathbf{r}}) , \tag{3.5}$$

with

$$h_{jlL}=\delta_{lL}-C^{(q)}_{jl}C^{(q)}_{jL} . \tag{3.6}$$

We next turn to the last term in (2.13). By Eq. (I.5.8),

$$(\nabla - ie\,\mathbf{A})\cdot[\mathbf{B}^2\boldsymbol{\phi}_{jj_z} - \mathbf{B}(\mathbf{B}\cdot\boldsymbol{\phi}_{jj_z})] = \left[\hat{\mathbf{r}}\frac{\partial}{\partial r} - \frac{i}{r^2}\mathbf{r}\times\mathbf{L}\right]\cdot\left[\frac{g^2}{r^4}\sum_{lL} h_{jlL} f_{jl}(r)\mathbf{Y}^{(q)}_{jLj_z}(\hat{\mathbf{r}})\right] . \tag{3.7}$$

In order to evaluate the last term here, we need relation (I.4.9). Invoking also Eq. (I.4.1), we can write Eq. (3.7) as

$$\sum_{lL} h_{jlL}\left[\frac{d}{dr}\left[\frac{g^2}{r^4}f_{jl}(r)\right]C^{(q)}_{jL}Y^{(q)}_{jj_z}(\hat{\mathbf{r}}) - \frac{ig^2}{r^5}f_{jl}(r)\hat{\mathbf{r}}\cdot[\mathbf{L}\times\mathbf{Y}^{(q)}_{jLj_z}(\hat{\mathbf{r}})]\right] = \sum_{lL} h_{jlL}C^{(q)}_{jL}\left[\frac{d}{dr}\left[\frac{g^2}{r^4}f_{jl}(r)\right] - \frac{ig^2}{r^5}f_{jl}(r)D_{jL}\right]Y^{(q)}_{jj_z}(\hat{\mathbf{r}})$$
$$\equiv v(r)Y^{(q)}_{jj_z}(\hat{\mathbf{r}}) . \tag{3.8}$$

Invoking again Eq. (I.5.8), we can thus write the quantity that appears in the last term of Eq. (2.13) as

$$(\nabla - ie\,\mathbf{A})\{(\nabla - ie\,\mathbf{A})\cdot[\mathbf{B}^2\boldsymbol{\phi}_{jj_z} - \mathbf{B}(\mathbf{B}\cdot\boldsymbol{\phi}_{jj_z})]\} = \left[\hat{\mathbf{r}}\frac{\partial}{\partial r} - \frac{i}{r^2}\mathbf{r}\times\mathbf{L}\right]v(r)Y^{(q)}_{jj_z}(\hat{\mathbf{r}})$$
$$= \sum_L\left[C^{(q)}_{jL}\left[\frac{d}{dr}v(r)\right] + A^{(q)}_{jjL}\sqrt{j(j+1)}\frac{1}{r}v(r)\right]\mathbf{Y}^{(q)}_{jLj_z}(\hat{\mathbf{r}}) . \tag{3.9}$$

In the last step, we have for the first term used Eq. (I.4.13), whereas for the second one, we have used Eqs. (I.4.14) and (I.4.5).

Let us now return to $v(r)$, as defined by Eqs. (3.8), (I.4.2), and (I.4.11),

$$v(r) = \sum_{lL} h_{jlL}C^{(q)}_{jL}\left[\frac{d}{dr}\left[\frac{g^2}{r^4}f_{jl}(r)\right] + \frac{g^2}{r^5}f_{jl}(r)\{1+\tfrac{1}{2}[L(L+1) - j(j+1)]\}\right] . \tag{3.10}$$

Invoking Eqs. (I.4.4) and (3.6), we find that

$$\sum_L h_{jlL}C^{(q)}_{jL} = 0 . \tag{3.11}$$

Therefore, the expression for $v(r)$ simplifies,

$$v(r) = \frac{g^2}{2r^5}\sum_{lL} f_{jl}(r)h_{jlL}C^{(q)}_{jL}L(L+1) . \tag{3.12}$$

By explicit evaluation, we have

$$v(r) = \frac{g^2}{r^5}\left[-(j+1)\lambda(j)\left[\frac{j}{2j+1}\right]^{1/2}f(r) + \frac{q}{\sqrt{j(j+1)}}g(r) - j\lambda(j+1)\left[\frac{j+1}{2j+1}\right]^{1/2}h(r)\right] , \tag{3.13}$$

with $\lambda(j) = [1-(q/j)^2]^{1/2}$ as defined by Eq. (I.4.3).

B. Coupled differential equations

The coupled radial equations are determined from (2.13), expanding everything in terms of $\mathbf{Y}^{(q)}_{jLj_z}(\hat{\mathbf{r}})$. The terms that are independent of λ are given by Eqs. (I.5.23)–(I.5.25), the new ones are found from (3.5) and (3.9). The results are

$$\left[-E^2 + m^2 - \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\frac{\partial}{\partial r}\right] + \frac{(j-1)j - q^2}{r^2}\right]f(r) + \frac{1+\kappa}{r^2}\left[\frac{q^2}{j}f(r) + q\lambda(j)\left[\frac{j+1}{2j+1}\right]^{1/2}g(r)\right]$$
$$-\frac{1-\kappa}{m^2}q\lambda(j)\left[\frac{j}{2j+1}\right]^{1/2}\left[u'(r) + \frac{j+1}{r}u(r)\right]$$
$$+\frac{\lambda q^2}{m^2r^4}\frac{1}{j\sqrt{(j+1)(2j+1)}}\left[[j(j+1)+q^2]\left[\frac{j+1}{2j+1}\right]^{1/2}f(r) + qj\lambda(j)g(r) + j(j+1)\lambda(j)\lambda(j+1)\left[\frac{j}{2j+1}\right]^{1/2}h(r)\right]$$
$$-\frac{\lambda e^2}{m^4}\lambda(j)\left[\frac{j}{2j+1}\right]^{1/2}\left[v'(r) + \frac{j+1}{r}v(r)\right] = 0 , \tag{3.14}$$

$$\left[-E^2+m^2-\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\frac{\partial}{\partial r}\right]+\frac{j(j+1)-q^2}{r^2}\right]g(r)$$
$$+\frac{1+\kappa}{r^2}\left[q\lambda(j)\left[\frac{j+1}{2j+1}\right]^{1/2}f(r)-\frac{q^2}{j(j+1)}g(r)+q\lambda(j+1)\left[\frac{j}{2j+1}\right]^{1/2}h(r)\right]$$
$$+\frac{1-\kappa}{m^2}\frac{q^2}{\sqrt{j(j+1)}}\left[u'(r)+\frac{1}{r}u(r)\right]+\frac{\lambda q^2}{m^2r^4}\frac{1}{j(j+1)}\left[qj\lambda(j)\left[\frac{j+1}{2j+1}\right]^{1/2}f(r)+[j(j+1)-q^2]g(r)\right.$$
$$\left.-q(j+1)\lambda(j+1)\left[\frac{j}{2j+1}\right]^{1/2}h(r)\right]$$
$$+\frac{\lambda e^2}{m^4}\frac{q}{\sqrt{j(j+1)}}\left[v'(r)+\frac{1}{r}v(r)\right]=0\ , \tag{3.15}$$

$$\left[-E^2+m^2-\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\frac{\partial}{\partial r}\right]+\frac{(j+1)(j+2)-q^2}{r^2}\right]h(r)+\frac{1+\kappa}{r^2}\left[q\lambda(j+1)\left[\frac{j}{2j+1}\right]^{1/2}g(r)-\frac{q^2}{j+1}h(r)\right]$$
$$+\frac{1-\kappa}{m^2}q\lambda(j+1)\left[\frac{j+1}{2j+1}\right]^{1/2}\left[u'(r)-\frac{j}{r}u(r)\right]$$
$$+\frac{\lambda q^2}{m^2r^4}\frac{1}{(j+1)\sqrt{j(2j+1)}}\left[j(j+1)\lambda(j)\lambda(j+1)\left[\frac{j+1}{2j+1}\right]^{1/2}f(r)-q(j+1)\lambda(j+1)g(r)\right.$$
$$\left.+[j(j+1)+q^2]\left[\frac{j}{2j+1}\right]^{1/2}h(r)\right]$$
$$+\frac{\lambda e^2}{m^2}\lambda(j+1)\left[\frac{j+1}{2j+1}\right]^{1/2}\left[v'(r)-\frac{j}{r}v(r)\right]=0\ . \tag{3.16}$$

It is convenient to introduce the functions $F(r)$, $G(r)$, and $H(r)$ defined by Eq. (I.5.26). Furthermore, in analogy with Eq. (I.5.27),

$$K(r)\equiv j\lambda(j)F(r)+[j(j+1)-q^2]G(r)-(j+1)\lambda(j+1)H(r)\ , \tag{3.17}$$

we define

$$M(r)\equiv e^2q\sqrt{j(j+1)}r^6v(r)\equiv q^2[-j(j+1)\lambda(j)F+q^2G-j(j+1)\lambda(j+1)H]\ . \tag{3.18}$$

The three radial equations (3.14)–(3.16) then take on a somewhat simpler appearance:

$$\left[r^2(m^2-E^2)-r^2\frac{d^2}{dr^2}+(j-1)j-q^2\right]F+\frac{q^2(1+\kappa)}{j}\left[F+\lambda(j)\frac{j(j+1)}{2j+1}G\right]-\frac{q^2(1-\kappa)}{(mr)^2}\frac{\lambda(j)}{2j+1}\left[r\frac{d}{dr}+j-3\right]K$$
$$+\frac{\lambda q^2}{(mr)^2}\frac{1}{j(2j+1)}\{[j(j+1)+q^2]F+q^2j\lambda(j)G+j(j+1)\lambda(j)\lambda(j+1)H\}-\frac{\lambda}{(mr)^4}\frac{\lambda(j)}{2j+1}\left[r\frac{d}{dr}+j-5\right]M=0\ , \tag{3.19}$$

$$\left[r^2(m^2-E^2)-r^2\frac{d^2}{dr^2}+j(j+1)-q^2\right]G+(1+\kappa)\left[\lambda(j)F-\frac{q^2}{j(j+1)}G+\lambda(j+1)H\right]+\frac{q^2(1-\kappa)}{(mr)^2}\frac{1}{j(j+1)}\left[r\frac{d}{dr}-3\right]K$$
$$+\frac{\lambda q^2}{(mr)^2}\frac{1}{j(j+1)}\{\lambda(j)F+[j(j+1)-q^2]G-(j+1)\lambda(j+1)H\}+\frac{\lambda}{(mr)^4}\frac{1}{j(j+1)}\left[r\frac{d}{dr}-5\right]M=0\ , \tag{3.20}$$

$$\left[r^2(m^2-E^2)-r^2\frac{d^2}{dr^2}+(j+1)(j+2)-q^2\right]H$$
$$+\frac{q^2(1+\kappa)}{j+1}\left[\lambda(j+1)\frac{j(j+1)}{2j+1}G-H\right]+\frac{q^2(1-\kappa)}{(mr)^2}\frac{\lambda(j+1)}{2j+1}\left[r\frac{d}{dr}-(j+4)\right]K$$
$$+\frac{\lambda q^2}{(mr)^2}\frac{1}{(j+1)(2j+1)}\{j(j+1)\lambda(j)\lambda(j+1)F-q^2(j+1)\lambda(j+1)G+[j(j+1)+q^2]H\}$$
$$+\frac{\lambda}{(mr)^4}\frac{\lambda(j+1)}{2j+1}\left[r\frac{d}{dr}-(j+6)\right]M=0\ . \quad (3.21)$$

IV. BEHAVIOR OF THE TYPE-A WAVE FUNCTIONS AT THE ORIGIN

At short distances we can neglect the small terms $\sim r^2(m^2-E^2)$ in Eqs. (3.19)–(3.21). It is furthermore convenient to introduce, as a new variable,

$$x=\frac{1}{(mr)^2}\ . \quad (4.1)$$

With

$$r\frac{d}{dr}=-2x\frac{d}{dx},\quad -r^2\frac{d^2}{dr^2}=-4x^2\frac{d^2}{dx^2}-6x\frac{d}{dx}\ , \quad (4.2)$$

Eqs. (3.19)–(3.21) can for large x (i.e., at short distances) be approximated as

$$\left[-4x^2\frac{d^2}{dx^2}-6x\frac{d}{dx}+(j-1)j-q^2\right]F+\frac{(1+\kappa)}{j^2(2j+1)\lambda(j)}(q^2jK-M)-\frac{q^2(1-\kappa)\lambda(j)}{2j+1}x\left[-2x\frac{d}{dx}+j-3\right]K$$
$$+\frac{\lambda}{j^2(2j+1)\lambda(j)}x(q^4K-jM)-\lambda\frac{\lambda(j)}{2j+1}x^2\left[-2x\frac{d}{dx}+j-5\right]M=0\ , \quad (4.3)$$

$$\left[-4x^2\frac{d^2}{dx^2}-6x\frac{d}{dx}+j(j+1)-q^2\right]G-\frac{1+\kappa}{q^2j(j+1)}M+\frac{q^2(1-\kappa)}{j(j+1)}x\left[-2x\frac{d}{dx}-3\right]K$$
$$+\frac{\lambda q^2}{j(j+1)}xK+\frac{\lambda}{j(j+1)}x^2\left[-2x\frac{d}{dx}-5\right]M=0\ , \quad (4.4)$$

$$\left[-4x^2\frac{d^2}{dx^2}-6x\frac{d}{dx}+(j+1)(j+2)-q^2\right]H+\frac{1+\kappa}{(j+1)^2(2j+1)\lambda(j+1)}[q^2(j+1)K+M)]$$
$$+\frac{q^2(1-\kappa)\lambda(j+1)}{2j+1}x\left[-2x\frac{d}{dx}-(j+4)\right]K-\frac{\lambda}{(j+1)^2(2j+1)\lambda(j+1)}x[q^4K+(j+1)M]$$
$$+\lambda\frac{\lambda(j+1)}{2j+1}x^2\left[-2x\frac{d}{dx}-(j+6)\right]M=0\ . \quad (4.5)$$

It is possible to obtain equations between K and M only. In order to do this, we first note that Eqs. (3.17) and (3.18) may be combined to yield

$$q^2(2j+1)\sqrt{\sigma(j)}F+q^2(j+1)\sigma(j)G=q^2jK-M\ , \quad (4.6a)$$

$$q^2j\sigma(j+1)G-q^2(2j+1)\sqrt{\sigma(j+1)}H=q^2(j+1)K+M\ , \quad (4.6b)$$

where we have introduced the notation

$$\sigma(j)\equiv[j\lambda(j)]^2\equiv j^2-q^2\ . \quad (4.7)$$

Forming now suitable linear combinations of Eqs. (4.3)–(4.5), one obtains

$$-2j^2\lambda(j)F-2(j+1)^2\lambda(j+1)H+\mathcal{D}K-\frac{1+\kappa}{q^2}M+q^2(1-\kappa)xK+\lambda q^2xK+\lambda x^2M=0\ , \quad (4.8)$$

and

$$2q^2 j(j+1)[j\lambda(j)F-(j+1)\lambda(j+1)H]+\mathcal{D}M-(1+\kappa)q^4K+q^4(1-\kappa)[j(j+1)-q^2]xK$$
$$+\lambda q^2 xM+\lambda q^2[j(j+1)-q^2]x^2M=0\ , \tag{4.9}$$

with

$$\mathcal{D}\equiv -4x^2\frac{d^2}{dx^2}-6x\frac{d}{dx}+j(j+1)-q^2\ . \tag{4.10}$$

Eliminating further F and H, one obtains, after some algebra,

$$\mathcal{D}[(\mathcal{D}-2j+\lambda q^2 x)(q^2jK-M)+(1+\kappa)(q^4K-jM)-(1-\kappa)q^4\sigma(j)xK-\lambda q^2\sigma(j)x^2M]+2(1+\kappa)\sigma(j)M$$
$$-2\lambda q^4\sigma(j)xK+2(1-\kappa)q^4\sigma(j)x\left[2x\frac{d}{dx}+3\right]K+2\lambda q^2\sigma(j)x^2\left[2x\frac{d}{dx}+5\right]M=0\ , \tag{4.11}$$

$$\mathcal{D}\{[\mathcal{D}+2(j+1)+\lambda q^2x][q^2(j+1)K+M]-(1+\kappa)[q^4K+(j+1)M]+(1-\kappa)q^4\sigma(j+1)xK+\lambda q^2\sigma(j+1)x^2M\}$$
$$-2(1+\kappa)\sigma(j+1)M+2\lambda q^4\sigma(j+1)xK-2(1-\kappa)q^4\sigma(j+1)x\left[2x\frac{d}{dx}+3\right]K-2\lambda q^2\sigma(j+1)x^2\left[2x\frac{d}{dx}+5\right]M=0\ . \tag{4.12}$$

It turns out that Eqs. (4.11) and (4.12) have solutions whose leading types of behavior are of the forms

$$\text{(I)}\quad K(x)\underset{x\to\infty}{\sim}K_0e^{-Ax}x^p,\quad M(x)\underset{x\to\infty}{\sim}M_0e^{-Ax}x^p\ , \tag{4.13a}$$

$$\text{(II)}\quad K(x)\underset{x\to\infty}{\sim}K_0x^\nu,\quad M(x)\underset{x\to\infty}{\sim}M_0x^\nu\ , \tag{4.13b}$$

$$\text{(III)}\quad K(x)\underset{x\to\infty}{\sim}K_1x^{\nu+1},\quad M(x)\underset{x\to\infty}{\sim}M_0x^\nu\ . \tag{4.13c}$$

For ansatz (I), Eqs. (4.11) and (4.12) yield

$$A=\pm\tfrac{1}{2}|q|\sqrt{\lambda[j(j+1)-q^2]}\ , \tag{4.14}$$

with the upper-sign solution acceptable for

$$\lambda>0\ . \tag{4.15}$$

In order to discuss the solutions whose leading behavior is powerlike, we introduce the notation

$$\mathcal{D}_\nu\equiv x^{-\nu}\mathcal{D}x^\nu=-4\nu(\nu-1)-6\nu+j(j+1)-q^2\ . \tag{4.16}$$

For ansatz (II), Eqs. (4.11) and (4.12) yield immediately

$$\mathcal{D}_{\nu+2}=2(2\nu+5)\ . \tag{4.17}$$

Substituting for $\mathcal{D}_{\nu+2}$ according to Eq. (4.16), we find the two solutions

$$\nu_{\substack{1\\2}}=-\tfrac{1}{4}\{11\pm\sqrt{1+4[j(j+1)-q^2]}\}\ . \tag{4.18}$$

For ansatz (III), Eqs. (4.11) and (4.12) yield, for the dominant terms,

$$\mathcal{D}_{\nu+2}=0\ , \tag{4.19}$$

or

$$\nu_{\substack{3\\4}}=-\tfrac{1}{4}\{9\pm\sqrt{1+4[j(j+1)-q^2]}\}\ , \tag{4.20}$$

By Eq. (4.1), the corresponding short-distance behavior is, for the power solutions,

$$M(r)\underset{r\to 0}{\sim}(mr)^{-2\nu}\ . \tag{4.21}$$

By Eqs. (4.8) and (4.9),

$$F\underset{x\to\infty}{\sim}x^2M\underset{r\to 0}{\sim}(mr)^{-2\nu-4}\ , \tag{4.22}$$

and similarly G and H.

In order to determine precisely which condition $F(r)$ must satisfy at the origin, we apply the analysis of paper I, Sec. VII. With the extra interaction term (2.3), the Hamiltonian takes the form [cf. (I.7.7)]

$$H=H_a+H_b+H_c+H_d+H_e+H_f+H_{\text{em}}\ , \tag{4.23}$$

with

$$H_f=\lambda\frac{q^2}{m^2}\int\frac{d^3r}{r^4}[|\boldsymbol{\phi}|^2-(\boldsymbol{\phi}\cdot\hat{\mathbf{r}})^2]\ , \tag{4.24}$$

and the other quantities given in Sec. VII of paper I. Here, H_b and H_c are given by integrals that involve ϕ^0. For the present interaction, ϕ^0 can be determined from Eq. (2.7) as [cf. Eq. (I.7.9)]

$$\phi^0=-\frac{i}{E}\left[(\boldsymbol{\nabla}-ie\,\mathbf{A})\cdot\boldsymbol{\phi}-\frac{ie(1-\kappa)}{m^2}\mathbf{B}\cdot[(\boldsymbol{\nabla}-ie\,\mathbf{A})\times\boldsymbol{\phi}]+\frac{\lambda}{m^2}\left[\frac{e}{m}\right]^2\{(\boldsymbol{\nabla}-ie\,\mathbf{A})\cdot(\mathbf{B}^2\boldsymbol{\phi})-(\boldsymbol{\nabla}-ie\,\mathbf{A})\cdot[(\mathbf{B}\cdot\boldsymbol{\phi})\mathbf{B}]\}\right]\ . \tag{4.25}$$

At short distances, the term proportional to λ is the most singular one. Indeed, using the results of Secs. IV and V of paper I, the last term can be written as

$$(\nabla - ie\,\mathbf{A})\cdot(\mathbf{B}^2\boldsymbol{\phi})-(\nabla - ie\,\mathbf{A})\cdot[(\mathbf{B}\cdot\boldsymbol{\phi})\mathbf{B}] = -i\sum_l r^{-5} f_{jl}(r)\left[D_{jl}-\sum_L D_{jL}(C_{jL}^{(q)})^2\right] C_{jl}^{(q)} Y_{jj_z}^{(q)}(\hat{\mathbf{r}}) \ . \tag{4.26}$$

With the radial functions behaving as

$$f(r)\sim r^p, \quad g(r)\sim r^p, \quad h(r)\sim r^p \quad \text{as } r\to 0 \ , \tag{I.7.27}$$

it follows from (4.25) and (4.26) that

$$\phi^0 \underset{r\to 0}{\sim} r^{p-5} \ . \tag{4.27}$$

As compared with the previous case of no λ term, ϕ^0 is now more singular as $r\to 0$, by a factor $\sim r^{-2}$ (because of the extra magnetic-field factor).

Requiring the various contributions to the Hamiltonian, $H_a, \ldots, H_f$ to be finite, one is led to conditions on p. As was the case for the interaction studied in paper I, we find the most restrictive term to be H_c, Eq. (I.7.8). By simple power counting, we obtain the condition

$$\operatorname{Re} p > \tfrac{9}{2} \ , \tag{4.28}$$

as compared with Re $p > \frac{1}{2}$ for the interaction of paper I. This difference of four units in the power p is due to the fact that ϕ^0 now is more singular, by two units in p.

By Eqs. (I.7.27) and (I.5.26),

$$F(r)\sim r^{p+1}, \quad G(r)\sim r^{p+1}, \quad H(r)\sim r^{p+1} \quad \text{as } r\to 0 \ . \tag{I.7.29}$$

Invoking Eq. (4.22), we convert (4.28) to a condition on ν:

$$-\nu > \tfrac{19}{4} \ . \tag{4.29}$$

In (4.18) and (4.20), only the upper-sign solutions ν_1 and ν_3 are of interest. Furthermore, since $|\nu_1|>|\nu_3|$, the most restrictive one is ν_3. For this case, Eq. (4.29) is satisfied when

$$\sqrt{1+4[j(j+1)-q^2]} > 10 \ . \tag{4.30}$$

Thus, for $\lambda > 0$ and with Eq. (4.30) satisfied, we have three linearly independent solutions that are acceptable as $r\to 0$. Hence, type-A bound states exist.

V. BEHAVIOR OF THE TYPE-B WAVE FUNCTIONS AT THE ORIGIN

For type-B states, $F=0$, and

$$j=|q|, \quad \lambda(j)=0 \ . \tag{5.1}$$

We therefore only need to consider Eqs. (4.4) and (4.5). Equivalently, $\sigma(j)=0$, and Eq. (4.9) becomes trivial. The remaining equation, (4.10), takes the form

$$\left[\left[-4x^2\frac{d^2}{dx^2}-6x\frac{d}{dx}+|q|\right]\left[-4x^2\frac{d^2}{dx^2}-6x\frac{d}{dx}+2|q|+2-|q|\kappa+(1-\kappa)q^2x+\lambda q^2x+\lambda|q|^3x^2\right] -2(1+\kappa)|q|+2\lambda q^2x-2(1-\kappa)q^2x\left[2x\frac{d}{dx}+3\right]-2\lambda|q|^3x^2\left[2x\frac{d}{dx}+5\right]\right]K=0 \ , \tag{5.2}$$

where we have used Eq. (4.6a) to obtain

$$M=|q|^3K \ . \tag{5.3}$$

The relation between H and K is

$$H=\frac{1}{2(|q|+1)\sqrt{2|q|+1}}\left[-4x^2\frac{d^2}{dx^2}-6x\frac{d}{dx}-\kappa|q| +(1-\kappa)q^2x+\lambda q^2x +\lambda|q|^3x^2\right]K \ . \tag{5.4}$$

In analogy with the type-A wave functions analyzed in Sec. IV, we find two exponential solutions

$$K \underset{x\to\infty}{\sim} \exp(\mp\tfrac{1}{2}|q|\sqrt{\lambda|q|}\,x)x^p \ , \tag{5.5}$$

where for $\lambda > 0$ the upper sign gives acceptable solutions.

Furthermore, there are two solutions whose leading types of behavior are powerlike:

$$K \underset{x\to\infty}{\sim} x^{\nu} \ , \tag{5.6}$$

with

$$\nu_{\substack{1\\2}} = -\tfrac{1}{4}(11\pm\sqrt{1+4|q|}) \ . \tag{5.7}$$

These correspond to the solutions (4.18) of Sec. IV, with $j=|q|$.

Here, we have

$$H \underset{x\to\infty}{\sim} x^2K \underset{r\to 0}{\sim} (mr)^{-2\nu-4} \ . \tag{5.8}$$

By Eqs. (4.29) and (5.7), we must require

$$|q|\geq 16 \ , \tag{5.9}$$

in order to have an acceptable short-distance behavior.

With two linearly independent, physically acceptable solutions at both large and small values of r, a matching is possible, and therefore type-B bound states exist.

VI. TYPE-C WAVE FUNCTIONS

For type-C states, $F=0, G=0$,

$$j=|q|-1, \quad \lambda(j+1)=0 \ , \tag{6.1}$$

and Eqs. (3.19) and (3.20) are trivially satisfied, whereas (3.21) takes the form

$$\left[r^2(m^2-E^2)-r^2\frac{d^2}{dr^2}-|q|\kappa+\lambda\frac{q^2}{(mr)^2}\right]H(r)=0 \ . \tag{6.2}$$

This equation is valid at *all* distances. We note that we must require

$$\lambda>0 \ , \tag{6.3}$$

in order to have solutions that vanish at the origin.

Because of (6.1), the most singular terms of (3.21) vanish identically. This great simplification is connected to the fact that the wave function for $j=|q|-1$ is transversal [see Eqs. (I.3.14), (I.4.1), and (I.4.2)],

$$\mathbf{r}\cdot\mathbf{Y}^{(q)}_{jj+1j_z}(\hat{\mathbf{r}})=0 \ . \tag{6.4}$$

It is instructive to simplify Eq. (6.2) further. Let us define, as an eigenvalue parameter,

$$\epsilon\equiv\frac{m^2-E^2}{2m^2}\underset{\epsilon\ll 1}{\simeq}\frac{m-E}{m} \ . \tag{6.5}$$

Different (discrete) eigenvalues, corresponding to the energies E_n will be labeled ϵ_n. Since $E_n^2>0$, it follows that

$$0<\epsilon_n<\tfrac{1}{2} \ . \tag{6.6}$$

Introducing then a new variable

$$z=mr\sqrt{2\epsilon} \ , \tag{6.7}$$

we can write Eq. (6.2) as

$$\left[\frac{d^2}{dz^2}-1+\frac{|q|\kappa}{z^2}-\frac{b}{z^4}\right]H=0 \ , \tag{6.8}$$

with

$$b\equiv 2\lambda\epsilon q^2 \ . \tag{6.9}$$

Hence, after this rescaling of the variable, the eigenvalue equation is written in terms of *two* parameters $|q|\kappa$ and b, one of which represents the eigenvalue. Thus, *the eigenvalue ϵ_n is inversely proportional to* λ. This is an *exact* result. For any $|q|$ and κ, it suffices to determine ϵ_n for one value λ_0. For other values λ it can be obtained by the scaling relation

$$\epsilon_n(\lambda)=\frac{\lambda_0}{\lambda}\epsilon_n(\lambda_0) \ . \tag{6.10}$$

Furthermore, the eigenfunctions $H(z)$ form a one-parameter family. An eigenfunction determined for one value of λ is therefore an eigenfunction for any λ, with appropriate rescaling of ϵ_n (keeping n fixed).

Equation (6.8) has an inversion symmetry. If we take out a factor

$$H\equiv\sqrt{z}\,\tilde{h}(z) \ , \tag{6.11}$$

then $\tilde{h}(z)$ is found to obey the equation

$$\left[\frac{d^2}{dz^2}+\frac{1}{z}\frac{d}{dz}-1-\frac{\nu^2}{z^2}-\frac{b}{z^4}\right]\tilde{h}(z)=0 \ , \tag{6.12}$$

where

$$\nu^2=\tfrac{1}{4}-|q|\kappa \ . \tag{6.13}$$

This equation is "inversion symmetric," in the sense that $\tilde{h}(z)$ and $\tilde{h}(\sqrt{b}/z)$ both satisfy this same equation. Since the boundary conditions also are consistent with this symmetry, the eigenfunctions must satisfy

$$\tilde{h}(z)=\pm\tilde{h}(\sqrt{b}/z) \ . \tag{6.14}$$

Because of (6.6), there is a lower value $\lambda_{\min}$, beyond which the interaction (10.1) does not yield bound states of type C. At this value, which is determined implicitly by the relation

$$\epsilon_n(\lambda_{\min})=\tfrac{1}{2} \ , \tag{6.15}$$

the energy vanishes:

$$E_n=0 \ ; \tag{6.16}$$

i.e., the binding energy equals the mass. This lower value depends on n.

VII. APPROXIMATE SOLUTIONS OF THE TYPE-C RADIAL EQUATION

Accurate approximate solutions to Eq. (6.12) can be found analytically. At large distances, for

$$z\gg\max\left[b^{1/4},\frac{\sqrt{b}}{|\nu|}\right] \ , \tag{7.1}$$

the last term (involving λ) can be neglected, and the solution is a modified Bessel function,

$$\tilde{h}(z)=\text{const}\times K_\nu(z) \ , \tag{7.2}$$

with ν given by Eq. (6.13). This is actually the solution given in Sec. VI of paper I (for the case of $\lambda=0$).

We thus have the solutions

$$H=C_{\text{ext}}\sqrt{z}\,K_\nu(z) \tag{7.3}$$

for large z and

$$H=C_{\text{int}}\sqrt{z}\,K_\nu(\sqrt{b}/z) \tag{7.4}$$

for small z.

The necessary condition that these solutions overlap is

TABLE I. Binding-energy eigenvalues $\lambda\epsilon$ for type-C states, where $\epsilon=(m^2-E^2)/(2m^2)$ for $|q|=1$ and $j=0$. Values larger than $\frac{1}{2}$ are only relevant for $\lambda>1$.

n \ κ	1	2	5	10
3	6.831×10^{-10} [a]	2.573×10^{-6} [a]	2.282×10^{-3}	8.769×10^{-2}
2	9.669×10^{-7} [a]	2.972×10^{-4}	4.058×10^{-2}	6.472×10^{-1}
1	1.363×10^{-3}	3.369×10^{-2}	6.919×10^{-1}	4.921[a]

[a]Approximate value given by Eq. (7.12).

that the term b/z^4 remains small at the "inversion point":

$$z_0=b^{1/4}\,, \tag{7.5}$$

i.e., that

$$b \ll (|q|\kappa-\tfrac{1}{4})^2 \sim q^2\kappa^2\,, \tag{7.6}$$

or

$$\epsilon_n\lambda \ll \tfrac{1}{2}\kappa^2\,. \tag{7.7}$$

There exists a range in which both solutions (7.3) and (7.4) are valid, provided Eq. (7.7) holds. We determine the eigenvalue from a matching of powers in this region.

For the modified Bessel functions we have[4]

$$K_\nu(z)\simeq\frac{\pi}{2\sin(\pi\nu)}\left[\frac{(\frac{1}{2}z)^{-\nu}}{\Gamma(1-\nu)}-\frac{(\frac{1}{2}z)^{\nu}}{\Gamma(1+\nu)}\right], \tag{7.8}$$

valid for $|z|\ll 1$. If we substitute

$$\nu=i\beta=i(|q|\kappa-\tfrac{1}{4})^{1/2}\,, \tag{7.9}$$

and match powers of (7.3) and (7.4) according to (7.8), we obtain

$$\epsilon_n^{i\beta}=e^{4i\phi}\left[\frac{\lambda q^2}{8}\right]^{-i\beta}, \tag{7.10}$$

where we have defined[5]

$$e^{2i\phi}=\frac{\Gamma(1+i\beta)}{\Gamma(1-i\beta)}\,. \tag{7.11}$$

The binding energies are thus given by

$$\epsilon_n=\frac{8}{\lambda q^2}\exp\left[-\frac{2}{\beta}(n\pi-2\phi)\right], \tag{7.12}$$

for $n=1,2,3,\ldots$. [It turns out that Eq. (6.12) has no eigenvalue corresponding to $n=0$.]

We show in Tables I and II the values of ϵ_n for the three lowest-energy states, for a few values of κ, and $|q|=1$ and $\frac{3}{2}$, respectively. Here we have taken $\lambda=1$. The variation of the binding energy with κ is reminiscent of that found for the case of spin $\frac{1}{2}$.[1,5,6] The values quoted in Tables I and II are exact. In this range of κ, those given by Eq. (7.12) are accurate to within 2%.

In Fig. 1 we show plots of the wave function. The general shape of these type-C wave functions is similar to that found for the case of spin $\frac{1}{2}$,[5,6] except that those states are described by two or four radial functions, as opposed to one for the present type-C states.

In order to determine the range of validity of this result (7.12), we note that the exterior solution, (7.3), can be approximated by the leading powers for

$$z\ll 1+|\nu|\,. \tag{7.13}$$

This must hold at the inversion point, and thus we must require that [see Eqs. (7.5) and (6.13)]

$$2\epsilon_n\lambda q^2\ll 1+q^2\kappa^2\,. \tag{7.14}$$

The formula (7.12) is therefore valid provided only that the product $\epsilon_n\lambda$ is small, ϵ_n might be of order unity. For $|q|\kappa>1$, this condition (7.14) reduces to (7.7)

We can use the approximate energies (7.12) to estimate the lowest value of λ at which the Lagrangian (2.2) produces bound states according to (6.15):

$$\lambda_{\min}\simeq\frac{16}{q^2}\exp\left[-\frac{2}{\beta}(n\pi-2\phi)\right]. \tag{7.15}$$

VIII. SUMMARY

The magnetic-moment interaction that for spin-$\frac{1}{2}$ particles leads to binding with magnetic monopoles,[3] was in the case of spin-1 particles found to lead to unsatisfactory behavior at short distances.[1] In the present paper, we

TABLE II. Binding-energy eigenvalues $\lambda\epsilon$ for type-C states, where $\epsilon=(m^2-E^2)/(2m^2)$ for $|q|=\frac{3}{2}$ and $j=\frac{1}{2}$. Values larger than $\frac{1}{2}$ are only relevant for $\lambda>1$.

n \ κ	1	2	5	10
3	6.110×10^{-8} [a]	3.352×10^{-5}	9.553×10^{-3}	2.302×10^{-1}
2	1.685×10^{-5}	1.480×10^{-3}	9.761×10^{-2}	1.190[a]
1	4.595×10^{-3}	6.338×10^{-2}	1.018[a]	6.113[a]

[a]Approximate value given by Eq. (7.12).

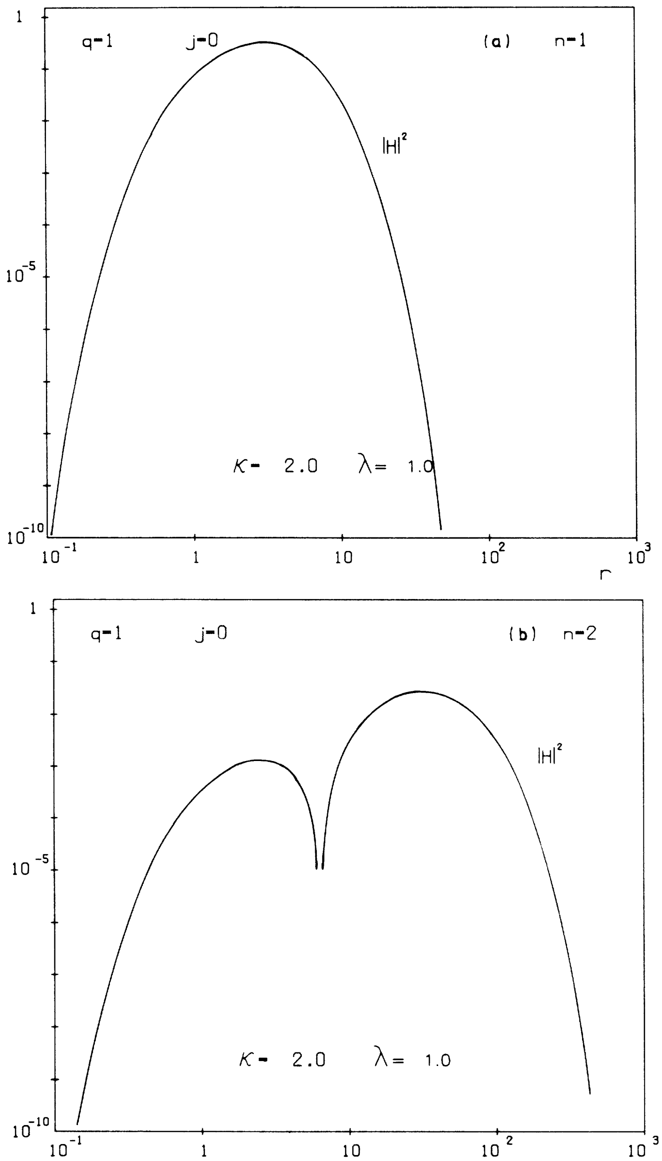

FIG. 1. Squares of the wave function for type-C bound states for $q=1$, $j=0$, $\kappa=2$, $\lambda=1$, and $m=1$: (a) $n=1$; (b) $n=2$. The corresponding binding energies are given in Table I.

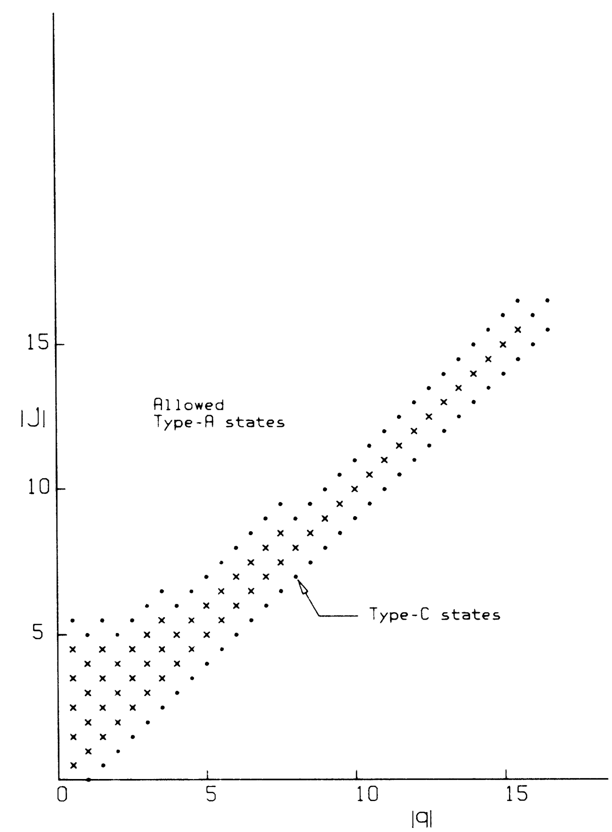

FIG. 2. Angular momentum states for which the interaction (2.2) leads to binding are indicated by filled circles. Some type-B and type-A states are not allowed (crosses). All states above the crosses are allowed.

have shown, as an example, how an additional induced-magnetization interaction can cure the divergent behavior at short distances, and lead to bound states.

For the case of lowest angular momentum, type-C states ($j=|q|-1$), binding energies and wave functions have been explicitly determined. These states exist *for all values of* q, and for all κ and λ such that [cf. Eqs. (7.9) and (7.15)]

$$|q|\kappa > \tfrac{1}{4}, \quad \lambda > \lambda_{\min} > 0 . \tag{8.1}$$

While the ratios of binding energies are given by the large-distance behavior of the interaction, in the familiar fashion,[3,5,6] the overall scale is found to depend crucially on the strength of the extra interaction term.

For the states of higher angular momentum, the situation is more complex, in the sense that the present interaction only leads to bound states when [cf. Eqs. (5.9) and (4.30)]

$$\text{for type-B states } (j=|q|):\quad |q| \geq 16 , \tag{8.2}$$

and

$$\text{for type-A states } (j \geq |q|+1):\quad j(j+1)-q^2 \geq 25 . \tag{8.3}$$

There are thus angular momentum states for which the present interaction cannot lead to radial wave functions whose short-range behavior is acceptable. They are indicated by crosses in Fig. 2. For those states, whether or not binding occurs, depends on interactions other than those analyzed in the present paper.

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