Dynamical mass generation in three-dimensional QED: Improved vertex function

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We study chiral-symmetry breaking for three-dimensional QED with N fermion flavors, just above the critical threshold. By analysis of a consistently truncated Schwinger-Dyson system for the fermion propagator and the fermion-boson vertex, we argue that the critical coupling must be strictly positive.

I. INTRODUCTION

In the context of current grand unified theories, elementary particles acquire masses that are much smaller than the unification scale. Three-dimensional QED (QED_3) with N fermion generations, a natural model for studying this hierarchy problem,[1] has received a great deal of attention in recent years.[2–10] The dimensional coupling strength sets the dynamical mass scale, and the theory is superrenormalizable. Perturbative expansions are not encumbered by ultraviolet divergences; however, in the massless theory the infrared behavior of the terms in the perturbative expansion steadily deteriorates. The remedy to this condition is to soften the free photon propagator through vacuum-polarization insertions, and to make an expansion in terms of the dimensionless parameter $1/N$. The Green's functions remain finite to every order in this $1/N$ expansion.[3]

The issue is whether the $1/N$ analysis is valid in a nonperturbative sense. In particular, is chiral symmetry nonperturbatively and dynamically broken for an arbitrarily large number of fermion species? Dynamical mass generation is a manifestation of a "strong coupling" regime, and accordingly it might not occur at large N.[3] The treatment of the infrared sector in QED_3 is an extremely delicate matter.[4–10] By contrast, in the corresponding supersymmetric version of QED_3, cancellations soften the infrared behavior of the fermion propagator, and fermions remain massless when there are more than $N=3$ (two-component) generations.[11] Some recent lattice simulations indicate that chiral symmetry is broken in QED_3 only for $N=1$, 2, or 3 fermion species.[12,13]

In the present paper, we examine chiral-symmetry breaking in QED_3 by studying truncated Schwinger-Dyson equations,[2–9] a standard approach.[14–18] In that context, using a bare vertex and neglecting wave-function renormalization, Appelquist *et al.*[3,7] showed that chiral symmetry was unbroken for $N>3$. It was shown subsequently that, if the vertex stays bare while wave-function renormalization is included, chiral symmetry is broken for arbitrary N.[4–6] Recently, Nash[8] established that chiral symmetry remains unbroken for $N>3$ when $O(1/N)$ improvements are made in both the vertex and wave-function renormalization. In the current study both the vertex and wave-function renormalization are treated nonperturbatively (but not exactly). In Sec. II the equation for the mass function and wave-function renormalization are studied and corrections are made, based on an expansion of the full vertex. In Sec. III a requirement of consistency is discussed: three different correction functions are examined to assess the impact of this requirement. In Sec. IV the Schwinger-Dyson equations for the three cases are analyzed and solved. In Sec. V the results are discussed and conclusions drawn. In particular, evidence is presented that chiral symmetry remains unbroken at large N.

II. VERTEX FUNCTION

Let us consider the case of N fermion flavors and investigate dynamical generation of fermion masses nonperturbatively by studying the Schwinger-Dyson equation for the fermion propagator. At Euclidean momenta, the propagator may be written as

$$S(p)=1/[-\not p\beta(p)+\Sigma(p)]\ , \tag{2.1}$$

with the dynamical mass function

$$m(p)=\Sigma(p)/\beta(p)\ . \tag{2.2}$$

We work in the Landau gauge, and take into account the one-loop fermion contribution to the vacuum polarization by making the following replacement in the free photon propagator:

$$\frac{1}{q^2}\to\frac{1}{q^2[1+\Pi(q)]}\to\frac{1}{q(q+\tilde\alpha)}\ , \tag{2.3}$$

where the dimensionful coupling

$$\tilde\alpha=\frac{e^2N}{8} \tag{2.4}$$

is the natural momentum state of the theory. The propagator is therefore

$$D_{\mu\nu}(q)=\frac{1}{q(q+\tilde\alpha)}L_{\mu\nu}\ , \tag{2.5}$$

with

$$L_{\mu\nu}(q)=\delta_{\mu\nu}-\frac{q_\mu q_\nu}{q^2}\ . \tag{2.6}$$

The Schwinger-Dyson equation for the fermion propagator is expressed in terms of the fermion-boson vertex function $\Gamma_\nu(p,p')$, where the photon momentum is related to the fermion momenta by $q=p-p'$. The longitudinal vertex is specified in terms of the fermion propagator via the Ward-Takahashi identity:

$$\Gamma^L_\mu(p,p')\equiv\frac{q_\mu q_\nu}{q^2}\Gamma_\nu(p,p') = \frac{q_\mu}{q^2}[S^{-1}(p')-S^{-1}(p)] . \tag{2.7}$$

Actually, in the Landau gauge only the "transverse" projection of the vertex into the plane perpendicular to q,

$$\Gamma^T_\mu(p,p')\equiv L_{\mu\nu}(q)\Gamma_\nu(p,p') , \tag{2.8}$$

contributes to the Schwinger-Dyson equation for the fermion propagator. In turn, the transverse vertex should be determined by study of its Schwinger-Dyson equation.[19] A phenomenological transverse contribution,

$$\Gamma^T_\mu(p,p')=L_{\mu\nu}\gamma_\nu f(p,p') , \tag{2.9}$$

with the scalar function f yet to be determined, is a naive approximation, and we will see later that it is misleading beyond lowest order.

With these choices, the Schwinger-Dyson equation for the fermion propagator reduces to the following coupled system of integral equations:

$$\beta(p)m(p)=\frac{2}{\pi^3N}\int d^3p'\frac{m(p')}{p'^2+m^2(p')}\frac{\tilde\alpha}{q(q+\tilde\alpha)}\frac{f(p,p')}{\beta(p')} , \tag{2.10}$$

$$\beta(p)=1+\frac{2}{\pi^3N}\int d^3p'\frac{(p\cdot q)(p'\cdot q)}{p'^2+m^2(p')}\frac{\tilde\alpha}{q(q+\tilde\alpha)}\frac{\tilde f(p,p')}{\beta(p')} , \tag{2.11}$$

where in this naive approximation, $\tilde f=f$.

The dimensionful coupling $\tilde\alpha$ is actually quite large in comparison with the momentum scale relevant for chiral-symmetry breaking. Accordingly, we cut the integrals off at $p'=\tilde\alpha$, make the replacement[3]

$$\frac{\tilde\alpha}{q+\tilde\alpha}\to 1 \tag{2.12}$$

in the integrals, and integrate over directions to obtain the simplified system

$$m(p)=\lambda\int_0^{\tilde\alpha}dp'\frac{p'^2m(p')}{p'^2+m^2(p')}\frac{1}{p_{\max}}\frac{f(p,p')}{\beta(p)\beta(p')} , \tag{2.13}$$

$$\beta(p)=1-\frac{\lambda}{3}\int_0^{\tilde\alpha}dp'\frac{p'}{p'^2+m^2(p')}\left[\frac{p_{\min}}{p}\right]^3\frac{f(p,p')}{\beta(p')} , \tag{2.14}$$

where $p_{\min}=\min(p,p')$, $p_{\max}=\max(p,p')$, and

$$\lambda=\frac{8}{\pi^2N} . \tag{2.15}$$

In Ref. 3, the vertex was taken to be free ($f\equiv1$), wave-function renormalization was neglected [$\beta\equiv1$ in Eq. (2.13); Eq. (2.14) ignored], and the resulting Schwinger-Dyson equation was shown to possess only the trivial solution for

$$\lambda<\lambda_c=\tfrac{1}{4} , \tag{2.16}$$

so that chiral symmetry was not broken dynamically for more than three fermion flavors. The replacements $f\equiv1$ and $\beta\equiv1$ were justified as being correct to leading order in the $1/N$ (or λ) expansion. However, such expansions of f and β were criticized as being unreliable in the infrared region that is responsible for chiral-symmetry breaking.[4-6]

One can obtain the first-order term in the β expansion by inserting $f=1$ and $\beta=1$ in the right side of (2.14), making the replacement $m(p')\to m(0)$, and taking p small:

$$\beta(p)\approx1+\frac{\lambda}{6}\ln[p^2+m^2(0)] . \tag{2.17}$$

The corresponding contribution to the function $f(p,p')$ can be determined by making a consistent expansion in the coupled system of Schwinger-Dyson equations for the fermion propagator and the fermion-boson vertex function, keeping all terms linear in λ and the mass function. One must truncate the coupled system by some procedure in which all such terms are included, as emphasized in Ref. 8. After such a truncation, the linearized equation for the fermion mass function has the structure (2.13), but the vertex function itself has a rather complicated tensor structure, and cannot be written in the simple form (2.9). However, such a complicated form does give rise to equations of the form (2.10) and (2.11), where in general $f\neq\tilde f$. In lowest nontrivial order these effective functions are equal,[9] but they are no longer derived from the simple-minded ansatz (2.9); and therefore they are not simply constrained by the Ward identifies. The "leading logarithm" result for vectors p and p' parallel is[9]

$$f(p,p')\simeq1+\frac{\lambda}{3}\frac{1}{p-p'}\{p\ln[p^2+m^2(0)] - p'\ln[p'^2+m^2(0)]\} . \tag{2.18}$$

The simplified form

$$f(p,p')\approx1+\frac{\lambda}{3}\ln[p^2_{\max}+m^2(0)] \tag{2.19}$$

adequately characterizes the function at infrared momenta. The effective vertex function f must be chosen in accordance with (2.19), to maintain consistency in the overall system. We discuss these matters more fully in Sec. IV.

III. ANOMALOUS DIMENSION

Let us start with the case without chiral-symmetry breaking. To first order in λ, we have the following behavior for the wave-function renormalization

$$\beta(p)=1+\frac{\lambda}{3}(\ln p+\text{ regular terms})+O(\lambda^2) . \tag{3.1}$$

This can be interpreted in terms of an anomalous dimension:[10]

$$\beta(p)=p^{\eta} \tag{3.2}$$

for small momenta p. There are of course corrections to this behavior, which are regular, and of order λ. To first order in λ, we have for the anomalous dimension η:

$$\eta=\frac{\lambda}{3} . \tag{3.3}$$

By leaving out the mass term in Eq. (2.14), we have the following simplified equation for the β function:

$$\beta(p)=1-\frac{\lambda}{3}\int_0^{\bar{\alpha}} dp'\frac{1}{p'}\left[\frac{p_{\min}}{p}\right]^3\frac{f(p,p')}{\beta(p')} . \tag{3.4}$$

If we suppose that there is an anomalous dimension, so that $\beta(p)$ behaves like (3.2), then we get, for small momenta,

$$p^{\eta}\simeq 1-\frac{\lambda}{3}\int_0^{\bar{\alpha}} dp'\,p'^{-1-\lambda/3}f(p,p') . \tag{3.5}$$

This equation leads to the conclusion that, to leading order,

$$f(p,p')\simeq(p_{\max})^{2\eta} , \tag{3.6}$$

with η as in (3.3).

Let us return to the case in which there is dynamical mass generation, and suppose that there is an anomalous dimension $\lambda/3$ for small momenta p:

$$S^{-1}(ap)=a^{1+\lambda/3}S^{-1}(p) . \tag{3.7}$$

This leads to the following set of equations:

$$\beta(ap)=a^{\lambda/3}\beta(p) , \tag{3.8}$$

$$m(ap)=am(p) . \tag{3.9}$$

Therefore, there is no anomaly in the dynamical mass: the mass function $m(p)$ has the naive dimension $+1$. The wave-function renormalization does have an anomalous dimension $\lambda/3$. This can be achieved by a behavior like (3.2), but a better expression for the leading-order behavior is

$$\beta(p)=(p^2+m^2)^{\lambda/6} . \tag{3.10}$$

This result is also in perfect agreement with the first-order result, Eq. (2.17).

This has important consequences for the effective correction function $f(p,p')$ in the set of Eqs. (2.13) and (2.14). First of all, after a study of the dimensions, we conclude that the effective correction function must have twice the anomalous dimension of the wave-function renormalization. From Eq. (2.14) we can learn something more: this equation is consistent if the effective correction function behaves like

$$f(p,p')=(p_{\max}^2+m^2)^{\lambda/3} . \tag{3.11}$$

Therefore, both in the case that there is no dynamical mass and in the case that there is dynamical mass generation, the effective correction function has to be quadratic in β.

In the next section, we shall consider the following three choices for the effective correction function:

$$f_1(p,p')=\beta(p_{\max}) , \tag{3.12}$$

$$f_2(p,p')=\beta(p)\beta(p') , \tag{3.13}$$

$$f_3(p,p')=[\beta(p_{\max})]^2 . \tag{3.14}$$

The choice (3.12) is inspired by the simple ansatz (2.9), according to which one would expect $f(p,p')$ to be constrained by the Ward identity and thus to be linear in β. However, as we remarked in Sec. II, $f(p,p')$ should rather be seen as an effective correction function that arises from a much more complicated vertex function. In view of the anomalous dimension, $f(p,p')$ should actually be proportional to β^2, rather than β. The choices (3.13) and (3.14) are quadratic in β; the former is easier to analyze, while the latter is chosen to be consistent with (2.19) and (3.11).

IV. ANALYSIS OF THREE CASES

We begin with the effective correction $\beta(p_{\max})$, given in (3.12), which is very similar in character to the case

$$f(p,p')=\beta(p') \tag{4.1}$$

that has been discussed extensively.[4-6] Our choice (3.12) represents a minor refinement of (4.1), in that f should be a symmetric function of p and p'. Let us consider solutions to the integral equations (2.13) and (2.14) at a particular coupling λ with f given by (3.12). These functions must satisfy the following differential equations in the variable $x=p^2$:

$$x\beta''(x)=[2B(x)-\tfrac{5}{2}]\beta'(x)+\frac{\lambda}{9}\frac{1}{x+m^2(x)}[B(x)-\tfrac{3}{2}]^2 \tag{4.2}$$

and

$$x\Sigma''(x)=\frac{C(x)-B(x)+\frac{3}{4}}{B(x)-\frac{1}{2}}\Sigma'(x)+\frac{\lambda}{2}\frac{m(x)}{x+m^2(x)}[B(x)-\tfrac{1}{2}] , \tag{4.3}$$

where we have defined

$$\Sigma(x)=\beta(x)m(x) , \tag{4.4}$$

$$B(x)=\frac{x\beta'(x)}{\beta(x)} , \tag{4.5}$$

$$C(x)=\frac{x^2\beta''(x)}{\beta(x)} . \tag{4.6}$$

In addition, these functions must have infrared limits $m(0)$ and $\beta(0)$ and satisfy the ultraviolet boundary conditions

$$[2xm'(x)+m(x)]|_{\tilde{\alpha}^2}=0 \ , \tag{4.7}$$

$$x\beta'(x)-\tfrac{3}{2}\beta(x)[1-\beta(x)]|_{\tilde{\alpha}^2}=0 \ . \tag{4.8}$$

By integrating the nonlinear differential equations (4.2) and (4.3), starting with particular infrared limits, and by varying those parameters until the solutions satisfy (4.7) and (4.8) (to within a relative accuracy of 10^{-8}) we obtain reliable solutions to the original integral equations.[17] The values of the dimensionless parameters $m(0)/\tilde{\alpha}$ and $\beta(0)$ at various couplings λ, as well as the dimensionless chiral condensate

$$\chi=\frac{1}{2\pi\tilde{\alpha}^2}\int d^3p\frac{m(p)}{p^2+m^2(p)}=\frac{1}{\tilde{\alpha}^2}\int_0^{\tilde{\alpha}^2}dx\frac{xm(x)}{x+m^2(x)} \ , \tag{4.9}$$

are presented in Table I.

A nontrivial solution is found down to $\lambda=0.09$, or $N\approx 9$ flavors. The mass function $m(x)$ decreases monotonically whereas the wave-function renormalization $\beta(x)$ increases. Because of a slightly larger correction function [i.e., $\beta(p_{\max})\geq\beta(p')$], the infrared limits $m(0)$ and $\beta(0)$ are somewhat greater than those obtained with (4.1) in Ref. 4; although they are in close agreement at small λ. These results, as well as those for the correction (4.1), are consistent at small λ with the formula

$$\frac{m(0)}{\tilde{\alpha}}\sim e^{-3/\lambda} \ . \tag{4.10}$$

In Fig. 1 a plot of $\ln\tilde{\alpha}/m(0)$ versus $3/\lambda$ is given. The chiral condensate χ decreases by 15 orders of magnitude as λ decreases from 0.30 ($N\sim 2.7$) to 0.09 ($N\sim 9$), and lattice simulations with $N>3$ flavors would be unable to detect such a feeble breaking of chiral symmetry. In going from $N=3$ to $N=4$, the chiral condensate decreases by a factor of 100, and even that is difficult to detect in practice. Finally, we note that at small λ the infrared wave function is fairly close to the perturbative value in (2.17):

$$\beta(0)\sim 1+\frac{\lambda}{3}\ln\frac{m(0)}{\tilde{\alpha}} \ . \tag{4.11}$$

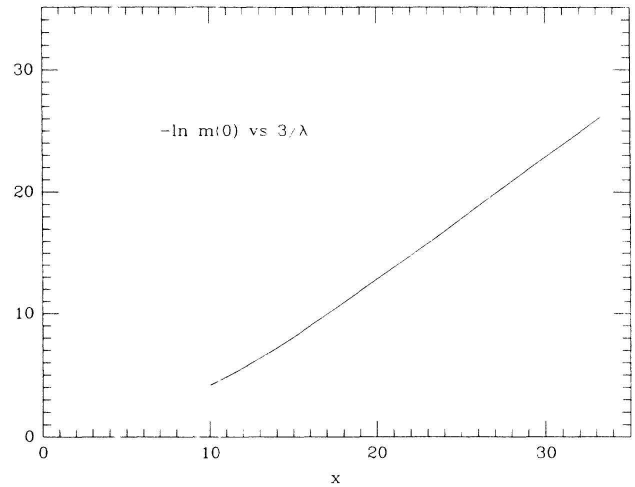

FIG. 1. The function $-\ln m(0)/\tilde{\alpha}$ is shown vs $3/\lambda$, for data corresponding to correction function (3.12). The curve is linear, with slope near 1.

Next we consider the effective correction (3.13), for which the integral equations become

$$m(x)=\frac{\lambda}{2}\int_0^{\tilde{\alpha}^2}dy\frac{m(y)}{y+m^2(y)}\left[\frac{y}{x_{\max}}\right]^{1/2} \tag{4.12}$$

and

$$\frac{1}{\beta(x)}=1+\frac{\lambda}{6}\int_0^{\tilde{\alpha}^2}dy\frac{1}{y+m^2(y)}\left[\frac{x_{\min}}{x}\right]^{3/2} \ . \tag{4.13}$$

This is a "decoupled" system, in that one can first solve (4.12) for $m(x)$, and then calculate $\beta(x)$ from $m(x)$ in (4.13). Equation (4.12) is identical with that considered previously by Appelquist;[3] it has nontrivial solutions for $\lambda>\lambda_c=0.25$. The values of $m(0)$, $\beta(0)$, and condensate χ for various couplings are presented in Table II. The in-

TABLE I. The infrared mass $m(0)$ (in units of $\tilde{\alpha}$), the infrared wave-function renormalization $\beta(0)$, and the dimensionless chiral condensate χ are tabulated for a range of couplings λ. These numbers are determined by solution of the system (4.2)–(4.8), corresponding to the effective correction function (3.12).

λ	$m(0)/\tilde{\alpha}$	$\beta(0)$	χ
0.30	0.1554×10^{-1}	0.5693	0.577×10^{-2}
0.28	0.9676×10^{-2}	0.5542	0.288×10^{-2}
0.25	0.3837×10^{-2}	0.5254	0.711×10^{-3}
0.22	0.1037×10^{-2}	0.4868	0.917×10^{-4}
0.20	0.3156×10^{-3}	0.4544	0.134×10^{-4}
0.18	0.6723×10^{-4}	0.4163	0.103×10^{-5}
0.16	0.8926×10^{-5}	0.3735	0.329×10^{-7}
0.14	0.6271×10^{-6}	0.3279	0.312×10^{-9}
0.12	0.1772×10^{-7}	0.2812	0.514×10^{-12}
0.11	0.1826×10^{-8}	0.2578	0.813×10^{-14}
0.10	0.1192×10^{-9}	0.2344	0.611×10^{-16}
0.09	0.4270×10^{-11}	0.2110	0.642×10^{-17}

TABLE II. The infrared limits $m(0)$ (in units of $\tilde{\alpha}$) and $\beta(0)$ and chiral condensate χ for a range of λ. These numbers are determined by solution of (4.12) and (4.13), corresponding to the effective correction function (3.13).

λ	$m(0)/\tilde{\alpha}$	$\beta(0)$	χ
0.30	0.2739×10^{-4}	0.5024	0.968×10^{-6}
0.295	0.1305×10^{-4}	0.4886	0.324×10^{-6}
0.290	0.5419×10^{-5}	0.4731	0.886×10^{-7}
0.285	0.1869×10^{-5}	0.4555	0.183×10^{-7}
0.280	0.4965×10^{-6}	0.4351	0.256×10^{-8}
0.275	0.8976×10^{-7}	0.4112	0.201×10^{-9}
0.270	0.8784×10^{-8}	0.3824	0.627×10^{-11}
0.265	0.2888×10^{-9}	0.3465	0.382×10^{-13}
0.262	0.1418×10^{-10}	0.3197	0.421×10^{-15}
0.260	0.1718×10^{-11}	0.3035	0.206×10^{-16}

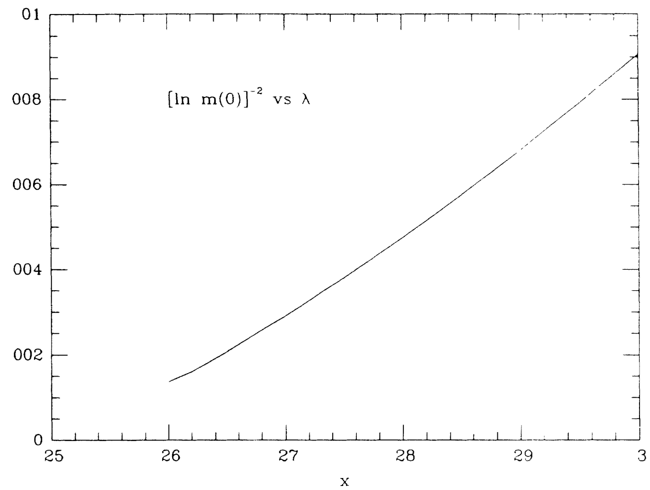

FIG. 2. The function $\ln^{-2}m(0)/\tilde{\alpha}$ is shown vs λ for data corresponding to correction function (3.13). This curve is roughly linear, with an intercept $\lambda_c = 0.25$.

frared mass function obeys the empirical relation

$$\frac{m(0)}{\tilde{\alpha}} \sim \exp\left[\frac{-K}{\sqrt{\lambda-\lambda_c}}\right] \tag{4.14}$$

for λ near λ_c, with $K \sim 2.7$. In Fig. 2 the function $\ln^{-2}m(0)/\tilde{\alpha}$ is plotted against λ. The value of $\beta(0)$ can be estimated by making the replacement $m(y) \to m(0)$ in the integrand of (4.13) to obtain

$$\frac{1}{\beta(0)} \simeq 1 - \frac{\lambda}{3}\ln\frac{m(0)}{\tilde{\alpha}} , \tag{4.15}$$

or

$$\beta(0) \simeq K'(\lambda-\lambda_c)^{1/2} \tag{4.16}$$

with $K' \sim 3.0$. In Fig. 3, $\beta^2(0)$ is plotted against λ. Relation (4.15) is different in character from the perturbative result (4.11) that applies to the first ansatz, because here the coupling parameter λ does not become small.

The third case, corresponding to the ansatz (3.14), will be considered next. The integral equations (2.13) and (2.14) are

$$m(x) = \frac{\lambda}{2}\int_0^{\tilde{\alpha}^2} dy \frac{m(y)}{y+m^2(y)} \left[\frac{y}{x_{\max}}\right]^{1/2} \frac{\beta(x_{\max})}{\beta(x_{\min})} \tag{4.17}$$

and

$$\beta(x) = 1 - \frac{\lambda}{6}\int_0^{\tilde{\alpha}^2} dy \frac{1}{y+m^2(y)} \left[\frac{x_{\min}}{y}\right]^{3/2} \frac{\beta^2(x_{\max})}{\beta(x_{\min})} . \tag{4.18}$$

They may be reduced to the following system of coupled, nonlinear differential equations:

$$x\beta'' = -[15 - 24B(x) + 8B^2(x)]\frac{\Sigma'(x)}{6} + \frac{\lambda}{36}\frac{\beta(x)}{x+m^2(x)}[4B(x)-3]^2 \tag{4.19}$$

and

$$x\Sigma''(x) = \frac{3 - 8B(x) + 8B^2(x) + 8C(x)}{4B(x)-1} + \frac{\lambda}{2}\frac{\Sigma(x)}{x+m^2(x)}[4B(x)-1] ; \tag{4.20}$$

cf. Eqs. (4.4)–(4.6). The ultraviolet conditions

$$m(x)\beta(x) + 2x[m'(x)\beta(x) - m(x)\beta'(x)]|_{\tilde{\alpha}^2} = 0 \tag{4.21}$$

and

$$x\beta'(x)[\beta(x)-2] + \tfrac{3}{2}\beta(x)[1-\beta(x)]|_{\tilde{\alpha}^2} = 0 \tag{4.22}$$

must be met, and β and m must have well-defined limits, to satisfy Eqs. (4.17) and (4.18). Again, we vary the in-

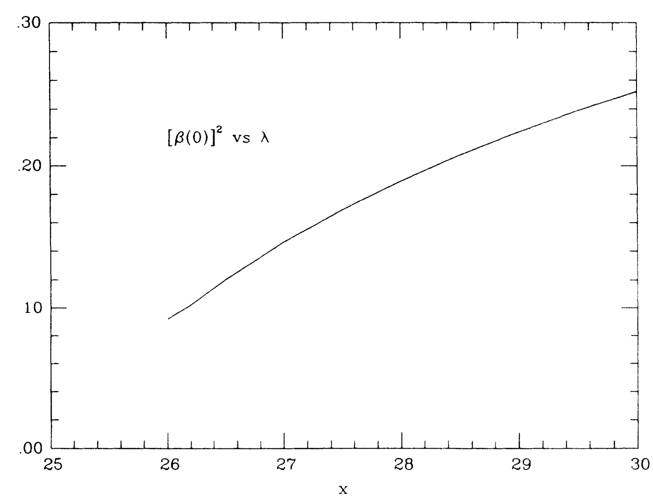

FIG. 3. The function $\beta^2(0)$ is shown vs λ for data corresponding to correction function (3.13). The curve approaches an asymptote, with an intercept $\lambda_c = 0.25$.

TABLE III. The parameters $m(0)/\tilde{\alpha}$, $\beta(0)$, and χ tabulated for a range of λ. These numbers are determined by solution of (4.17) and (4.18), corresponding to the effective correction function (3.14).

λ	$m(0)/\tilde{\alpha}$	$\beta(0)$	χ
0.30	0.8912×10^{-3}	0.5017	0.146×10^{-3}
0.29	0.4653×10^{-3}	0.4828	0.571×10^{-4}
0.28	0.2086×10^{-3}	0.4602	0.178×10^{-4}
0.27	0.7510×10^{-4}	0.4326	0.400×10^{-5}
0.26	0.1929×10^{-4}	0.3977	0.542×10^{-6}
0.25	0.2816×10^{-5}	0.3523	0.315×10^{-7}
0.245	0.7560×10^{-6}	0.3239	0.448×10^{-8}
0.24	0.1381×10^{-6}	0.2902	0.357×10^{-9}
0.235	0.1366×10^{-7}	0.2495	0.114×10^{-10}
0.232	0.2121×10^{-8}	0.2206	0.705×10^{-12}
0.23	0.4529×10^{-9}	0.1991	0.702×10^{-13}
0.228	0.6828×10^{-10}	0.1754	0.415×10^{-14}
0.227	0.2227×10^{-10}	0.1627	0.776×10^{-15}
0.226	0.7896×10^{-11}	0.1520	0.173×10^{-15}

frared limits $\beta(0)$ and $m(0)$ until the ultraviolet conditions are met to sufficient accuracy. The dimensionless parameters $m(0)/\tilde{\alpha}$, $\beta(0)$, and χ are presented for various coupling strengths λ in Table III. A stable numerical solution is found for $\lambda \geq 0.226$, and over this region the infrared mass obeys the empirical asymptotic relation

$$\frac{m(0)}{\tilde{\alpha}} \sim \exp\left[\frac{-K}{\sqrt{\lambda-\lambda_c}}\right], \tag{4.23}$$

where $\lambda_c \simeq 0.215$ and $K \simeq 2.7$. In Fig. 4, the function $\ln^{-2} m(0)/\tilde{\alpha}$ is plotted against λ. The infrared limit $\beta(0)$ goes to zero as λ approaches the critical coupling λ_c in the following way:

$$\beta(0) \simeq \left[\frac{m(0)}{\tilde{\alpha}}\right]^{K'\lambda}, \tag{4.24}$$

where $K' \simeq 0.33$, as we see from Fig. 5, where $[\ln\beta(0)]/[\ln m(0)/\tilde{\alpha}]$ is plotted against λ. This is consistent with the assumption that $\beta(p)$ has an anomalous dimension $\lambda/3$ and behaves like (3.10).

The perturbative form (2.17) is not a good description of the infrared behavior of $\beta(p)$ in our case—it is not expected to be since the critical coupling λ_c is not small. Nevertheless, we can use (2.17) to establish that a critical coupling $\lambda_c > 0$ does occur. Assume the contrary, and at "small" λ insert the expression (2.17) for β into the integral equation (4.18). Because of the bound

$$\frac{\beta(x_{\max})}{\beta(x_{\min})} \leq \left[\frac{x_{\max}+m^2(0)}{x_{\min}+m^2(0)}\right]^{3\lambda} \leq \left[\frac{x_{\max}}{x_{\min}}\right]^{3\lambda}, \tag{4.25}$$

we can conclude that $\lambda_c \geq \bar{\lambda}$, where $\bar{\lambda}$ is the smallest value of λ at which the equation

$$m(x) = \frac{\lambda}{2}\int_0^{\tilde{\alpha}^2} dy \frac{m(y)}{y}\left[\frac{y}{x_{\max}}\right]^{1/2}\left[\frac{x_{\max}}{x_{\min}}\right]^{3\lambda} \tag{4.26}$$

has a nontrivial solution. One may easily show that

$$\bar{\lambda} = \tfrac{3}{14} = 0.2143, \tag{4.27}$$

in contradiction with the assumption of a perturbative regime of small coupling. [Note: It does not follow from this argument that $\lambda_c \geq \bar{\lambda}$, since the integral equations (4.17) and (4.18) are intrinsically nonperturbative, and the bound (4.25) is not satisfied by them.] The coupling (4.27) is quite close to the numerical estimate for λ_c, and it does involve a consistent incorporation of first-order perturbative effects on the vertex and the wave-function renormalization.[8]

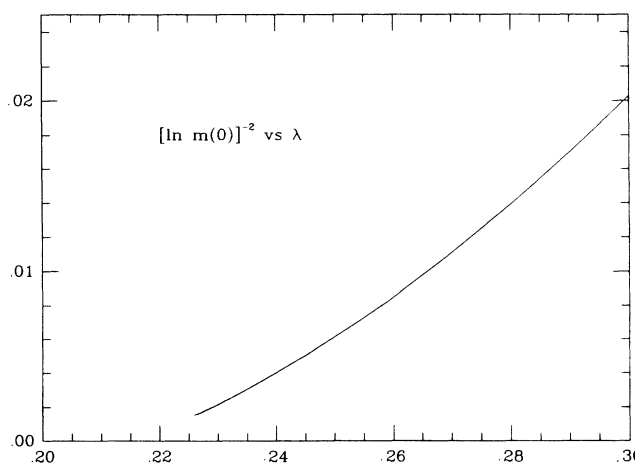

FIG. 4. The function $\ln^{-2} m(0)/\tilde{\alpha}$ is shown vs coupling λ for data corresponding to correction function (3.14). The curve approaches an asymptote, with an intercept $\lambda_c \approx 0.215$.

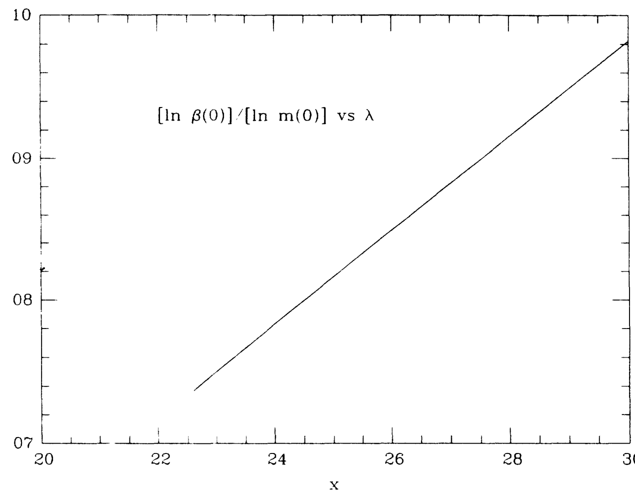

FIG. 5. The function $[\ln\beta(0)]/[\ln m(0)/\tilde{\alpha}]$ is shown vs coupling λ for data corresponding to correction function (3.14). The curve is linear, with slope 0.33.

V. DISCUSSION

The Ward-Takahashi identity (2.7) plays an important role in the truncation of Schwinger-Dyson equations, even though the longitudinal vertex actually drops out of the Schwinger-Dyson equation for the fermion propagator. The differential Ward identity

$$\Gamma^\mu(p,p) = \frac{\partial}{\partial p^\mu} S^{-1}(p), \tag{5.1}$$

which follows from the Ward-Takahashi identity if the vertex function has a well-defined limit as the photon momentum k approaches zero, does involve the transverse vertex as well. Both (2.7) and (5.1) should be consistently met, because of the basic requirements of gauge covariance and rotational invariance. It might appear that only the correction function (3.12) is consistent with the Ward identity, but we shall argue that it is, *in fact*, incorporated at a more basic level in (3.14), and that (3.12) has certain deficiencies in that regard.

Let us define the full vertex function implicitly in terms of a given mass function as

$$\Gamma^{\mu}(p',p,m)=\gamma^{\mu}+\tilde{\alpha}\int\frac{d^3q}{(2\pi)^3}\gamma^{\rho}S_0(p'-q)\times\gamma^{\mu}S_0(p+q)\gamma^{\sigma}D_{\rho\sigma}(q)\ , \qquad (5.2)$$

where

$$S_0(p)=[-\not p+m(p)]^{-1}=\frac{\not p+m(p)}{p^2+m^2(p)} \qquad (5.3)$$

is the renormalized fermion propagator. This expression corresponds to the first iteration of the Schwinger-Dyson vertex equation[19] starting with a free vertex with the photon propagator kept at (2.5). The vertex (5.2), substituted into the Schwinger-Dyson equation for the fermion propagator, forms a closed equation for the fermion propagator, which incorporates all $1/N$ effects, except those relating to the photon propagator, which are expected to be relatively unimportant. One could in principle solve that equation, but instead we consistently neglect the higher-order terms in the numerators of that propagator equation. We find that the fermion mass function satisfies an equation of the form (2.10), with the effective correction given by Eq. (2.18) in the infrared. The wave-function renormalization is given by (2.17) to this order. We are thus led to consider the effective correction (3.14), because it is consistent with the perturbative result (2.19) and with the anomalous behavior of the wave-function renormalization (3.11).

We will next show that the vertex (5.2) is consistent with the Ward-Takahashi identity. Taking the longitudinal projection, and dropping m^2 terms in the numerator, we obtain

$$(p'-p)_{\mu}\Gamma^{\mu}(p,p',m)=[\bar{S}(p')]^{-1}-[\bar{S}(p)]^{-1}\ , \qquad (5.4)$$

where

$$[\bar{S}(p)]^{-1}=\not p-\tilde{\alpha}\int\frac{d^3q}{(2\pi)^3}\gamma^{\rho}S_0(p+q)\gamma^{\sigma}D_{\rho\sigma}(q)\ . \qquad (5.5)$$

If the function $m(p)$ is taken to satisfy (4.12), the "leading-order" equation considered by Appelquist *et al.*,[3] then $\bar{S}=S_0$ and the Ward-Takahashi identity is satisfied. Since the vertex (5.2) is manifestly covariant under rotations, a uniform limit exists as $k\to 0$, and it will satisfy the differential Ward identity to the same order as (5.4).

It is crucial to include the implicit dependence of (5.2) upon the mass function $m(p)$. If one sets $m(p)$ to zero, it is a direct exercise to show that, in the infrared

$$\Gamma^{\mu}(p',p,0)\sim\gamma^{\mu}\left[1+\frac{\lambda}{6}\ln p^2\right]\ , \qquad (5.6)$$

corresponding to the ansatz (3.12) for $p\gg m(a)$. In the longitudinal projection (5.4) and (5.5), the propagator S_0 is replaced by the free propagator, so that the function $\bar{S}$ cannot be identified with either the free propagator or S_0. In other words, neglecting the dependence of (5.2) upon the mass function is actually inconsistent with the Ward-Takahashi relation (2.7), and the ansatz (3.12) must be regarded as deficient.

We have shown that when $O(1/N^2)$ effects are included in a consistent way, the effective correction function f is quadratic in β. This is in agreement with the requirement of consistency, using the concept of an anomalous dimension. With such an effective correction function, the chiral symmetry remains unbroken for a sufficiently large number of fermion species.

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