On predicting correlations from Wigner functions

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A proposal for obtaining predictions of correlations between positions and momenta from the peaks of the Wigner function is studied. An implementation of this proposal based on approximate Wigner functions of a special form obtained from semiclassical wave functions is shown to be unreliable because the exact Wigner functions do not have this form. This approach has been used to argue that classical general relativity is predicted from the semiclassical wave function of the Universe, and this must now be reconsidered. A new measure of correlation, involving projection onto coherent states, is proposed for predicting classical correlations from a general Wigner function. In the harmonic oscillator, it predicts a correlation of position and momentum given by the Hamiltonian equated to the classical energy.

Halliwell[1] has made the proposal that correlations of coordinates and momenta can be predicted from the peaks of the Wigner function of an isolated quantum system. This proposal is valuable because a prediction of a classical correlation could be used to recover classical behavior from a quantum system. Several authors[1,2] have considered the application of Wigner functions to quantum cosmology, and Halliwell[1] has argued that classical general relativity can be recovered from the Wigner function of the semiclassical wave function for the Universe. This result is very important. Indeed, one of the criteria proposed for defining the wave function of the Universe[3] is that classical correlations are predicted when the Universe is large (and therefore, by presumption, semiclassical).

This paper examines the problem of predicting correlations from Wigner functions. Results in quantum cosmology are not subject to experimental verification, so their only test is mathematical consistency. It is especially important to check approximations of unknown accuracy against exact results to verify their validity, as a bad approximation may corrupt otherwise good mathematics. The exactly soluble example of the harmonic oscillator is used to show that the prediction in Ref. 1 of classical correlations from semiclassical wave functions is unjustified. The difficulty is that a measure of correlation applicable to general Wigner functions is not proposed, but instead a crude approximation is used to produce Wigner functions in a highly peaked (δ-function) form. Since these approximate Wigner functions bear no resemblance to the exact Wigner functions, the reliability of their predictions is in doubt. In particular, the conclusion that classical general relativity is predicted from the Wigner function of the semiclassical wave function of the Universe is not justified by the argument of Ref. 1.

Halliwell's suggestion of predicting correlations from the peaks of the Wigner function is a good one. To implement it, a measure of the correlation of coordinates and momenta is proposed. As motivation, the meaning of a correlation between conjugate variables in a Wigner function is considered. The correlation coefficient,[4] familiar from probability and statistical mechanics, is behind the prescription for correlation in Ref. 1, but it is shown not to be the desired measure of correlation. A measure is needed which is sensitive to the localization of the Wigner function and which approximates the classical ideal of interchangeable measurements of conjugate variables. To meet these criteria, the overlap integral between a Gaussian (coherent-state) Wigner function and the Wigner function under consideration is offered as a measure of correlation.

This measure of correlation is applied to interpret the exact Wigner functions of the harmonic-oscillator energy eigenstates. It is found that, with a decreasing probability at higher energy, the position and momentum are correlated by the Hamiltonian equated to the classical energy. This differs from the result predicted in Ref. 1.

The Wigner function[5,6] is a joint quasiprobability distribution on phase space obtained from the wave function by the transformation

$$F(x,p,t)=\frac{1}{2\pi\hbar}\int_{-\infty}^{\infty}du\,\psi^*(x-u/2,t)\times e^{-ipu/\hbar}\psi(x+u/2,t)\ . \quad (1)$$

It carries information about both the coordinate and momentum probability distributions as shown from the properties

$$\int_{-\infty}^{\infty}dp\,F(x,p,t)=|\psi(x,t)|^2\ ,\qquad \int_{-\infty}^{\infty}dx\,F(x,p,t)=|\tilde{\psi}(p,t)|^2\ , \quad (2)$$

where $\tilde{\psi}(p,t)$ is the wave function in momentum space. The Wigner function is positive definite for Gaussian wave functions, but is not in general and is therefore not a true joint probability distribution. The prefix "quasi" is used to reflect this. Furthermore, the coordinate and momentum wave functions are related by the Fourier transform, so not all joint (quasi)probability distributions are Wigner functions. This indicates that interpretation of the Wigner function is not simply the interpretation of a classical joint probability distribution.

Reference 1 does not give a precise criterion for determining correlation of coordinates and momenta. Rather, the general remarks are made that if the Wigner function F is separable as a function of x and p, then it predicts no correlation while if F is strongly peaked around $p=g(x)$, then it predicts this as a correlation. This prescription has apparent success because only Wigner functions which are highly peaked on a single hypersurface in phase space are considered. In general, such Wigner functions are obtained from an approximation. The fact that these approximate Wigner functions do not agree qualitatively with the exact Wigner functions, together with the inability to interpret a general Wigner function, seriously undermines this approach.

In addition, the examples in Ref. 1 are atypical because several involve Wigner functions which are not normalizable: The integral of F over all phase space diverges. Strictly, these are not true Wigner functions. This happens even in the harmonic-oscillator example because the wave function considered is essentially the Green's function (without the Heaviside function in time) and not the normalizable energy eigenstates. The failure of normalizability is symptomatic of this approach, but it is not the main difficulty.

The central result of Ref. 1 is the prediction obtained from semiclassical (WKB) wave functions of the form

$$\psi(x,t)=C(x,t)e^{iS(x,t)/\hbar} . \tag{3}$$

Evaluating (1) after expanding S in a Taylor series about x to first order in u, the approximate Wigner function[1,6]

$$F_{\rm app}(x,p,t)=|C(x,t)|^2\delta(p-\partial S/\partial x) \tag{4}$$

is found. The δ-function peak predicts the correlation

$$p=\partial S/\partial x . \tag{5}$$

This equation describes a family of classical trajectories, and Ref. 1 concludes that the semiclassical wave function (3) predicts a classical correlation of position and momentum.

This result is attractive, but it is cast into doubt by the realization that approximate Wigner function (4) bears no resemblance to the exact Wigner function for the system. Consider the harmonic oscillator as an example. The semiclassical wave function is

$$\psi_{\rm sc}=(2E-\omega^2x^2)^{-1/4} \times\exp\left[i\int(2E-\omega^2x^2)^{1/2}dx/\hbar\right] . \tag{6}$$

From the above analysis, this gives the approximate Wigner function

$$F_{\rm app}(x,p,t)=|2E-\omega^2x^2|^{-1/2} \times\delta(p-(2E-\omega^2x^2)^{1/2}) . \tag{7}$$

This Wigner function makes the prediction

$$E=\tfrac{1}{2}p^2+\tfrac{1}{2}\omega^2x^2 . \tag{8}$$

The position and momentum have a classical correlation, given by the Hamiltonian equated to the quantum energy $E=(n+1/2)\hbar\omega$.

The exact Wigner function for the harmonic oscillator is known,[7] and it can be used to check the approximate Wigner function (7). The exact harmonic-oscillator energy eigenfunctions are well known to be

$$\psi_n=\left[\frac{\kappa}{2^n\pi^{1/2}n!}\right]^{1/2} \times e^{-\kappa x^2/2}H_n(\kappa x)e^{-i(n+1/2)\omega t} , \tag{9}$$

where $\kappa=(\omega/\hbar)^{1/2}$ and H_n is the nth Hermite polynomial. The exact Wigner function is[7]

$$F_{\rm ex}(x,p,t)=\frac{(-1)^n}{\pi\hbar}e^{-2H(x,p)/\hbar\omega} \times L_n(4H(x,p)/\hbar\omega) , \tag{10}$$

where $H(x,p)=p^2/2+\omega^2x^2/2$ is the Hamiltonian and L_n is the nth Laguerre polynomial.

It is instructive to look at a particular example. The Wigner function for $n=5$ is shown in Fig. 1. The Wigner function has its maximum absolute value at $H=0$. This is generally true: For the nth energy eigenfunction, the Wigner function at zero has the value $(-1)^n/\pi\hbar$. As H increases, the Wigner function oscillates with a decreasing amplitude but with increasingly broad peaks. For $n=5$, after $H\simeq4.5$, it dies off exponentially. The approximate Wigner function (7) is a δ function at $H=5.5$.

Clearly the approximate result does not agree with the exact result and does not even capture the qualitative flavor. What has gone wrong? One's first suspicion is that the trouble lies with the semiclassical wave function. It might not be surprising that the prediction $p=\partial S/\partial x$ is made. In constructing the WKB wave function (3), the ansatz $\psi=\exp(i\mathcal{S}/\hbar)$ is made, and $\mathcal{S}$ is computed as a power series in $\hbar$. The leading term of this series is found by solving the classical Hamilton-Jacobi equation. In the semiclassical approximation as $\hbar\rightarrow0$, $\mathcal{S}$ becomes $S+\hbar\ln C$, where S is a solution of the Hamilton-Jacobi

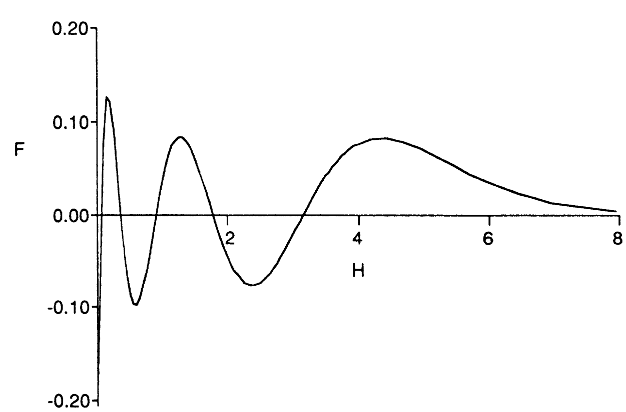

FIG. 1. The Wigner function F for the $n=5$ harmonic-oscillator energy eigenstate, where $H=p^2/2+\omega^2x^2/2$.

equation. The prediction $p = \partial S/\partial x$ made by the approximate Wigner function is not surprising because S is a solution of the Hamilton-Jacobi equation by construction. This may be cause for future concern, but it is not the trouble here.

Berry[8] has also calculated an approximate Wigner function using the semiclassical wave function (3), but he evaluates the transform (1) with a uniform approximation. His result in the harmonic-oscillator example is

$$F_{\rm Ber}(x,p,t)=\frac{[\frac{3}{2}A(H)]^{1/6}{\rm Ai}\left[-\left[\frac{3}{2\hbar}A(H)\right]^{2/3}\right]}{\pi 2^{1/2}\hbar^{2/3}[H(E-H)]^{1/4}}\ , \tag{11}$$

where

$$A(H)=\frac{E\pi}{\omega}-\frac{2}{\omega}[H(E-H)]^{1/2} -\frac{2E}{\omega}\arcsin[(H/E)^{1/2}] \tag{12}$$

and $H=p^2/2+\omega^2x^2/2$ is the Hamiltonian. For H nearly equal to or greater than E, this expression takes the simpler form

$$F_{\rm Ber}(x,p,t)=\frac{1}{\pi(2E)^{1/3}}{\rm Ai}(2^{2/3}(H-E)H^{-1/3})\ . \tag{13}$$

The difference between Berry's approximation and the exact Wigner function for $E=5.5$ is less than 3×10^{-3} except near the origin where it diverges like $H^{-1/4}$. This shows that the approximate Wigner function (7) is an artifact of the approximation of the transform (1).

The exact Wigner function of the harmonic oscillator for $n=5$ does not have a single peak which is obviously more dominant than the rest. The prescription of Ref. 1 for determining the correlation fails, and one might conclude that no prediction is made. Before doing so, however, a more serious effort to find a measure of correlation should be made. After all, the harmonic oscillator is among the most classical of quantum systems, and if one cannot predict correlations from its Wigner function, what hope would there be for more complex systems like those in quantum cosmology?

The first step toward finding a measure of correlation is to clarify what is meant by "correlation." From the prescription of Ref. 1 for determining correlation, it is evident that two variables are taken to be correlated if they bear a functional relation. The correlation coefficient familiar from classical probability[4] is tacitly in mind[9] as a measure of this correlation. The correlation coefficient is intended as a measure of dependence, and it vanishes if two random variables are independent, as they are when the joint probability distribution separates. This is the origin of the prescription that a separable Wigner function makes no prediction.

There are two difficulties with this approach to correlation. First, the correlation coefficient is not a general measure of dependence, but only of linear dependence. In particular, the correlation coefficient can vanish even if one variable is a function of the other.[4] Second, "correlation" as it is intended to apply to quantum mechanics is not a consequence of dependence of random variables.

In a (classical) experimental context, two variables x and p are said to be correlated if, knowing the result of a measurement of x, the result of a measurement of p can be predicted and vice versa. A separable joint probability distribution that is localized predicts this form of correlation. From the standpoint of probability, a related notion is that x and p are correlated if the most probable outcome $(\bar{x},\bar{p})$ chosen from their joint probability distribution consists of the most probable x and the most probable p chosen from their separate distributions. The strength of the correlation depends on the various probabilities. Again, the central feature is localization of the joint probability distribution. The proposal that a sufficiently strong peak in the Wigner function makes a prediction springs from this conception of correlation.

There are two difficulties which complicate the identification of classical correlations in quantum mechanics. First, the uncertainty principle restricts the amount of localization that a Wigner function can have. This limits the strength a correlation in quantum mechanics can have. Second, the measurement of one variable collapses the wave function thereby affecting the measurement of the conjugate variable. Classically, the predictions which follow from correlation are independent of the order of measurement. So, if the measurement $x=\bar{x}$ predicts $p=\bar{p}$, then correlation requires that $p=\bar{p}$ predicts $x=\bar{x}$. This condition cannot be perfectly realized in quantum mechanics, so an approximate version of it must be found.

The most classical of quantum-mechanical wave functions is the Gaussian wave packet, or coherent state:[10]

$$\psi_c(x)=(\pi\hbar)^{-1/4}e^{-(x-\bar{x})^2/2\hbar+i\bar{p}x/\hbar}\ . \tag{14}$$

[In conventional notation for coherent states, $\alpha=2^{1/2}(\bar{x}+i\bar{p})$.] This is the most classical of wave functions because it is equally localized in coordinate and momentum space ($\Delta x=\Delta p$) to the minimum allowed by the Heisenberg uncertainty principle:

$$\Delta x\,\Delta p=\frac{\hbar}{2}\ .$$

Measurements of x or p will return the outcome of $\bar{x}$ or $\bar{p}$ to within the same uncertainty. The localization in one variable cannot be improved without losing localization in the other. All Gaussians are minimum uncertainty wave packets, but an inequality between Δx and Δp departs from the classical ideal that x and p are on equal footing and equally knowable.

The Wigner function of the coherent state is

$$F_c(x,p;\bar{x},\bar{p})=\frac{1}{\pi\hbar}e^{-(x-\bar{x})^2/\hbar-(p-\bar{p})^2/\hbar} \tag{15}$$

and is the most localized of Wigner functions. Treated as a classical joint probability distribution, it makes the strongest possible prediction of a correlation between the measurement outcomes $\bar{x}$ and $\bar{p}$ compatible with the uncertainty principle.

A quantum measurement by projection onto position or momentum eigenstates would destroy the correlation between $\bar{x}$ and $\bar{p}$ before it could be verified. This is inevitable with quantum measurements. The attempt to localize one variable (to obtain a measurement result) necessarily delocalizes the conjugate variable. The classical ideal that coordinates and momenta are equally known is most closely achieved when they are equally localized. For this reason, it is proposed that a measurement which is to return a classical result be defined as a projection onto a coherent state. The prediction made by the Wigner function of the coherent state is then the most classical prediction possible from a quantum wave function.

The measure of the correlation of $\bar{x}$ and $\bar{p}$ predicted by a Wigner function F is given by the overlap integral

$$P_F(\bar{x},\bar{p})=2\pi\hbar\int dx\,dp\,F_c(x,p;\bar{x},\bar{p})F(x,p) \tag{16}$$

in which the Wigner function of interest is projected onto the Wigner function of the coherent state (15). This measure gives the probability that a measurement will produce a classical correlation of $\bar{x}$ and $\bar{p}$. The most classical correlation predicted from the Wigner function F is found by varying $\bar{x}$ and $\bar{p}$ to maximize the overlap with the Wigner function of the coherent state. By working with the overlap integral, one is able to compare the probabilities of different correlations. This would not be possible if one tried to use expectation values to define the measure of correlation.

The physical (positive-definite) nature of the probability obtained from (16) can be understood by reducing the overlap integral to one in coordinate space. Using the definition of the Wigner function (1), one has

$$P_F=(2\pi\hbar)^{-1}\int du\,dv\,dx\,dp\,\psi_c^*(x-u/2)e^{-ipu/\hbar}\times\psi_c(x+u/2)\psi^*(x-v/2)\times e^{-ipv/\hbar}\psi(x+v/2)\ . \tag{17}$$

Performing the p integration gives a δ function which allows the v integration to be done. This gives

$$P_F=\int du\,dx\,\psi_c^*(x-u/2)\psi(x-u/2)\psi_c(x+u/2)\times\psi^*(x+u/2)\ . \tag{18}$$

After changing variables, this reduces to

$$P_F(\bar{x},\bar{p})=\left|\int dx\,\psi_c^*(x)\psi(x)\right|^2 . \tag{19}$$

Thus, the overlap integral (16) gives the familiar probability of quantum mechanics for a transition from the state ψ to the coherent state ψ_c.

This procedure may be applied to interpret the Wigner function for the harmonic-oscillator energy eigenstates. Let $\omega=1$ for convenience. The probability of obtaining a coherent state located at $\bar{x},\bar{p}$ from a harmonic-oscillator energy eigenstate is given by

$$P_n(\bar{x},\bar{p})=\frac{2(-1)^n}{\pi}\int dx\,dp\,\exp[-(x-\bar{x})^2-x^2-(p-\bar{p})^2-p^2]\times L_n[2(p^2+x^2)]\ , \tag{20}$$

where L_n is the nth Laguerre polynomial. The result (25) is well known from the right-hand side of (19),[10] but the overlap integral of Wigner functions may be computed directly. Changing variables to polar coordinates in the phase-space plane, one has

$$P_n(\bar{x},\bar{p})=\frac{2(-1)^n}{\pi}\int r\,dr\,d\theta\times\exp[-2r^2+2r(\bar{x}\sin\theta+\bar{p}\cos\theta)]\times L_n(2r^2)\ . \tag{21}$$

The θ integration may be done using the identity[11]

$$\int_0^{2\pi}d\theta\exp[2r(\bar{x}\sin\theta+\bar{p}\cos\theta)]=2\pi I_0(2r(\bar{p}^2+\bar{x}^2)^{1/2})\ , \tag{22}$$

where I_0 is a modified Bessel function. The r integration may now be done using[11]

$$\int_0^\infty re^{-r^2}L_n(r^2)J_0(ry)dr=\frac{2^{-2n-1}}{n!}y^{2n}e^{-y^2/4}\ . \tag{23}$$

The result is

$$P_n(\bar{x},\bar{p})=\frac{1}{n!}\left[\frac{\bar{p}^2+\bar{x}^2}{2}\right]^n e^{-(\bar{p}^2+\bar{x}^2)/2}\ . \tag{24}$$

Letting $H=(\bar{p}^2+\bar{x}^2)/2$, this may be more simply written as

$$P_n=\frac{H^n}{n!}e^{-H}\ . \tag{25}$$

To find the predicted classical correlation, one maximizes (25). One finds that $H=n$ with probability $P_n=n^ne^{-n}/n!$. This is an interesting result because it says that the most probable classical measurement will find the position and momentum correlated by the classical Hamiltonian equated to the classical energy. The probability that a classical result is obtained falls rapidly with increasing energy. Note that this is a different result from that of Ref. 1 that the position and momentum are correlated by the classical Hamiltonian equated to the quantum energy.

In this paper it has been shown that the prediction of correlations from semiclassical wave functions in Ref. 1 is unreliable because the approximate Wigner functions used bear no resemblance to the exact Wigner functions. The trouble was traced to a crude approximation of the transform (1) producing the Wigner functions. When faced with analyzing exact Wigner functions which do not have obviously dominant peaks, the prescription for predicting correlations given in Ref. 1 fails.

The prescription of Ref. 1 for making predictions is based on a definition of the correlation between two random variables in terms of dependence while a more physical definition would be in terms of localization of the joint probability distribution. The Wigner function of the coherent state was observed to be the most localized of Wigner functions. It is also the most classical in the sense that position and momenta are equally well known.

Since quantum measurement involves projection onto a basis of eigenstates and the uncertainty principle prevents conjugate variables from being specified with arbitrary accuracy, it is proposed that a classical measurement be defined as a projection onto a coherent state because this gives the result with the position and momenta most equally known. The overlap integral between the Wigner function of a coherent state and the Wigner function of interest is then a natural measure of correlation. By maximizing the overlap with respect to the location of the coherent state, the most classical correlation predicted by the Wigner function is found. In the harmonic oscillator, the Hamiltonian equated to the classical energy is recovered as the predicted correlation. This result encourages the hope that this new method will reliably predict the classical correlations of general relativity from the semiclassical wave function of the Universe.

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