Fock-space representation of coupled Abelian Chern-Simons theory

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The theory in $2+1$ dimensions described by the Abelian Chern-Simons gauge field minimally coupled to a spinor field is given a Fock-space representation. The states which satisfy the constraint of Gauss's law are explicitly constructed for both the weak and strong implementation. The charged states are shown to be coherent states of disallowed ghost operators. The S matrix for the theory is explicitly constructed and the gauge field is shown to decouple from the theory. The charged states are reduced to exhibit a manifestly gauge-invariant form. The angular momentum operator for the theory is derived from Noether's theorem and given a decomposition in terms of Fock-space operators. The charged eigenstates of the angular momentum are shown to have the induced spin $Q^2/4\pi\kappa$, where Q is the total charge operator of the theory and κ is the Chern-Simons action strength, leading to the possibility of exotic statistics for the charged particles.

I. INTRODUCTION

Chern-Simons theory[1] in $2+1$ dimensions has recently become the object of extensive investigation. The interest is threefold. First, the theory, despite the absence of the metric from the measure of the action, is covariant under general coordinate diffeomorphisms, and has been shown to be related to conformal field theories in $1+1$ dimensions.[2] Second, the generally covariant observables of pure (uncoupled) Chern-Simons theories are Wilson loops, and the expectation values of these have been shown to give various topological properties of closed paths and knots, such as linking numbers, in an arbitrary three-manifold.[3] Third, a relationship between coupled Chern-Simons theory and high-T_c superconductors has been argued, and a relation to the statistical mechanics of anyons has been developed.[4]

The analyses of Chern-Simons theories to date have been in terms of functional formalisms, path-integral techniques, or holomorphic quantization.[5] There are many advantages to functional methods to field theories, particularly when nonperturbative results are sought. However, there are also some advantages to be gained from an explicit construction of the Fock-space representation of a field theory. In order to formulate a scattering matrix it is necessary to understand the structure of the asymptotic states, and this is readily accomplished in a Fock-space formulation of the problem. It is also possible to analyze arbitrary operator products in either the physical or unphysical sector once the Fock space has been identified. Although many calculations require only the behavior of the fields and their equations of motion, a Fock-space formulation allows actual construction of the eigenstates of operators and an explicit representation of the S matrix and evolution operator.

It is the intent of this paper to present such an analysis for $(2+1)$-dimensional Abelian Chern-Simons gauge fields coupled to a spinor field on the topologically trivial spatial manifold R^2. Of course, adding a minimally coupled spinor field to the theory immediately requires the presence of the metric to maintain general covariance, and thus this property of pure Chern-Simons theory is lost. However, there are interesting aspects to be gained by coupling spinors. It has been argued that Chern-Simons theories with sources lead to Fermi-Bose transmutation through the appearance of induced spin.[6] Such a property leads to the possibility of exotic statistics,[7] and may explain the nature of the high-T_c superconductors. However, previous work has focused on c-number external sources, and some calculations of the induced spin have been done by either using definitions of the angular momentum which are not derived from Noether's theorem or by deducing the presence of induced spin from Wilson loop arguments. In this paper the sources will be quantized and the angular momentum will be derived from Noether's theorem. This definition of angular momentum will be shown to be gauge invariant when applied to states which satisfy the strong implementation of the constraint of Gauss's law. The explicit form of such states will be developed and used to verify the presence and value of the induced spin by direct application of the angular momentum operator. Additional assumptions regarding the form of the Green's function are discussed. Mixtures of Chern-Simons and Maxwell actions for the gauge field will not be considered.

The outline of the remainder of this paper is straightforward. In Sec. II the uncoupled Abelian Chern-Simons theory for the manifold $R\times R^2$ is given a Fock-space representation for the choice of the temporal gauge. It is shown how the constraint of Gauss's law, or equivalently gauge invariance, can be implemented both weakly and strongly. It is shown that the physical states in the weak implementation of the constraint are zero-norm gauge-invariant ghosts. In the strong implementation the physical states are zero-norm gauge-invariant ghosts dressed with gauge particle pairs. The operator version of Stokes's theorem is demonstrated for the uncoupled theory, showing that the Wilson loops for paths lying entirely in the R^2 manifold are trivial. In Sec. III the Chern-Simons gauge field is minimally coupled to a

(2+1)-dimensional spinor field. The spinor field is quantized such that, in the absence of coupling, its basis states would obey Fermi-Dirac statistics. The constraint of Gauss's law for the coupled theory is met by dressing the charged states with ghosts which are not gauge invariant. In effect the charged states of the coupled theory are coherent states.[8] Again, the constraint is realized both weakly and strongly in a manner similar to the uncoupled case. The Wilson loops for the coupled theory are shown to satisfy the operator version of Stokes's theorem when applied to the physically allowed states. The charged states develop a nontrivial holonomy which depends on the charge distribution of the state and the path chosen for the Wilson loop. In Sec. IV the scattering matrix for the charged particles is developed. The states weakly satisfying the constraint are used, and the evolution operator or S matrix is shown to be trivial for the allowed ghost states of the coupled theory. The gauge field decouples from the theory and leaves behind a velocity-dependent potential for the spinor field, which, in the classical limit, affects the angular momentum of the charged particles. In Sec. V the angular momentum operator, derived from Noether's theorem, is analyzed for a charged state. Given a particular choice of representation for the Green's functions of the theory, the charged states which satisfy the strong form of the constraint are gauge-invariant eigenstates of this total angular momentum operator with an anomalous eigenvalue. It is shown that a state containing a pair of identical spinor particles picks up an induced phase when rotated through π radians using the angular momentum operator, indicating the presence of anyon statistics in the charged sector in agreement with results obtained by other methods and definitions.

II. UNCOUPLED CHERN-SIMONS THEORY

The Abelian version of pure Chern-Simons theory in 2+1 dimensions is described by the action

$$\mathcal{L}=\frac{\kappa}{2}\epsilon^{\mu\nu\lambda}A_\mu\partial_\nu A_\lambda\ , \tag{2.1}$$

where $\epsilon^{\mu\nu\lambda}$ is the standard Levi-Civita tensor density. Unlike the non-Abelian case, the constant κ is not required to be an integer to maintain gauge invariance. The constant κ can be related to other parameters in the coupled theory if the Chern-Simons action is derived from decoupling other fields[9] or if the theory is to be invariant under large gauge transformations and quantized on a genus one surface.[10] In this paper κ will be considered arbitrary. Varying the action (2.1) leads to the equations of motion

$$\kappa(\partial_1 A_0-\partial_0 A_1)=0\ , \tag{2.2a}$$

$$\kappa(\partial_2 A_0-\partial_0 A_2)=0\ , \tag{2.2b}$$

$$\kappa(\partial_1 A_2-\partial_2 A_1)=0\ . \tag{2.2c}$$

At this point the temporal gauge is chosen, and A_0 is set to zero. The equations of motion reduce to

$$\dot{A}_1=\dot{A}_2=0\ , \tag{2.3}$$

and

$$G=\kappa(\partial_1 A_2-\partial_2 A_1)=0\ . \tag{2.4}$$

Equation (2.4) is the equivalent of Gauss's law for the Chern-Simons theory. In the temporal gauge the theory still possesses an invariance under the time-independent gauge transformations given by

$$A_i\to A_i-\partial_i\Lambda,\ \dot{\Lambda}=0\ . \tag{2.5}$$

The momenta canonically conjugate to A_1 and A_2, denoted Π_1 and Π_2, respectively, are given by

$$\Pi_1=\kappa A_2 \tag{2.6a}$$

and

$$\Pi_2=-\kappa A_1\ . \tag{2.6b}$$

The Hamiltonian density of the theory is given by

$$\mathcal{H}=\frac{\kappa}{2}(A_2\dot{A}_1-A_1\dot{A}_2)-\kappa A_0(\partial_1 A_2-\partial_2 A_1)\ . \tag{2.7}$$

Using the equations of motion (2.3) and (2.4) shows that the Hamiltonian vanishes in the temporal gauge, a standard feature of all generally covariant theories. This is, in turn, consistent with the equations of motion (2.3) when they are expressed as a commutator with the Hamiltonian. Therefore, the remaining fields A_1 and A_2 are time independent.

The canonical quantization of the theory requires that the equal-time commutators satisfy

$$[A_1(\mathbf{x}),\Pi_1(\mathbf{y})]=[A_2(\mathbf{y}),\Pi_2(\mathbf{x})]$$
$$=\kappa[A_1(\mathbf{x}),A_2(\mathbf{y})]=i\delta^2(\mathbf{x}-\mathbf{y})\ . \tag{2.8}$$

The Dirac delta appearing on the right-hand side of (2.8) is understood to possess the test function space consisting of all piecewise integrable functions defined on the manifold R^2, hence the previous statement regarding the space-time manifold being $R\times R^2$. The commutators of (2.8) are therefore satisfied by the decompositions

$$A_1(\mathbf{x})=\int\frac{d^2k}{2\pi}[k_1a_1(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}}+k_1a_1^\dagger(\mathbf{k})e^{-i\mathbf{k}\cdot\mathbf{x}}] \tag{2.9a}$$

and

$$A_2(\mathbf{x})=-i\int\frac{d^2k}{2\pi}[k_2a_2(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}}-k_2a_2^\dagger(\mathbf{k})e^{-i\mathbf{k}\cdot\mathbf{x}}]\ , \tag{2.9b}$$

along with the algebra

$$[a_1(\mathbf{k}),a_2^\dagger(\mathbf{p})]=[a_2(\mathbf{p}),a_1^\dagger(\mathbf{k})]$$
$$=(2\kappa p_1p_2)^{-1}\delta^2(\mathbf{k}-\mathbf{p})\ , \tag{2.10a}$$

$$[a_1(\mathbf{k}),a_1^\dagger(\mathbf{p})]=[a_2(\mathbf{k}),a_2^\dagger(\mathbf{p})]=0\ . \tag{2.10b}$$

The reason for the presence of the k_i factors in the expansions of (2.9) will become apparent shortly.

A Fock-state representation for the theory is obtained by interpreting $a_i^\dagger$ as a creation operator and introducing a state, denoted $|0\rangle$, which is cylic under the operators

a_i. It is easy to show that only states with equal numbers of both type of particle, i.e., equal numbers of the operators $a_1^\dagger$ and $a_2^\dagger$, have a nonzero norm, all others having zero norm. Thus, the single-particle states of either type are ghosts.

The physical states are selected by implementing Gauss's law (2.4) as a constraint. That this is equivalent to the demand of gauge invariance is easy to show. Using the commutation relations (2.8) it follows that the unitary operator defined by

$$U=e^{iQ_\Lambda}\ , \tag{2.11}$$

where

$$Q_\Lambda=\int d^2x\ \Lambda(\mathbf{x})G(\mathbf{x}),\quad \dot{\Lambda}=0\ , \tag{2.12}$$

generates the gauge transformations given by

$$UA_jU^{-1}=A_j-\partial_j\Lambda\ . \tag{2.13}$$

Thus, instituting Gauss's law $G=0$ is equivalent to demanding that states or transition amplitudes be gauge invariant. This is quite similar to manifestly covariant formulations of quantum electrodynamics.[11]

There are several ways to implement a constraint in quantum field theory. In the first method it is implemented strongly. If $|P\rangle_s$ is a physically allowed state in the strong implementation, then

$$G|P\rangle_s=0\ . \tag{2.14}$$

In the second approach the constraint is implemented weakly. For this case, if $|P\rangle_w$ and $|P'\rangle_w$ are any two physically allowed states in the weak implementation, then

$$_w\langle P|G|P'\rangle_w=0\ . \tag{2.15}$$

It is critical to verify that a state selected by either procedure has nonvanishing S matrix amplitudes to other physically allowed states only if the state has a nonzero positive norm. In standard quantum electrodynamics this ensures that unitarity is preserved. This will be verified in the next section.

Substituting the expansions of (2.9) into (2.4) gives

$$G(\mathbf{x})=\kappa\int\frac{d^2k}{2\pi}k_1k_2[g(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}}+g^\dagger(\mathbf{k})e^{-i\mathbf{k}\cdot\mathbf{x}}]\ , \tag{2.16}$$

where

$$g(\mathbf{k})=a_2(\mathbf{k})-ia_1(\mathbf{k})\ . \tag{2.17}$$

It follows from (2.10) that

$$[g(\mathbf{k}),g^\dagger(\mathbf{p})]=0\ , \tag{2.18}$$

so that $g^\dagger$ is the creation operator for a ghost state. It is apparent from the algebra (2.18) that the weak version of the constraint, i.e., (2.15), is satisfied by states composed solely of g-type ghosts, i.e., states of the form

$$|P\rangle=f(\mathbf{p}_1,\ldots,\mathbf{p}_n)g^\dagger(\mathbf{p}_1)\cdots g^\dagger(\mathbf{p}_n)|0\rangle\ , \tag{2.19}$$

where f is some function of the momenta. Physical states of the form (2.19) will be denoted $|P\rangle_w$, and, apart from the vacuum, all have the property that

$$_w\langle P|P\rangle_w=0\ . \tag{2.20}$$

In order to satisfy the strong version of the constraint, (2.14), it is first necessary to introduce the ghost operator

$$b(\mathbf{k})=a_2(\mathbf{k})+ia_1(\mathbf{k})\ . \tag{2.21}$$

Like the operator (2.17), the b operator satisfies

$$[b(\mathbf{k}),b^\dagger(\mathbf{p})]=0\ . \tag{2.22}$$

However, it follows from (2.10) that

$$[b(\mathbf{k}),g^\dagger(\mathbf{p})]=i(\kappa p_1p_2)^{-1}\delta^2(\mathbf{p}-\mathbf{k}) \tag{2.23a}$$

and

$$[g(\mathbf{k}),b^\dagger(\mathbf{p})]=-i(\kappa p_1p_2)^{-1}\delta^2(\mathbf{k}-\mathbf{p})\ . \tag{2.23b}$$

The states satisfying the strong form of the constraint can now be derived from the physical states $|P\rangle_w$, i.e., those states of the form (2.19), by dressing them with the transformation

$$D=\exp\left[-i\kappa\int d^2p\ p_1p_2b^\dagger(\mathbf{p})g^\dagger(-\mathbf{p})\right]\ . \tag{2.24}$$

In order to demonstrate that the strong form of the constraint is satisfied by applying D to the states of the form (2.19) the constraint is first rewritten as

$$G=\int\frac{d^2k}{2\pi}k_1k_2[g(\mathbf{k})+g^\dagger(-\mathbf{k})]e^{i\mathbf{k}\cdot\mathbf{x}}\ . \tag{2.25}$$

It is straightforward to show that

$$[g(\mathbf{k})+g^\dagger(-\mathbf{k})]D=Dg(\mathbf{k})\ . \tag{2.26}$$

It follows that the states which satisfy the strong version of the constraint, denoted $|P\rangle_s$, are given by

$$|P\rangle_s=D|P\rangle_w\ . \tag{2.27}$$

It is to be noted that D is not a unitary transformation and for this reason the dressed vacuum $D|0\rangle$ has an infinite norm. This is tedious but straightforward to verify by expanding the exponential (2.24) in a formal power series to obtain

$$\langle 0|D^\dagger D|0\rangle=e^P\ , \tag{2.28}$$

where P is the dimensionless and divergent volume of phase space:

$$P=\frac{1}{4\pi^2}\int d^2x\int d^2p\ . \tag{2.29}$$

Expression (2.28) is obtained in the large but finite P limit. Using a similar argument it can be shown that all the physically allowed gauge field strong states, apart from the dressed vacuum, are still ghosts in the finite P case.

It follows from the form of the fields (2.9) that gauge transformations of the form (2.5) are generated by the replacements

$$a_1(\mathbf{k})\to a_1(\mathbf{k})-i\lambda(\mathbf{k})\ , \tag{2.30a}$$

$$a_2(\mathbf{k})\to a_2(\mathbf{k})+\lambda(\mathbf{k})\ , \tag{2.30b}$$

where the corresponding gauge transformation function is given by

$$\Lambda(\mathbf{x})=\int \frac{d^2k}{2\pi}[\lambda(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}}+\lambda^*(\mathbf{k})e^{-i\mathbf{k}\cdot\mathbf{x}}] . \tag{2.31}$$

This shows that the physically allowed ghost $g(\mathbf{k})$ is invariant under gauge transformations, while the disallowed ghost $b(\mathbf{k})$ transforms as

$$b(\mathbf{k})\rightarrow b(\mathbf{k})+2\lambda(\mathbf{k}) . \tag{2.32}$$

Property (2.32) will be used to verify the form of the spatial part of the gauge field angular momentum in Sec. V.

For a classical field configuration which satisfies the equations of motion it follows that, for a closed loop L in the R^2 space, Stokes's theorem predicts

$$\oint_L dx^i A_i=\kappa^{-1}\int_{S(L)} d^2x\, G=0 . \tag{2.33}$$

The operator version of this statement can be demonstrated by use of expansions (2.9) and integrating around a rectangular path whose x_1 and x_2 limits are $\pm L_1$ and $\pm L_2$, respectively. Direct substitution yields

$$\oint_L dx^i A_i=4\int \frac{d^2k}{2\pi}[g(\mathbf{k})+g^\dagger(-\mathbf{k})] \times\sin(k_1L_1)\sin(k_2L_2) . \tag{2.34}$$

Using the identity

$$\Theta(x+L)-\Theta(x-L)=\frac{1}{\pi}\int dk\, e^{ikx}\frac{\sin(kL)}{k} , \tag{2.35}$$

where $\Theta(x)$ is the standard step function given by

$$\Theta(x)=\begin{cases}1 & \text{if } x>0 ,\\ 0 & \text{if } x<0 ,\end{cases} \tag{2.36}$$

immediately shows that (2.34) is the operator version of (2.33). When evaluated in the physical sector of states it follows that the Wilson loop is trivial for the uncoupled theory defined over this spatial manifold.

III. COUPLED ABELIAN CHERN-SIMONS THEORY

The Abelian gauge field of Sec. II can be coupled to a spinor field. The Lagrangian density of such a theory takes the form[12]

$$\mathcal{L}=\frac{\kappa}{2}\epsilon^{\mu\nu\lambda}A_\mu\partial_\nu A_\lambda+\overline{\Psi}[i\gamma^\mu(\partial_\mu-ieA_\mu)]\Psi-m\overline{\Psi}\Psi , \tag{3.1a}$$

which has the Abelian gauge invariance

$$\Psi\rightarrow e^{-ie\Lambda}\Psi ,\qquad A_\mu\rightarrow A_\mu-\partial_\mu\Lambda . \tag{3.1b}$$

For 2+1 dimensions the algebra of the γ matrices is satisfied by the Pauli spin matrices,

$$\gamma^0=\sigma^3,\gamma^1=i\sigma^2,\gamma^2=i\sigma^1 , \tag{3.2a}$$

and the uncoupled spinor field, denoted ψ, solves

$$(i\gamma^\mu\partial_\mu-m)\psi=0 . \tag{3.2b}$$

Equation (3.2b) has two solutions corresponding to positive and negative energy. The positive-energy solutions take the form

$$u_pe^{-ipx}=[2m(m+p_0)]^{-1/2}\begin{bmatrix}p_0+m\\ p_1-ip_2\end{bmatrix}e^{-ipx} , \tag{3.3a}$$

while the negative-energy solutions take the form

$$v_pe^{ipx}=[2m(m+p_0)]^{-1/2}\begin{bmatrix}-p_1-ip_2\\ -p_0-m\end{bmatrix}e^{ipx} . \tag{3.3b}$$

These solutions satisfy the standard normalizations

$$\bar{u}_pu_p=1,\quad \bar{v}_pv_p=-1 ,\qquad u_p^\dagger u_p=v_p^\dagger v_p=\frac{p_0}{m} . \tag{3.4}$$

For the massive case the free spinor field is given the Fock decomposition

$$\psi(x)=\int \frac{d^2p}{2\pi}\left[\frac{m}{p_0}\right]^{1/2}(c_pu_pe^{-ipx}+d_p^\dagger v_pe^{ipx}) . \tag{3.5}$$

Using the properties of (3.4) and the anticommutation relations

$$\{c_p,c_k^\dagger\}_+=\delta^2(\mathbf{k}-\mathbf{p}) ,\qquad \{d_p,d_k^\dagger\}_+=\delta^2(\mathbf{p}-\mathbf{k}) , \tag{3.6}$$

it is straightforward to show that the equal-time anticommutation relation

$$\{\psi_a^\dagger(\mathbf{x},t),\psi_b(\mathbf{y},t)\}_+=\delta_{ab}\delta^2(\mathbf{x}-\mathbf{y}) \tag{3.7}$$

is satisfied.

The Fock states associated with the algebra (3.6) are the charged particle states of the uncoupled theory. In the coupled theory these states must be dressed in order to satisfy Gauss's law. In the coupled theory described by the Lagrangian density (3.1a) Gauss's law takes the form

$$G_c=\kappa(\partial_1A_2-\partial_2A_1)+e\Psi^\dagger\Psi=0 . \tag{3.8}$$

Clearly, the equations of motion (2.2a) and (2.2b) are also altered in form for the coupled theory, but in the temporal gauge these equations of motion need not be imposed as constraints. The coupled equations for the time development of A_1 and A_2 will be enforced dynamically. Equation (3.8) then constitutes the constraint placed on the interacting theory and, as in the case of the free gauge field, may be implemented in either the weak or strong sense.

In the weak sense two physical states $|P\rangle_w$ and $|P'\rangle_w$ must satisfy (2.15) with G replaced by G_c. If the theory was uncoupled, the states of the theory would be given by the tensor product Fock space of the free physical states of both the gauge theory and the spinor theory. However, because of the presence of the spinor charge density in (3.8), such simple tensor product states will fail to obey Gauss's law for the coupled theory if there are charged particles present. The solution is to take the tensor product states and dress them with a unitary transformation involving the charge density and the disallowed ghosts. Since the goal is to create asymptotic or interaction picture states, the fields appearing in (3.8) will be replaced by the respective free fields. For the states satisfying the weak form of the constraint, the unitary transformation is then given by

$$V^{-1}(t)=\exp\left[-\tfrac{1}{2}ie\int d^2p[b(\mathbf{p})j_0(-\mathbf{p},t)+b^\dagger(\mathbf{p})j_0(\mathbf{p},t)]\right], \quad (3.9)$$

where

$$j_0(\mathbf{k},t)=\int\frac{d^2x}{2\pi}e^{i\mathbf{k}\cdot\mathbf{x}}\psi^\dagger(\mathbf{x},t)\psi(\mathbf{x},t)=j_0^\dagger(-\mathbf{k},t) . \quad (3.10)$$

It is straightforward to show that

$$V(t)[\kappa(\partial_1A_2-\partial_2A_1)+e\psi^\dagger\psi]V^{-1}(t)=\kappa(\partial_1A_2-\partial_2A_1) \quad (3.11)$$

if the decompositions (2.9) for the gauge field are used. As a result, in the interaction picture, the physical states of the coupled theory can be constructed from the physical states of the uncoupled theory by applying the V transformation. The physical states of the coupled theory in the weak sense are denoted $|P\rangle_w$, and, in the interaction picture, are given at time t by

$$|P\rangle_w=V^{-1}(t)|P_0\rangle_w , \quad (3.12)$$

where the state $|P_0\rangle_w$ is a physical state of the uncoupled theory of the form (2.19), and thus contains only g-type ghosts and possibly charged particles. Using (3.11) and the properties of the uncoupled physical states immediately establishes that the states (3.12) satisfy the coupled constraint weakly, i.e.,

$$_w\langle P|G_c|P'\rangle_w=0 , \quad (3.13)$$

when G_c is written in terms of the free fields. In effect, the unitary transformation shifts the allowed ghost operator $g(\mathbf{p})$ by the charge density in the manner

$$V(t)g(\mathbf{p})V^{-1}(t)=g(\mathbf{p})-\frac{e}{2\kappa p_1p_2}j_0(\mathbf{p},t) . \quad (3.14)$$

The disallowed b-type ghosts are unaffected.

It is straightforward to show that the same transformation can be applied to the uncoupled physical states (2.27) to obtain states which satisfy

$$G_c|P\rangle_s=0 , \quad (3.15)$$

where

$$|P\rangle_s=V^{-1}(t)|P_0\rangle_s , \quad (3.16)$$

and the states $|P_0\rangle_s$ are the physical states defined by (2.27) tensor producted with the charged states.

The equivalent form for Stokes's theorem may now be demonstrated for the coupled case. Clearly, a loop integral for a classical solution of Gauss's law should satisfy

$$\kappa\oint_L dx^iA_i=\kappa\int d^2x(\partial_1A_2-\partial_2A_1)=-e\int_{S(L)}d^2x\,\psi^\dagger\psi , \quad (3.17)$$

so that the loop integral should give the charge enclosed by the loop. It is easy to see from (2.34), (2.35), and (3.14) that the expectation value of the loop integral between physical states is given by

$$_w\langle P|\oint_L dx^iA_i|P'\rangle_w=-\frac{e}{\kappa}\int_{S(L)}d^2x_w\langle P|\psi^\dagger\psi|P'\rangle_w . \quad (3.18)$$

Thus, charged states develop a nontrivial Abelian holonomy. Since the gauge field is a connection for the spinor field it is possible to interpret the Wilson loop around a charge as the gauge field contribution to dragging a similar charge around a loop enclosing the first charge. Therefore the additional phase factor

$$\exp\left[-ie\oint_L dx^iA_i\right]=\exp\left[i\frac{e^2}{\kappa}\right] \quad (3.19)$$

should appear on the state containing two particles. This indicates that a phase of half that amount should appear on a single-particle state when rotated through 2π radians. This will be verified in Sec. V by the standard methods of quantum mechanics.

IV. S MATRIX AND EVOLUTION OPERATOR

In this section the physical states satisfying the weak form of the constraint, defined in the preceding two sections, will be used as asymptotic particles in formulating a scattering matrix. This is similar to manifestly covariant formulations of quantum electrodynamics where the asymptotic particles, in addition to possessing nontrivial infrared structure, must satisfy a Gupta-Bleuler condition which is identical in effect to Gauss's law.[11]

In this section only the charged states will be reduced, since it will be seen that the S-matrix elements of allowed ghosts are zero. The charged states may be reduced in a manner similar to the Lehmann-Symanzik-Zimmermann (LSZ) construction[13] while taking into account the presence of the unitary transformation V necessary to define the asymptotic states. The starting point is the observation that

$$V(t)\psi(\mathbf{x},t)V^{-1}(t)=e^{ieB(\mathbf{x})}\psi(\mathbf{x},t) , \quad (4.1)$$

where

$$B(\mathbf{x})=\tfrac{1}{2}\int\frac{d^2k}{2\pi}[b(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}}+b^\dagger(\mathbf{k})e^{-i\mathbf{k}\cdot\mathbf{x}}] . \quad (4.2)$$

The transformed spinor field of (4.1) is manifestly invariant under a gauge transformation, since (2.32) shows that

$$B(\mathbf{x})\to B(\mathbf{x})+\Lambda(\mathbf{x}) , \quad (4.3)$$

thus canceling the phase induced on the spinor field by a gauge transformation. Thus, since reducing spinor particle states requires commuting ψ past V, the resulting reduction formulas are manifestly gauge invariant. To give an explicit form, reducing a c-type particle from the in state and ignoring possible forward-scattering terms gives the standard time-ordered product

$$ {}_{\text{out}}\langle\alpha|T\{\cdots\}|\mathbf{p},\beta\rangle_{\text{in}} = i\int d^3x\, u_p e^{ipx}(-i\gamma^\mu\partial_\mu - m) \times {}_{\text{out}}\langle\alpha|T[\cdots\Psi^\dagger(x)e^{-ieB(\mathbf{x})}]|\beta\rangle_{\text{in}} , \quad (4.4) $$

where the in and out states are assumed to coincide with the physical states (3.12) in the interaction picture for the weak implementation of the constraint at suitably remote times in the past and future. Implicit in (4.4) is the assumption that the interpolating field Ψ approaches ψ, the free field, at these asymptotic times, modulo a wave-function renormalization factor.

A perturbative form for the reduced S matrix is achieved by the standard assumption that[13]

$$ \psi(\mathbf{x},t) = U(t)\Psi(\mathbf{x},t)U^{-1}(t) . \quad (4.5) $$

It follows that

$$ \dot U(t)U^{-1}(t) = i(H_0 - H) = -iH_{\text{int}}(t) , \quad (4.6) $$

where the interaction Hamiltonian is written in terms of the free fields ψ and A_j. It follows for this theory in the temporal gauge that

$$ H_{\text{int}} = -e\int d^2x\, \mathbf{j}\cdot\mathbf{A} , \quad (4.7) $$

where j_μ is the standard current density given by

$$ j_\mu = \bar\psi\gamma_\mu\psi , \quad (4.8) $$

and which satisfies

$$ \partial_\mu j^\mu = 0 . \quad (4.9) $$

The time-ordered product appearing in (4.4) can be given a perturbative representation through the replacement

$$ {}_{\text{out}}\langle\alpha|T[\cdots\Psi(x)e^{ieB(\mathbf{x})}]|\beta\rangle_{\text{in}} = {}_w\langle\alpha_0|V(t_+)U^{-1}(t_+)V^{-1}(t_+) \times T[V(t_+)U(t_+)\cdots\times U^{-1}(t)V^{-1}(t)\psi(x)V(t)U(t)U^{-1}(t_-)V^{-1}(t_-)] \times V(t_-)U(t_-)V^{-1}(t_-)|\beta_0\rangle_w , \quad (4.10) $$

where t_+ and t_- are times in the remote future and past used to define the in and out states which are now explicitly written in the form (3.12). Using the asymptotic properties of the fields it can be shown that the unitary operators outside the square brackets must reduce to simple factors. It follows that the evolution operator has the interaction picture representation

$$ E(t,t') = V(t)U(t)U^{-1}(t')V^{-1}(t') . \quad (4.11) $$

The evolution operator is most easily evaluated by noting that it solves the first-order differential equation

$$ \dot E(t,t') = -iH^I_{\text{eff}}(t)E(t,t') , \quad (4.12) $$

where

$$ H^I_{\text{eff}}(t) = i[V(t)\dot U(t)U^{-1}(t)V^{-1}(t) + \dot V(t)V^{-1}(t)] . \quad (4.13) $$

Integrating and iterating (4.12) gives

$$ E(t,t') = T\exp\left[-i\int_{t'}^{t} d\tau\, H^I_{\text{eff}}(\tau)\right] . \quad (4.14) $$

H^I_{eff} is the effective interaction due to the dressing and is easily evaluated. First, the interaction is written in terms of ghost operators to give

$$ H^I_g(t) = -i\dot U(t)U^{-1}(t) = e\int d^2x\, \mathbf{j}\cdot\mathbf{A} = \tfrac{1}{2}ie\int d^2k[\mathbf{k}\cdot\mathbf{j}(-\mathbf{k},t)b^\dagger(\mathbf{k}) - \mathbf{k}\cdot\mathbf{j}(\mathbf{k},t)b(\mathbf{k})] + \tfrac{1}{2}ie\int d^2k[k_1 j_1(\mathbf{k},t) - k_2 j_2(\mathbf{k},t)][g(\mathbf{k}) + g^\dagger(-\mathbf{k})] , \quad (4.15a) $$

where

$$ \mathbf{j}(\mathbf{p},t) = \int \frac{d^2x}{2\pi}\mathbf{j}(\mathbf{x},t)e^{-i\mathbf{p}\cdot\mathbf{x}} . \quad (4.15b) $$

It follows directly from the properties of the V transformation that the unitary transform of (4.15) is given by

$$ -iV(t)\dot U(t)U^{-1}(t)V^{-1}(t) = H^I_g(t) - \frac{e^2}{\kappa}\int d^2x\, d^2y[j_1(\mathbf{x},t)S_2(\mathbf{x}-\mathbf{y})j_0(\mathbf{y},t) - j_2(\mathbf{x},t)S_1(\mathbf{x}-\mathbf{y})j_0(\mathbf{y},t)] , \quad (4.16) $$

where the Green's functions S_j are given by

$$S_j(\mathbf{x}-\mathbf{y})=\int \frac{d^2k}{4\pi^2}\frac{-i}{k_j}e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})} . \tag{4.17}$$

Clearly, the function S_j satisfies

$$\frac{\partial}{\partial x_j}S_j(\mathbf{x}-\mathbf{y})=\delta^2(\mathbf{x}-\mathbf{y}) . \tag{4.18}$$

It is shown in the Appendix that the distribution defined by (4.17) has the representation

$$S_j(\mathbf{x}-\mathbf{y})=\frac{1}{\pi}\frac{x_j-y_j}{|\mathbf{x}-\mathbf{y}|^2} . \tag{4.19}$$

This is true when the distribution is restricted to a test function space of symmetric functions, i.e., functions which are invariant under $x_1 \Longleftrightarrow x_2$ exchange symmetry and when the integration of the distribution against the space of test functions covers a square patch of R^2. Such a test function space includes rotationally invariant functions.

The effective interaction can now be found by noting that

$$\dot{V}(t)V^{-1}(t)=\tfrac{1}{2}ie\int d^2k[b(\mathbf{k})\partial_0 j_0(-\mathbf{k},t) + b^\dagger(\mathbf{k})\partial_0 j_0(\mathbf{k},t)] . \tag{4.20}$$

From the conservation of current (4.9) it follows that

$$\partial_0 j_0(\mathbf{p},t)=-\int \frac{d^2x}{2\pi}e^{-i\mathbf{p}\cdot\mathbf{x}}\partial_k j_k(\mathbf{x},t) , \tag{4.21}$$

which is easily integrated by parts to yield

$$\partial_0 j_0(\mathbf{p},t)=-i\mathbf{p}\cdot\mathbf{j}(\mathbf{p},t) . \tag{4.22}$$

Upon substitution of (4.22) into (4.20) and subsequent substitution of (4.20) into (4.13) along with forms (4.15a) and (4.16) it follows that the terms in H^I_{eff} proportional to b-type ghosts, originally present in H^I_g, are canceled. From the algebra (2.18) the effective interaction therefore commutes with all g-types ghosts, and the S matrix for ghost states is therefore zero. This is identical in outcome to quantum electrodynamics when the constraint is correctly implemented.[11] Furthermore, the part of H^I_{eff} which contains the g-type ghosts gives a zero contribution to any physical process and can therefore simply be dropped.

The sole remaining term which can contribute to physical scattering processes is given by

$$H^I_{\text{eff}}=-\frac{e^2}{2\kappa}\int d^2x\, d^2y\, \epsilon^{\mu\nu\rho}j_\mu(\mathbf{x},t)S_\nu(\mathbf{x}-\mathbf{y})j_\rho(\mathbf{y},t) , \tag{4.23}$$

where $S_0=0$ is implicit. The interaction (4.23) has arisen in much the same way the instantaneous Coulomb interaction appears in quantum electrodynamics, by applying the constraint correctly and decoupling the unphysical ghosts from the S matrix.[11]

Interaction (4.23) is a velocity-dependent potential and some insight into its nature may be gained by treating the currents appearing in (4.23) as if they described two classical (spinless) point particles in relative motion. For such a case the interaction (4.23) becomes, after subtracting off the self-energy of each particle,

$$H^I_{\text{eff}}=-\frac{e^2}{\kappa}\frac{|\mathbf{v}\times\mathbf{r}|}{\pi r^2} , \tag{4.24}$$

where $\mathbf{v}$ is the relative velocity of the two particles and $\mathbf{r}$ is their relative separation in the plane. For the especially simple case that one of the particles is at rest at the origin expression (4.24) reduces to

$$\mathcal{L}_{\text{eff}}=-H^I_{\text{eff}}=\frac{e^2}{\pi\kappa}\dot{\theta} \tag{4.25}$$

when expressed in polar coordinates. It follows that for such a classical situation the angular momentum of the moving particle is given by

$$p_\theta=\frac{\partial\mathcal{L}}{\partial\dot{\theta}}=mr^2\dot{\theta}+\frac{e^2}{\pi\kappa} , \tag{4.26}$$

which is the first indication that the angular momentum and thus possibly the statistics of the particles, has been altered by the gauge interaction. In the next section this argument is extended to the quantized theory.

V. TOTAL ANGULAR MOMENTUM AND STATISTICS

The intent of this section is quite simple. The total angular momentum operator for the coupled theory will be identified and evaluated for states which satisfy the strong form of the constraint. It is necessary to use the strong form of the constraint since the states must be eigenstates of total angular momentum in order to evaluate the effect of rotations. Two very minor subtleties arise. The first is in identifying the total angular momentum operator. This is done by using the Noether current associated with rotations about the timelike direction in the plane and demanding that this total angular momentum operator be both conserved and gauge invariant, at least when applied to a physically allowed state. It will be seen that the demand for gauge invariance is equivalent to demanding that the state satisfy the strong version of the constraint. The second subtlety arises in evaluating the angular momentum operator using the Fock decompositions of the fields, since ambiguities associated with integrating by parts can develop. These ambiguities are removed by demanding that the final result be gauge invariant over physical states. Using these rather mild criteria and the Green's function (4.19) it will be seen that physically allowed charged states develop the induced angular momentum $Q^2/4\pi\kappa$, where Q is the total charge operator.

The total angular momentum operator is defined as the generator of rotations about the timelike direction in the theory. The contribution to the total angular momentum from the spinor fields is given by

$$\begin{aligned} L_s &= -i\int d^2x\, \Psi^\dagger(x_1\partial_2-x_2\partial_1)\Psi , \\ S_s &= -\tfrac{1}{2}\int d^2x\, \Psi^\dagger\gamma_0\Psi , \end{aligned} \tag{5.1}$$

where L_s and S_s are the orbital and spin angular momenta, respectively, of the spinor field. For the uncoupled theory the sum L_s+S_s is time independent, a fact which follows directly from the equations of motion (3.2b) for the free spinor field.

Likewise, the angular momentum for the gauge field, defined from the usual Noether current, is given by

$$L_g=\frac{\kappa}{2}\int d^2x[A_1(x_1\partial_2-x_2\partial_1)A_2-A_2(x_1\partial_2-x_2\partial_1)A_1]\ ,$$
$$S_g=-\frac{\kappa}{2}\int d^2x(A_1^2+A_2^2)\ , \tag{5.2}$$

where L_g and S_g are the orbital and spin angular momenta, respectively. Again, for the uncoupled theory, the sum L_g+S_g is trivially conserved.

For the coupled theory only the total angular momentum $J=L_g+S_g+L_s+S_s$ is conserved. It is tedious but straightforward to verify this using the equations of motion in the temporal gauge

$$\dot{A}_j=-\frac{ie}{\kappa}\Psi^\dagger\gamma_j\Psi\ , \tag{5.3a}$$

$$[i\gamma^\mu(\partial_\mu-ieA_\mu)-m]\Psi=0\ , \tag{5.3b}$$

and the property of the γ matrices that $(\gamma_j)^\dagger=-\gamma_j$.

It is also necessary to verify the invariance of J under time-independent gauge transformations of the form (3.1b). It is straightforward to show that a gauge transformation on the total angular momentum J induces the change

$$J\to J+\int d^2x(x_2\partial_1\Lambda-x_1\partial_2\Lambda)\times[\kappa(\partial_1A_2-\partial_2A_1)+e\Psi^\dagger\Psi]\ , \tag{5.4}$$

so that J is gauge invariant only when applied to states which satisfy the strong form of the constraint. Thus the candidate eigenstates of J will be the states dressed with the unitary transformation V^{-1} and the transformation D defined by (2.24), i.e., the physical states (3.16) of the strong implementation.

The total angular momentum J will be evaluated in the interaction picture or, equivalently, for the asymptotic fields of the theory. This is done by using the free field expansions for the fields. It follows that the gauge field angular momentum L_g+S_g, when expanded using (2.9), reduces to

$$L_g+S_g=-\frac{\kappa}{4}\int d^2p(p_1^2+p_2^2)\times[g(-\mathbf{p})+g^\dagger(\mathbf{p})][g(\mathbf{p})+g^\dagger(-\mathbf{p})]+\frac{\kappa}{4}\int d^2p(p_1^2-p_2^2)[g(-\mathbf{p})+g^\dagger(\mathbf{p})]\times[b(\mathbf{p})+b^\dagger(-\mathbf{p})]\ . \tag{5.5}$$

In evaluating (5.5) the following symmetrization convention has been used. If $f(\mathbf{x})$ and $g(\mathbf{x})$ have the Fourier transforms $f(\mathbf{p})$ and $g(\mathbf{p})$, respectively, then the integral of these functions against x_j is given by

$$\int d^2x\, f(\mathbf{x})x_jg(\mathbf{x})=-\tfrac{1}{2}i\int d^2k\, f(\mathbf{k})\left[\frac{\overrightarrow{\partial}}{\partial k_j}-\frac{\overleftarrow{\partial}}{\partial k_j}\right]g(-\mathbf{k})\ . \tag{5.6}$$

The symmetrization condition (5.6) alleviates ambiguities in evaluating the spatial integrals which can arise from integration by parts. That (5.5) is indeed the correct form in terms of the modes of the fields can be determined by performing a gauge transformation on (5.5). Using (2.32) it is straightforward to show that the resulting expression is identical to the spatial form of the transformed gauge field angular momentum. The second integral appearing in (5.5) can be written as

$$\frac{\kappa}{4}\int d^2p(p_1^2-p_2^2)[g(-\mathbf{p})+g^\dagger(\mathbf{p})][b(\mathbf{p})+b^\dagger(-\mathbf{p})]=-\kappa\int d^2x(\partial_1A_2-\partial_2A_1)(x_1\partial_2B-x_2\partial_1B)\ ,$$

where B is the field operator (4.2) composed of b-type ghosts.

Now these operators, as well as the spinor angular momentum, must be commuted past both the V^{-1} operator and the D operator before they reach the bare Fock state. The action of the V dressing operator is presented first. Using (3.14) the gauge field angular momentum becomes (suppressing the time arguments)

$$V(L_g+S_g)V^{-1}=L_g+S_g+e\int d^2x\,\psi^\dagger\psi[x_1\partial_2B(\mathbf{x})-x_2\partial_1B(\mathbf{x})]+\frac{e}{2}\int d^2p\frac{p_1^2+p_2^2}{p_1p_2}[g(-\mathbf{p})+g^\dagger(\mathbf{p})]j_0(\mathbf{p},t)-\frac{e^2}{4\kappa}\int d^2p\frac{p_1^2+p_2^2}{p_1^2p_2^2}j_0(\mathbf{p},t)j_0(-\mathbf{p},t)\ . \tag{5.7}$$

Next the effect of the V^{-1} operator on the spinor field angular momentum is calculated. From property (4.1) it follows that the V^{-1} operator is like a gauge transformation, so that

$$V(L_s+S_s)V^{-1}=L_s+S_s-e\int d^2x[x_1\partial_2B(\mathbf{x})-x_2\partial_1B(\mathbf{x})]\psi^\dagger\psi\ . \tag{5.8}$$

Clearly, the third term on the right-hand side of (5.8) cancels the same term in (5.7). Since neither L_s nor S_s has any reference to gauge field operators they both commute with the D operator. From Sec. II the action of the D operator is simply to remove the presence of any creation operators in the combinations $g(-p)+g^\dagger(\mathbf{p})$ and $b(-\mathbf{p})+b^\dagger(\mathbf{p})$. Therefore, the form for J after commuting it through both V^{-1} and D is given by

$$D^{-1}V(t)JV^{-1}(t)D=L_s+S_s-\frac{\kappa}{4}\int d^2p(p_1^2+p_2^2)g(\mathbf{p})g(-\mathbf{p})+\frac{\kappa}{4}\int d^2p(p_1^2-p_2^2)b(\mathbf{p})g(-\mathbf{p})$$
$$+\frac{e}{2}\int d^2p\frac{p_1^2+p_2^2}{p_1p_2}j_0(\mathbf{p},t)g(-\mathbf{p})-\frac{e^2}{4\kappa}\int d^2p\frac{p_1^2+p_2^2}{p_1^2p_2^2}j_0(\mathbf{p},t)j_0(-\mathbf{p},t)\ . \tag{5.9}$$

It is clear that the terms which contain g-type annihilation operators will be zero over the physically allowed Fock states. Therefore, after commuting the total angular momentum past the two dressing operators, only the spinor angular momentum L_s and S_s, written in terms of the free field and acting on a Fock state, will contribute, along with the last term in (5.9). Therefore, only charged states possess angular momentum. The last term in (5.9) is the angular momentum induced by the gauge field necessary to dress a charged state. Denoted J_{ind}, it can be written in a more transparent way as

$$J_{\text{ind}}=-\frac{e^2}{4\kappa}\int d^2x\, d^2y\, j_0(\mathbf{x},t)j_0(\mathbf{y},t)\int\frac{d^2p}{4\pi^2}\left[\frac{1}{p_1^2}+\frac{1}{p_2^2}\right]e^{-i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}\ . \tag{5.10}$$

Using the identity

$$\frac{1}{p_j^2}=-\frac{\partial}{\partial p_j}\left[\frac{1}{p_j}\right] \tag{5.11}$$

and integrating by parts, it follows that

$$J_{\text{ind}}=\frac{ie^2}{4\kappa}\int d^2x\, d^2y\, j_0(\mathbf{x},t)j_0(\mathbf{y},t)\int\frac{d^2p}{4\pi^2}\left[\frac{(x_1-y_1)}{p_1}+\frac{(x_2-y_2)}{p_2}\right]e^{-i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}\ . \tag{5.12}$$

Using (4.17) as the Fourier representation of the Green's function it follows that

$$J_{\text{ind}}=\frac{e^2}{4\kappa}\int d^2x\, d^2y\, j_0(\mathbf{x},t)(x_j-y_j)S_j(\mathbf{x}-\mathbf{y})j_0(\mathbf{y},t)\ , \tag{5.13}$$

so that, if the form (4.19) for the Green's function is used, then the final result for the induced angular momentum is given by

$$J_{\text{ind}}=\frac{e^2}{4\pi\kappa}\int d^2x\, d^2y\, j_0(\mathbf{x},t)j_0(\mathbf{y},t)=\frac{Q^2}{4\pi\kappa}\ , \tag{5.14}$$

which is the result obtained by other authors by various means.[14] It shows that a state with two similarly charged particles, $\langle Q\rangle=2e$, contains the additional angular momentum $e^2/\pi\kappa$, in complete agreement with the classical result (4.26). The exchange of these particles, for the case that the two particles are represented by plane-wave states, is effected in two spatial dimensions by rotating the state through π radians. This is accomplished by applying the operator $\exp(i\pi J)$. When this is done the state of two similarly charged particles picks up the additional phase e^2/κ, which can give rise to fractional statistics. The state of a single charged particle picks up the phase $e^2/2\kappa$ when rotated through 2π radians in agreement with the Wilson loop argument of Sec. III.

It would be of interest to generalize these results in three ways. The first would be to develop a Fock-space representation of the non-Abelian version of the Chern-Simons theory and verify the presence of induced spin. The second would be to apply the uncoupled Abelian case to nontrivial spatial manifolds. The third would be to take the second result and understand the role of conformal invariance in the structure of the states.

APPENDIX

In this appendix the form (4.17) for the Green's function of the theory will be evaluated. The starting point is the Fourier representation

$$S_j(\mathbf{x}-\mathbf{y})=\int\frac{d^2k}{4\pi^2}\frac{-i}{k_j}e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}\ . \tag{A1}$$

This Fourier representation defines a distribution which satisfies the defining differential equation

$$\partial_jS_j(\mathbf{x}-\mathbf{y})=\delta^2(\mathbf{x}-\mathbf{y})\ . \tag{A2}$$

The goal of this appendix is to construct a Coulomb-like distribution which satisfies this differential equation. The form (A1) will be replaced by the Fourier representation

$$S_j(\mathbf{x}-\mathbf{y})\to\int\frac{d^2k}{4\pi^2}\frac{-2ik_j}{k_1^2+k_2^2}e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}\ . \tag{A3}$$

Clearly, (A3) and (A1) are formally different distributions and can only coincide for a restricted class of test functions. In effect, (A3) can be viewed as a regularization of (A1) for this class of test functions by softening its singularity structure. The class of test functions will now be made explicit. If $f(\mathbf{x})$ is a member of the space of symmetric test functions, i.e., those functions with the property that $f(x_1,x_2)=f(x_2,x_1)$, then the function $g_j(\mathbf{x})$, defined as

$$g_j(\mathbf{x})=\int d^2y\, S_j(\mathbf{x}-\mathbf{y})f(\mathbf{y})\ , \tag{A4}$$

where the definition (A3) is used for S_j, satisfies the expression

$$\frac{\partial}{\partial x_j} g_j(\mathbf{x}) = f(\mathbf{x}) \ , \tag{A5}$$

where there is no summation over the index j. The proof is straightforward and uses the fact that the Fourier transform of the function f must also be symmetric in its arguments. For this class of test functions the two distributions (A1) and (A3) coincide in the sense that both satisfy the differential form (A2).

The evaluation of the function (A3) is elementary, giving

$$S_j(\mathbf{x}-\mathbf{y}) = \frac{1}{\pi} \frac{x_j - y_j}{|\mathbf{x}-\mathbf{y}|^2} \ . \tag{A6}$$

As a final demonstration of the fact that (A6) is the properly normalized Green's function for ∂_j, the argument $\mathbf{y}$ is set to zero and the expression $\partial_2 S_2$ is integrated over a square patch of side $2L$ centered on the origin. This gives, by elementary results,

$$\int_{-L}^{L} dx_1 \int_{-L}^{L} dx_2 \frac{\partial}{\partial x_2} \frac{1}{\pi} \frac{x_2}{x_1^2 + x_2^2} = \int_{-L}^{L} dx_1 \frac{1}{\pi} \frac{2L}{x_1^2 + L^2} = \frac{4}{\pi} \arctan(1) = 1 \ . \tag{A7}$$

The result is independent of the size of the patch, demonstrating the presence of the normalized Dirac delta at the origin of the derivative. It is critical that a square patch be used since any other shape will yield a result different in value from (A7). Thus, given these restrictions, the Green's function takes the form (A6).

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