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Energy density of relic gravity waves from inflation

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We evaluate both the spectral energy density and the total energy density for relic gravity waves produced during the transition from an early inflationary phase to a matter-dominated Friedmann-Robertson-Walker-type expansion: $a \sim t^c$ $(c < 1)$. We find that for power-law inflation the spectral energy density for gravity waves has more power on larger scales than for purely exponential inflation. Evaluating the energy density of created massless particles (both gravitons and massless scalars) we find that in the case of exponential inflation the ratio of the density of created particles to the total density of matter is a constant, if $c \geq \frac{1}{2}$. This unusual behavior is a consequence of the fact that the equation of state for created particles mimics the equation of state for matter driving the expansion of the Universe. As a result, self-consistent solutions of the Einstein equations can be found, in which the expansion of the Universe is sustained solely by the ongoing production of massless particles, so that $G_{\mu\nu} = 8\pi G \langle T_{\mu\nu} \rangle$. In the case of power-law and quasiexponential inflation we find that the ratio of the energy density of gravity waves to the background matter density increases with time, as gravity waves with longer wavelengths and larger amplitudes enter the horizon at successively later epochs. This could lead to the energy density of gravity waves becoming comparable to the energy density of matter at late times, if inflation commenced at Planckian energies.

I. INTRODUCTION

A fundamental consequence of inflation, first investigated by Starobinsky,[1] is the production of relic gravity waves, which in the case of exactly exponential inflation have a scale-invariant spectrum and enter our horizon with a dimensionless amplitude $h \sim GH_0$ (H_0 being the Hubble constant for de Sitter space). In the case of power-law inflation, gravity waves are created with a non-scale-invariant spectrum, and upon horizon crossing have a time-dependent amplitude $h \sim GH_{\rm inf}(t_*)$, $H_{\rm inf}(t_*)$ being the Hubble parameter at a time t_*, when a wave reentering the Friedmann-Robertson-Walker (FRW) horizon today crossed the horizon during the inflationary era. As shown in Refs. 2 and 3, interesting upper limits on the dimensionless amplitude of gravity waves just entering the FRW horizon can be obtained by evaluating the distortion to the cosmic-microwave-background radiation (MBR) caused by these waves via the Sachs-Wolfe effect,[4] and comparing this distortion with the observed upper limits on $\Delta T/T$. As a result, a model-independent bound on h can be obtained, $h < 10^{-4}$, corresponding to $H_{\rm inf}(t_*) < 10^{-4} m_p$ for the Hubble parameter during inflation.[3]

We extend previous work on this subject by evaluating the energy density of gravity waves created during the transition from inflation to an almost arbitrary FRW expansion. One of our main results is that the energy density of gravity waves falls off slower than the corresponding background energy density of matter driving the FRW expansion. This is true both in the case of power-law inflation and for quasiexponential inflation. In the idealized case of a purely exponential inflation, the ratio of the resulting graviton energy density to that of the background matter density remains fixed during the course of expansion, if the equation of state of matter is weaker than that of radiation: $p < \epsilon/3$, confirming earlier results by Allen[5] and Starobinsky.[6]

II. THE ENERGY DENSITY AND SPECTRUM OF GRAVITONS FROM GENERALIZED INFLATION

The quantum gravitational creation of gravitons in expanding FRW cosmologies was first demonstrated by Grishchuk[7] who also showed that in the linearized approximation, the behavior of gravity waves propagating in a FRW universe becomes identical to that of a massless, minimally coupled scalar field, with each of the two polarization states of the metric tensor satisfying the Klein-Gordon equation: $h_{ik;l}^{;l} = 0$. The quantization of gravity waves and that of minimally coupled massless scalar fields also proceeds along identical lines as was demonstrated by Ford and Parker.[8]

In order to study the production of gravitons from inflation we shall assume the background space-time geometry to be described by a spatially flat FRW metric

$$ds^2 = dt^2 - a^2(t) d\mathbf{x}^2 = a^2(\eta)(d\eta^2 - d\mathbf{x}^2) , \quad (1)$$

where $dt = a\, d\eta$. For $t < t_0$ the expansion of the Universe is assumed to be inflationary, so that $a \propto e^{Ht}$ in the case of exponential inflation, or $a \propto t^c$, $c > 1$ in the case of power-law inflation. As demonstrated by Ford,[9] towards the end of inflation, during the epoch of reheating, the equation of state of the Universe is very model dependent, and can vary all the way from $p = 0$ (dust) to $p = \epsilon$ (stiff matter). After reheating, the Universe enters a radiation-dominated epoch, and later still, the present dust-dominated epoch. Since the effective equation of state of matter in the Universe varies considerably from epoch to epoch after inflation, we shall not specialize to any particular expansion law during the FRW stage, but shall work instead with the flexible assumption: $a \propto t^c$ for $t > t_0$, with $c < 1$.

The Klein-Gordon equation in a FRW universe yields to a separation of variables so that solutions of $h_{mn;l}^{;l}=0$ in metric (1) satisfy $h_{mn}=\phi_k(\eta)e^{-i\mathbf{k}\cdot\mathbf{x}}e_{mn}$, where e_{mn} is the polarization tensor, and $\phi_k(\eta)$ satisfies the second-order differential equation

$$\ddot{\phi}_k+2\frac{\dot{a}}{a}\dot{\phi}_k+k^2\phi_k=0 , \tag{2}$$

where the differentiation is carried out with respect to the conformal time coordinate η. k is the comoving wave number $k=2\pi a/\lambda$, λ being the physical wavelength of the gravitational wave. For power-law expansion $a=(t/t_0)^c \equiv (\eta/\eta_0)^{(1-2\nu)/2}$, $[\nu=(1-3c)/2(1-c)]$, Eq. (2) can be solved exactly, resulting in the following solution for the adiabatic vacuum state:

$$\phi_k^+(\eta)=\left[\frac{\pi\eta_0}{4}\right]^{1/2}\left[\frac{\eta}{\eta_0}\right]^{\nu}H_\nu^{(2)}(k\eta) \tag{3}$$

($\nu \geq \frac{3}{2}$ if $c>1$). It may be noted that the fairly narrow range $\frac{3}{2} \leq \nu \leq \frac{5}{2}$ reflects a broad range of powers: $c > =2$. For $\nu=\frac{3}{2}$ ($c=\infty$) we recover the adiabatic vacuum in de Sitter space[10]

$$\phi_k^+(\eta)=\left[\frac{\pi\eta_0}{4}\right]^{1/2}\left[\frac{\eta}{\eta_0}\right]^{3/2}H_{3/2}^{(2)}(k\eta) . \tag{4}$$

At early times $\eta<\eta_0<0$, Eqs. (3) and (4) define the scalar field in its vacuum state. At late times $\eta>|\eta_0|$, the scalar field will no longer be in its vacuum state and will in general be described by a linear superposition of positive- and negative-frequency solutions of (2):

$$\tilde{\phi}_k(\eta)=c_1\tilde{\phi}_k^{(+)}+c_2\tilde{\phi}_k^{(-)} , \tag{5}$$

where

$$\tilde{\phi}_k^{(+,-)}(\eta)=\left[\frac{\pi\eta_0}{4}\right]^{1/2}\left[\frac{\eta}{\eta_0}\right]^{\mu}H_{|\mu|}^{(2,1)}(k\eta)$$

and $\mu \leq 0$ corresponding to $\frac{1}{3} \leq c < 1$, $(\eta \in \{|\eta_0|,\infty|\})$. $\tilde{\phi}(\eta)$ satisfies the Wronskian normalization condition $W_\eta(\tilde{\phi},\tilde{\phi}^*)=i/a^2$, so that $|c_1|^2-|c_2|^2=1$. In order to determine the Bogoliubov coefficients c_1 and c_2 we follow Starobinsky[1] and Grishchuk[7] and note that for wavelengths greater than the horizon scale ($k|\eta|<2\pi$) at any given time, Eq. (2) has the simple solution

$$\phi(\eta) \underset{k\to 0}{\sim} A(k)+B(k)\int\frac{d(\eta/\eta_0)}{a^2} , \tag{6}$$

where A and B are constants. For modes in the "in" region defined by (3) and (4), we may determine A and B after using the small-argument limit of the Hankel functions:

$$\begin{aligned} H_\nu^{(2)}(k\eta)&=J_\nu(k\eta)-iY_\nu(k\eta)\\ &\underset{k\to 0}{\sim}\frac{(k\eta/2)^\nu}{\Gamma(1+\nu)}+\frac{i}{\pi}\Gamma(\nu)\left[\frac{k\eta}{2}\right]^{-\nu} ;\\ H_\nu^{(1)}(k\eta)&=H_\nu^{(2)*}(k\eta) , \end{aligned} \tag{7}$$

as a result we find

$$\begin{aligned} \phi_k^+ &\underset{k\to 0}{\sim} A+B\int\frac{d(\eta/\eta_0)}{a^2} ,\\ A(k\eta_0)&=\frac{i}{\pi}\Gamma(\nu)\left[\frac{k\eta_0}{2}\right]^{-\nu}\left[\frac{\pi\eta_0}{4}\right]^{1/2} ,\\ B(k\eta_0)&=\frac{2\nu}{\Gamma(1+\nu)}\left[\frac{k\eta_0}{2}\right]^{\nu}\left[\frac{\pi\eta_0}{4}\right]^{1/2} . \end{aligned} \tag{8a}$$

For de Sitter space, $\nu=\frac{3}{2}$ so that $A=i\sqrt{\eta_0/2}(k\eta_0)^{-3/2}$ and $B=\sqrt{\eta_0/2}(k\eta_0)^{3/2}$. Likewise, for modes in the "out" region defined by (5),

$$\tilde{\phi}^{\pm}(\eta)\underset{k\to 0}{\sim}\tilde{A}(k)\pm\tilde{B}(k)\int\frac{d(\eta/\eta_0)}{a^2} , \tag{8b}$$

where

$$\begin{aligned} \tilde{A}(k\eta_0)&=\left[\frac{\pi\eta_0}{4}\right]^{1/2}\frac{1}{\Gamma(1+|\mu|)}\left[\frac{k\eta_0}{2}\right]^{|\mu|} ,\\ \tilde{B}(k\eta_0)&=\left[\frac{\pi\eta_0}{4}\right]^{1/2}\frac{2i}{\pi}\Gamma(1+|\mu|)\left[\frac{k\eta_0}{2}\right]^{-|\mu|}\\ &\qquad (\mu<0) . \end{aligned}$$

As a result

$$\tilde{\phi}(\eta)\underset{k\to 0}{\sim}(c_2+c_1)\tilde{A}+(c_2-c_1)\tilde{B}\int\frac{d(\eta/\eta_0)}{a^2} . \tag{8c}$$

c_1 and c_2 can now be determined by matching ϕ^+ and $\tilde{\phi}$ on wavelength scales greater than that of the horizon: $\lim_{k\to 0}\phi^+=\tilde{\phi}$, so that, for $\mu<0$,

$$\begin{aligned} c_2+c_1&=\frac{A}{\tilde{A}}=i\gamma\left[\frac{k\eta_0}{2}\right]^{-(\nu+|\mu|)} ,\\ c_2-c_1&=\frac{B}{\tilde{B}}=-i\gamma^{-1}\left[\frac{k\eta_0}{2}\right]^{(\nu+|\mu|)} , \end{aligned} \tag{9}$$

giving

$$c_{2,1}=\frac{i}{2}\left[\gamma\left[\frac{k\eta_0}{2}\right]^{-(\nu+|\mu|)}\mp\gamma^{-1}\left[\frac{k\eta_0}{2}\right]^{(\nu+|\mu|)}\right] \tag{10a}$$

for $k\eta_0<2\pi$, where $\gamma=\pi^{-1}\Gamma(\nu)\Gamma(1+|\mu|)$, and $|c_1|^2-|c_2|^2=1$. For $k\eta_0>2\pi$ the adiabatic theorem[11] gives $c_1 \simeq 1, c_2 \simeq 0$. For $\mu=0$, corresponding to matter with a stiff equation of state $p=\epsilon$,

$$c_{2,1}=\frac{i}{2}\left[\gamma\left[\frac{k\eta_0}{2}\right]^{-\nu}\mp\gamma^{-1}\left[1\mp\frac{2i}{\pi}\ln(k\eta_0)\right]\left[\frac{k\eta_0}{2}\right]^{\nu}\right]. \quad (10b)$$

Setting $\nu=\frac{3}{2}$ in (10) defines the Bogoliubov coefficients corresponding to the transition from de Sitter space to a FRW-type expansion.

From (10a) and (5) we finally obtain, for modes defined in the "out" region,

$$\tilde{\phi}_k(\eta)=\left[\frac{\pi\eta_0}{4}\right]^{1/2}\left[\frac{\eta}{\eta_0}\right]^{-|\mu|}\left[i\gamma\left[\frac{k\eta_0}{2}\right]^{-(\nu+|\mu|)}J_{|\mu|}(k\eta)+\gamma^{-1}\left[\frac{k\eta_0}{2}\right]^{\nu+|\mu|}Y_{|\mu|}(k\eta)\right], \quad (10c)$$

where $J_{|\mu|}(k\eta)$ and $Y_{|\mu|}(k\eta)$ are Bessel functions of the first and second kind, respectively.

The energy density of the created gravitons is given by[8,12]

$$\epsilon_g=\langle T^0_0\rangle =\frac{1}{2\pi^2a^2}\int_{2\pi\eta^{-1}}^{\eta_0^{-1}}dk\, k^2(|\dot{\tilde{\phi}}_k|^2+k^2|\tilde{\phi}_k|^2), \quad (11)$$

where we have included contributions arising from both polarization states of the graviton.

There are physical reasons for the momentum cutoffs at high and low k in (11), which have been previously discussed in a similar context by Allen.[5] The low-frequency cutoff corresponds to the scale of the horizon today ($k\eta=2\pi$), and appears because gravity waves with wavelengths longer than the horizon scale appear locally to be a gauge transformation and so do not contribute to the local energy density. The high-frequency cutoff in (11) arises because particle creation is adiabatically suppressed on small scales.[7,11] For gravitons created during the transition of the Universe from an inflationary regime to a FRW-style expansion, a natural high-frequency cutoff exists, and is set by the value of the Hubble parameter at the time of transition: $H\simeq\eta_0^{-1}$. The exact value of the high-frequency cutoff is in fact somewhat ambiguous, depending among other things on the duration of the reheating phase after inflation. For this reason previous authors have sometimes assumed slightly different values for the high-frequency cutoff scale. However, this ambiguity in the choice of the high-frequency cutoff does not affect the results of the present calculation since the main contribution to ϵ_g comes from the lower frequency cutoff in (11) which is well defined.

Substituting $\tilde{\phi}_k$ from (10c) into (11) we obtain[13] (for $\mu<0$)

$$\epsilon_g=\epsilon_g^{(1)}+\epsilon_g^{(2)},$$

where

$$\epsilon_g^{(1)}=\frac{\eta_0\gamma^2}{8\pi a^2}\left[\frac{\eta}{\eta_0}\right]^{-2|\mu|}\int_{2\pi\eta^{-1}}^{\eta_0^{-1}}dk\, k^4\left[\frac{k\eta_0}{2}\right]^{-2(\nu+|\mu|)}[J^2_{|\mu|}(k\eta)+J^2_{|\mu|+1}(k\eta)],$$

$$\epsilon_g^{(2)}=\frac{\eta_0\gamma^{-2}}{8\pi a^2}\left[\frac{\eta}{\eta_0}\right]^{-2|\mu|}\int_{2\pi\eta^{-1}}^{\eta_0^{-1}}dk\, k^4\left[\frac{k\eta_0}{2}\right]^{2(\nu+|\mu|)}[Y^2_{|\mu|}(k\eta)+Y^2_{|\mu|+1}(k\eta)], \quad (12)$$

where $\gamma=\pi^{-1}\Gamma(\nu)\Gamma(1+|\mu|)$.

For the range of interest $\nu+|\mu|\geq 2$, the main contribution to ϵ_g comes from $\epsilon_g^{(1)}$ which after the transformation $x=k\eta$ can be recast in the form

$$\epsilon_g^{(1)}=2^{2(\nu+|\mu|)}\frac{\gamma^2}{(\nu-\frac{1}{2})^2}\frac{H^2_{\rm inf}(\eta)}{8\pi a^2\eta^2}\times\int_{2\pi}^{\eta/\eta_0}dx\, x^{4-2(\nu+|\mu|)}\times[J^2_{|\mu|}(x)+J^2_{|\mu|+1}(x)], \quad (13)$$

where $H_{\rm inf}$ is the Hubble parameter during inflation:

$$H_{\rm inf}(\eta)=\frac{\frac{1}{2}-\nu}{\eta}\left[\frac{\eta}{\eta_0}\right]^{\nu-1/2}, \quad \nu\geq\tfrac{3}{2}\ \ (\eta<\eta_0<0).$$

The integral in (13) can be evaluated exactly if we note that $Z^2_{|\mu|}(x)+Z^2_{|\mu|+1}(x)=2/\pi x$, $(Z_{|\mu|}\equiv J_{|\mu|},Y_{|\mu|})$, for the range of interest $|\mu|\leq\frac{3}{2}$, corresponding to physically realistic equations of state for ordinary matter: $p=\alpha\epsilon$, $0\leq\alpha<1$. Since the main contribution to the integral in (13) comes from the lower limit $x=2\pi$ we obtain

$$\epsilon_g\simeq C\frac{H^2_{\rm inf}(\eta)}{8\pi^2a^2\eta^2}, \quad (14a)$$

where, for $\nu+|\mu|>2$, $C=[16\gamma^2/(\nu-\frac{1}{2})^2](\pi^{-2\lambda}/\lambda)=\text{const}$, $\lambda=\nu+|\mu|-2>0$, and $\gamma=\pi^{-1}\Gamma(\nu)\Gamma(1+|\mu|)$.

For $\nu+|\mu|=2$ [as, for instance, in the case of exponential inflation ($\nu=\frac{3}{2}$) followed by a radiation-dominated FRW expansion ($\mu=-\frac{1}{2}$)],

$$C(\eta)=2^{1+2(\nu+|\mu|)}\frac{\gamma^2}{(\nu-\frac{1}{2})^2}\left[\ln\left[\frac{\eta}{\eta_0}\right]-\ln 2\pi\right]. \quad (14b)$$

This result can be rewritten in terms of the cosmological parameter $\Omega_g\simeq\epsilon_g/\epsilon_m$ where $\epsilon_m\simeq 3H^2/8\pi G$

$=3(|\mu|+\frac{1}{2})^2(8\pi G a^2\eta^2)^{-1}$ is the background matter density ($\Omega_m \simeq 1$ is assumed). We obtain, for $\nu+|\mu|>2$,

$$\Omega_g=\tilde{C}GH^2_{\rm inf}(\eta) , \tag{14c}$$

where

$$\tilde{C}=\frac{16\gamma^2}{3\lambda}\frac{\pi^{-2\lambda-1}}{(|\mu|+\frac{1}{2})^2(\nu-\frac{1}{2})^2} .$$

This result is both interesting and unusual since it indicates that $\Omega_g=\epsilon_g/\epsilon_m$ can increase with time. (A similar growth in Ω_g arises even in the absence of inflation, if the equation of state for the background matter is stiffer than that of radiation: $p>\epsilon/3$, as first noted by Grishchuk[7].)

The origin of this behavior lies in the fact that for $\nu+|\mu|>2$, the dominant contribution in $\epsilon_g^{(1)}$ arises from the vicinity of the lower limit $k=2\pi\eta^{-1}$, corresponding to those modes which are entering the horizon today. It is well known that once a given gravitational-wave mode leaves the inflationary horizon at some time $\eta=\eta_{\rm HC}<0$, its amplitude freezes out to a constant value determined by the Hubble parameter at the time of horizon crossing, so that

$$h^2=16\pi G|\Phi|^2=16\pi G\frac{k^3}{2\pi^2}|\phi_k|^2\Bigg|_{k|\eta|=2\pi}$$
$$=16\pi G A^2H^2_{\rm inf}(\eta_{\rm HC}) , \tag{15}$$

where $A=\Gamma(\nu)\pi^{-\nu}(\nu-\frac{1}{2})^{-1}$. h is the dimensionless amplitude for gravity waves, and following Ref. (3), $\eta_{\rm HC}$ denotes the time of horizon crossing: $k|\eta_{\rm HC}|=2\pi$. [(15) is readily derived from (4), (6), and (8)].

Later, after undergoing superadiabatic amplification, this mode reenters the horizon at a time $\eta=|\eta_{\rm HC}|$, and contributes to the energy density an amount $\epsilon_k\propto\omega^2h^2/16\pi G$ where $\omega=k/a=2\pi(a\eta)^{-1}$ is the associated frequency at horizon crossing. This establishes the origin of the $H^2_{\rm inf}(\eta)/a^2\eta^2$ term in (13) and (14).

As pointed out in Refs. 2 and 3 present observational constraints on the anisotropy of the microwave-background radiation indicate that $H_{\rm inf}(\eta_{\rm HC})<10^{-4}m_p$ thereby constraining the gravity wave energy density: $\Omega_g=\epsilon_g/\epsilon_m<10^{-8}$. However, since $H_{\rm inf}(\eta)$ grows with time as newer waves reenter the cosmological particle horizon after undergoing superadiabatic amplification, it follows that Ω_g too will grow giving rise to the possibility that eventually, $\epsilon_g\approx\epsilon_m$ when waves which left the inflationary horizon during the Planck era, reenter the FRW horizon with Planckian amplitudes: $h={\rm const}\times m_p^{-1}H^{\rm Pl}_{\rm inf}\approx 1$. The net effect of Ω_g growing, will be to speed up the rate of expansion of the Universe at late times. It is easy to see that if inflation commences in the Planck domain, then the FRW scale factor will approach the stable asymptotic form $a\propto t$ at late times, for which no new waves enter the horizon, signaling the end of further graviton creation. (However, because of the small value of Ω_g today, it will be many Hubble times before $\epsilon_g\approx\epsilon_m$, and the back reaction of newly created gravitons on the background space-time geometry becomes appreciable.[14])

From (14) we find that for exponential inflation $H_{\rm inf}=|\eta_0|^{-1}={\rm const}$, so that the ratio of the gravity wave density to that of the background matter is a fixed quantity whose value (for a given background equation of state) is determined solely by the Hubble constant for de Sitter space. For a dust-dominated universe,

$$\Omega_g=\frac{\epsilon_g}{\epsilon_m}=\frac{3}{16\pi^3}(m_p^{-1}H_{\rm inf})^2={\rm const} . \tag{16}$$

(Since $H_{\rm inf}<10^{-4}m_p$ this indicates $\Omega_g<10^{-8}$.) Ω_g as defined in (16) is in fact smaller by a factor of π^2 than a recent evaluation of Ω_g by Allen.[5] This discrepancy arises because the lower-frequency cutoff scale used by us in Eq. (13) and that used by Allen[5] are slightly different. In Ref. 5 the value of the lower-frequency limit is taken to be $\omega=k/a=H(\eta)$. $H(\eta)$ being the present value of the Hubble parameter. Since $H(\eta)=2/a\eta$ in a dust-dominated universe, this corresponds to a cutoff scale $k=2/\eta$ which is smaller by a factor of π than the wave number corresponding to the present-day horizons scale $k=2\pi/\eta$, at which we impose our long-wavelength cutoff. Since the spectral energy density of gravitons for a dust-dominated universe—$kd\epsilon_g/dk$—goes as $1/k^2$ for small k, this results is an overall enhancement of π^2 for the energy density of gravitons obtained in Ref. 5.

The result that Ω_g is independent of time is however strictly true only for exponential inflation when the inflating space-time geometry is exactly described by de Sitter space. For the more realistic case of quasiexponential inflation when

$$a(t)\propto\exp\left[\int H_{\rm inf}(t)dt\right] ,$$

and $H_{\rm inf}(t)$ decreases slowly with time, one should expect, in keeping with our preceding analysis, $\epsilon_g/\epsilon_m\propto H^2_{\rm inf}(\eta)$ to be a monotonically increasing function of time as in the case of power-law inflation, so that as in that case, $\epsilon_g\approx\epsilon_m$ eventually, when waves originating during Planck-epoch inflation begin reentering the horizon.

From (14a) we also find that, for a particular choice of μ and ν, $\nu+\mu=3$, the energy density of gravity waves does not depend upon time, thereby behaving like an effective cosmological constant. For a dust-dominated universe ($\mu=-\frac{3}{2}$) this arises if $\nu=\frac{9}{2}$, which corresponds to very weak power-law inflation: $a\propto t^{4/3}$. Reheating constraints, however, seem to rule out this rather intriguing possibility.[3]

From (13) we obtain the following spectral energy density for gravity waves (in units of erg/cm^3):

$$\epsilon(\tilde{\omega})=k\frac{d\epsilon_g}{dk}=b^2s^2\epsilon_m\tilde{\omega}^{4-2(\nu+|\mu|)} , \tag{17}$$

where

$$b=4(\pi)^{-(\nu+\mu|)}\left[\frac{2\pi}{3}\right]^{1/2}\frac{\Gamma(\nu)\Gamma(1+|\mu|)}{(\nu-\frac{1}{2})(|\mu|+\frac{1}{2})} ,$$

and $s=m_p^{-1}H_{\rm inf}(\eta_{\rm HC})$ is the dimensionless parameter first introduced by Starobinsky,[1] which is related to the amplitude of gravity waves entering our horizon today (η_{HC} stands for horizon crossing time as in Ref. 3), $\tilde{\omega}=k\eta/2\pi$ is the dimensionless wave number expressed in units of the horizon scale: $\tilde{\omega}=\omega/\omega_h=\lambda_h/\lambda$ (λ being the physical wavelength and λ_h being the present scale of the horizon), so that $\tilde{\omega}\geq 1$. ($\lambda_h \simeq 2\times 10^{28}h^{-1}$ cm, where $h=H/50$, H being the present value of the Hubble parameter expressed in units of km sec^{-1}Mpc^{-1}.) This result may be rewritten in terms of the spectral cosmological parameter: $\Omega_g(\omega)=\epsilon(\tilde{\omega})/\epsilon_{\rm cr}$, so that

$$\Omega_g(\omega)=b^2s^2\Omega_m\tilde{\omega}^{4-2(\nu+|\mu|)} ,$$

where $\Omega_m=\epsilon_m/\epsilon_{\rm cr}$ is the ratio of the energy density of matter to its critical value. For exponential inflation followed by a radiation-dominated universe ($\nu=\frac{3}{2},\mu=-\frac{1}{2}$), we recover the scale-invariant spectrum first obtained by Starobinsky:[1]

$$\epsilon(\tilde{\omega})=\frac{2}{3\pi}s^2\epsilon_r , \tag{18}$$

where ϵ_r is the total energy density in massless particles (excluding gravitons).

We see from (17) that for power-law inflation ($\nu>\frac{3}{2}$), the resulting gravity-wave spectrum has greater power at lower frequencies than if the inflation was exponential. For both exponential and for power-law inflation, the long-wavelength power in the graviton spectrum is directly related to the equation of state for matter during the epoch following inflation, with the spectrum displaying greater power at longer wavelengths for matter having softer equations of state. In particular for exponential inflation followed by a dust-dominated epoch of expansion: $a\propto t^{2/3}$ [$\nu=\frac{3}{2},\mu=-\frac{3}{2}$ in (17)], we obtain

$$\epsilon(\tilde{\omega})=\frac{3}{8\pi^3}\frac{s^2\epsilon_m}{\tilde{\omega}^2} . \tag{19}$$

A similar spectrum is also produced if "domain-wall-driven inflation" $a\propto t^2$, is followed by a radiation-dominated FRW expansion: $a\propto\sqrt{t}$, in which case $\nu=\frac{5}{2}$, $\mu=-\frac{1}{2}$ in (17) and once again

$$\epsilon(\tilde{\omega})=\frac{3}{8\pi^3}\frac{s^2\epsilon_r}{\tilde{\omega}^2} . \tag{20}$$

So far we have confined our attention to the simplest possible case of simple inflation followed by single-component FRW expansion. We do not, however, expect any of our main conclusions (such as the slow growth in ϵ_g/ϵ_m) to change if one considers more complex scenarios, such as double inflation followed by a multicomponent FRW expansion.[15] In this case, one would expect the spectrum of created particles to be somewhat modified due to the presence of additional particles created during the change in the equation of state of the Universe, as its expansion is now governed by different components at different times.

We evaluate in the next section the spectral energy density for gravitons in a multicomponent FRW universe. Some work in this direction has previously been reported in Refs. 5 and 16.

III. THE GRAVITON SPECTRAL ENERGY DENSITY IN A MULTICOMPONENT FRW UNIVERSE

In general, if the expansion of the Universe is governed by different components at different times, then the spectrum for gravity waves will show a characteristic change in slope at a frequency determined by the horizon scale at the time when the equation of state for matter changes. To treat the problem most generally, we use the methods developed earlier in this paper to evaluate the Bogoliubov coefficients for a universe undergoing successive inflationary stages, as well as for multicomponent FRW-style expansion.

In the case of double inflation, let us assume that the Universe underwent two successive stages of inflation characterized, respectively, by the expansion indices λ and ν. The solutions to the wave equation (2) are then, for $\eta<\eta_1<0$,

$$\phi_k^+(\eta)=\left[\frac{\pi\eta_1}{4}\right]^{1/2}\left[\frac{\eta}{\eta_1}\right]^{\lambda}H_\lambda^{(2)}(k\eta),\quad \lambda>0 , \tag{21}$$

corresponding to the adiabatic vacuum state, and, for $\eta_1<\eta<0$,

$$\tilde{\phi}_k(\eta)=c_1^\nu\tilde{\phi}_k^{(+)}+c_2^\nu\tilde{\phi}_k^{(-)} ,$$

where

$$\tilde{\phi}_k^{(+,-)}(\eta)=\left[\frac{\pi\eta_1}{4}\right]^{1/2}\left[\frac{\eta}{\eta_1}\right]^{\nu}H_\nu^{(2,1)}(k\eta),\quad \nu>0 . \tag{22}$$

c_2^ν and c_1^ν can once more be determined by requiring that both ϕ and $\tilde{\phi}$ have the same asymptotic behavior given by (6). As a result we obtain, using $\lim_{k\to 0}\phi=\tilde{\phi}$ and the low-frequency limits of ϕ and $\tilde{\phi}$ given in (8a),

$$c_{2,1}^\nu=\frac{1}{2}\left[\gamma\left[\frac{k\eta_1}{2}\right]^{\lambda-\nu}\mp\gamma^{-1}\left[\frac{k\eta_1}{2}\right]^{\nu-\lambda}\right] \tag{23}$$

for $k\eta_1<2\pi$, where $\gamma=\Gamma(\nu)/\Gamma(\lambda)$ and $|c_1^\nu|^2-|c_2^\nu|^2=1$. For $k\eta_1>2\pi$, $c_1^\nu\simeq 1$, $c_2^\nu\simeq 0$. For exponential inflation ($\lambda=\frac{3}{2}$) followed by domain-wall-driven driven inflation ($\nu=\frac{5}{2}$) we get

$$c_{2,1}^{(5/2)}=\frac{1}{2}\left[\frac{3}{k\eta_1}\mp\frac{k\eta_1}{3}\right] . \tag{24}$$

If, for $\eta>\eta_0$ the Universe undergoes conventional FRW-type expansion characterized by the expansion index μ: $\mu<0$, then the final Bogoliubov coefficients corresponding to the overall transition $\lambda\to\nu\to\mu$ will be

$$\begin{bmatrix} c_1 & c_2 \\ c_2^* & c_1^* \end{bmatrix}_{\rm final}=\begin{bmatrix} c_1^\nu & c_2^\nu \\ c_2^{\nu *} & c_1^{\nu *} \end{bmatrix}_{\lambda\to\nu}\begin{bmatrix} c_1 & c_2 \\ c_2^* & c_1^* \end{bmatrix}_{\nu\to\mu} , \tag{25}$$

where $c_{1(2)}^\nu$ are given by (23). The Bogoliubov coefficients describing the transition from inflation to a FRW expansion, $(c_2,c_1)_{\nu\to\mu}$, have been previously evaluated in (10a) and (10b). [For the noninflationary range $0<\nu<\frac{1}{2}$, Eq. (23) also gives the Bogoliubov coefficients corresponding to the transition from an inflationary expansion (21), with $\lambda\geq\frac{3}{2}$, to a FRW expansion Eq. (22), with $0<\nu<\frac{1}{2}$ (corresponding to $0<c<\frac{1}{3}$ in $a\propto t^c$)].

Similarly, the case of a multicomponent FRW universe is important to consider, if only for the reason that the Universe we live in has at least two components—dust and radiation. Furthermore, the equation of state of the Universe during reheating is very model dependent and must also be taken into account in order to obtain the correct behavior of the spectral energy density at small wavelengths.[9,16]

In order to evaluate the Bogoliubov coefficients in this case, we assume as before that the adiabatic vacuum at a time $\eta<\eta_1$ is described by (21) but with $\lambda<0$, since the expansion of the Universe is assumed to be noninflationary. Similarly, for $\eta>\eta_1$ solutions to the wave equation are described by (22), again with $\nu<0$. Once more using $\lim_{k\to 0}\phi=\tilde{\phi}$, and (6) and (8b) for the low-frequency behavior of ϕ and $\tilde{\phi}$ we obtain (for $k\eta_1<2\pi$)

$$c_{2,1}^{\nu}=\frac{1}{2}\left[\gamma\left[\frac{k\eta_1}{2}\right]^{|\lambda|-|\nu|}\mp\gamma^{-1}\left[\frac{k\eta_1}{2}\right]^{|\nu|-\lambda|}\right], \tag{26}$$

where $\gamma=\Gamma(1+|\nu|)/\Gamma(1+|\lambda|)$, and $|c_1^{\nu}|^2-|c_2^{\nu}|^2=1$. For $k\eta_1>2\pi$ the adiabatic theorem gives $c_1^{\nu}\simeq 1, c_2^{\nu}\simeq 0$.

If a period of inflation preceded the FRW expansion then the final Bogoliubov coefficients corresponding to the overall transition $\tilde{\nu}\to\lambda\to\nu$ will be given by

$$\begin{bmatrix} c_1 & c_2 \\ c_2^* & c_1^* \end{bmatrix}_{\text{final}}=\begin{bmatrix} c_1 & c_2 \\ c_2^* & c_1^* \end{bmatrix}_{\tilde{\nu}\to\lambda}\begin{bmatrix} c_1^{\nu} & c_2^{\nu} \\ c_2^{\nu *} & c_1^{\nu *} \end{bmatrix}_{\lambda\to\nu}, \tag{27}$$

where c_2^{ν} and c_1^{ν} are defined in (26), and the Bogoliubov coefficients for the transition from inflation to FRW expansion $(c_2,c_1)_{\tilde{\nu}\to\lambda}$ are given in (10a) and (10b) with ν and μ in those equations being replaced, respectively, by $\tilde{\nu}$ and λ. For the important case of a two-component universe consisting of *radiation* and *dust*, we obtain, from (27) noting that $\lambda=-\frac{1}{2}$ for radiation and $\nu=-\frac{3}{2}$ for dust,

$$c_{2,1}^{\text{final}}=\frac{i}{2}\left[\frac{3\gamma}{k\eta_1}\left[\frac{k\eta_0}{2}\right]^{-(\tilde{\nu}+1/2)}\mp\frac{k\eta_1}{3\gamma}\left[\frac{k\eta_0}{2}\right]^{(\tilde{\nu}+1/2)}\right], \tag{27a}$$

where $\gamma=\Gamma(\tilde{\nu})/2\sqrt{\pi}$. $[a=(t/t_0)^c\equiv(\eta/\eta_0)^{(1-2\tilde{\nu})/2}$, so that $\tilde{\nu}=(1-3c)/2(1-c)$; for exponential inflation $\tilde{\nu}=\frac{3}{2}$ and $\gamma=\frac{1}{4}$.]

Knowing the value of c_2, the corresponding value of the spectral energy density for gravitons can be easily obtained from the relation $\epsilon(\omega)=\omega d\epsilon_g/d\omega=\frac{\omega^4}{\pi^2}|c_2|^2$. Combining the value of c_2 obtained in (27a) for long wavelengths with the intermediate wavelength value given in (10a) we obtain

$$\epsilon(\tilde{\omega})=b_2^2\tilde{\omega}^{1-2\tilde{\nu}}s^2\epsilon_m \quad \text{for } 1<\tilde{\omega}<\frac{3}{4\pi}\Omega_r^{-1/2}.$$

and

$$\epsilon(\tilde{\omega})=b_1^2\tilde{\omega}^{3-2\tilde{\nu}}s^2\epsilon_r$$

$$\text{for } \frac{3}{4\pi}\Omega_r^{-1/2}<\tilde{\omega}<\frac{3}{4\pi}\Omega_r^{-1/2}\frac{H_{\text{inf}}}{H_{\text{rad}}}, \tag{27b}$$

where

$$b_1^2=\frac{8\pi^2}{3}\frac{\Gamma(\tilde{\nu})^2}{(\tilde{\nu}-\frac{1}{2})^2}\pi^{-2\tilde{\nu}-1},\quad b_2^2=\frac{9}{16\pi^2}b_1^2.$$

H_{inf} and H_{rad} are the values of the Hubble parameter at the end of the inflationary epoch and at the end of the radiation-dominated epoch, respectively. ϵ_m and ϵ_r are the energy densities of matter and radiation, respectively; $\epsilon_m\approx\epsilon_{\text{cr}}=4.2.10^{-9}h^2$ ergs/cm^3, $\Omega_r=\epsilon_r/\epsilon_m\simeq 10^{-4}h^{-2}$, where $h=H/50$, H being the present value of the Hubble parameter (expressed in units of km sec^{-1}Mpc^{-1}). $s=m_p^{-1}H_{\text{inf}}(\eta_{\text{HC}})$ is related to the dimensionless amplitude of gravity waves entering the cosmological horizon today; $s<10^{-4}$ is indicated by the present bounds on the anisotropy of the cosmic-microwave-background radiation[2,4] (see Fig. 1).

As before $\tilde{\omega}=k\eta/2\pi=\lambda_h/\lambda$, is the dimensionless wave number expressed in units of the horizon scale (λ being the physical wavelength and λ_h being the present scale of the horizon, $\lambda_h\simeq 2\times 10^{28}h^{-1}$ cm), so that $\tilde{\omega}\geq 1$.

For exponential inflation ($\tilde{\nu}=\frac{3}{2}$), Eq. (27b) agrees with the spectrum of gravitons obtained in Ref. 5 [Eq. (4.8)], if we note that $\tilde{\omega}=\omega/\pi H(t_p)$ in terms of notations used in that paper. The upper threshold frequency in (27b) has been assigned assuming that the transition from inflation to a radiation-dominated epoch was instantaneous. As pointed out by Ford,[9] however, the equation of state of matter during reheating may differ from that of radiation, in which case the duration of the reheating phase will have to be taken into account in assigning the upper limit

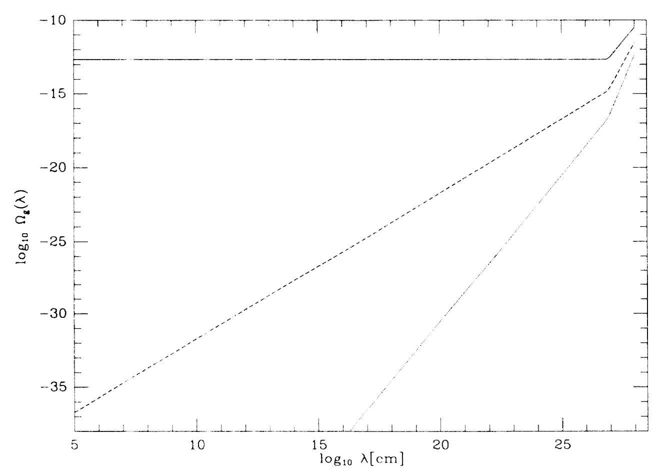

FIG. 1. The spectral energy density of gravity waves (in units of the critical energy density) is shown as a function of the wavelength, for exponential inflation (solid line), and for power-law inflation with $a\propto t^3$ (dashed line), and $a\propto t^2$ (dotted line). The spectral energy density has been evaluated under the assumption that $s=m_p^{-1}H_{\text{inf}}(\eta_{\text{HC}})\simeq 10^{-4}$. A lower value of s will result in a smaller amplitude for gravity waves.

to $\tilde{\omega}$. A more realistic upper threshold frequency may therefore be $\tilde{\omega} < (3/4\pi)\Omega_r^{-1/2}(H_{\rm reheat}/H_{\rm rad})$, where $H_{\rm reheat}$ is the Hubble parameter just after reheating. The related question of graviton production during the reheating phase has recently been addressed by Ressell and Turner.[16]

For the important case of a spatially flat universe containing both radiation and dust, the Klein-Gordon equation (2) can be solved exactly, with the adiabatic vacuum at $\eta \simeq \infty$ being described by[17]

$$\Psi = [a_0\eta_1 a(\eta)]^{-1/2} u(\eta) ,$$

where $a(\eta) = a_0\eta(\eta+\eta_1)$, and

$$u(\eta) = \sqrt{\omega} S_1^{1(4)}(\omega,x) \underset{k\eta\to\infty}{\sim} \left[\frac{\pi\eta_1}{4\eta}\right]^{1/2} H_{3/2}^{(2)}(k\eta) , \quad (28)$$

where $\omega = k\eta_1/2$, $x = 1+2\eta/\eta_1$, and $S_1^{1(4)}(\omega,x)$ is the radial prolate spheroidal function.[18] If at early times $\eta < \eta_0$ the Universe underwent a period of inflation, then, the "out" modes in Eq. (28) will be modified to

$$\tilde{u} = c_1 u + c_2 u^* . \quad (29)$$

Once again, for $k\eta_0 < 2\pi$, c_1 and c_2 can be evaluated using (6) and the well-known asymptotic form for prolate spheroidal functions:[17,18]

$$S_1^{1(4)}(\omega,x) \underset{\omega\to 0}{\sim} \frac{\omega}{3} P_1^1(x) - \frac{3i}{2\omega^2} Q_1^1(x) , \quad (30)$$

where $P_1^1(x)$ and $Q_1^1(x)$ are Legendre functions of the first and second kind, respectively. Once more requiring $\lim_{k\to 0}\phi^+ = \Psi$, with ϕ^+ defined in (3), and (8a), we obtain (for $k\eta_0 < 2\pi$)

$$c_{2,1} = \frac{i}{2}\left[\gamma\left[\frac{k\eta_0}{2}\right]^{-(\nu+3/2)} \mp \gamma^{-1}\left[\frac{k\eta_0}{2}\right]^{\nu+3/2}\right] , \quad (31)$$

where $\gamma = 3\Gamma(\nu)/4\sqrt{\pi}$, and $|c_1|^2 - |c_2|^2 = 1$. For $\nu = \frac{3}{2}$ corresponding to exponential inflation,

$$c_{2,1} = \frac{i}{2}\left[\frac{3}{(k\eta_0)^3} \mp \frac{(k\eta_0)^3}{3}\right] .$$

The Klein-Gordon equation can also be exactly solved in a FRW universe consisting of stiff matter ($\epsilon_m \propto A/a^6$) and radiation ($\epsilon_r \propto B/a^4$). In this case $a(\eta) = \sqrt{a_1\eta(\eta+\eta_1)}$, where $a_1 = B/3$ and $\eta_1 = 2\sqrt{3A}/B$. The adiabatic and the conformal vacuua being identical are described by[17]

$$\Psi(\eta) = \left[\frac{\omega}{a_1\eta_1}\right]^{1/2} S_0^{0(4)}(\omega,x)$$

$$\underset{k\eta\to\infty}{\sim} \frac{1}{\sqrt{2k}}\frac{1}{a(\eta)}\exp(-ik\eta) , \quad (32)$$

where $\omega = k\eta_1/2$. As in the previous case, if the expansion of the Universe was inflationary at early times, then the late time behavior of Ψ will be given by

$$\tilde{\Psi} = c_1\Psi + c_2\Psi^* . \quad (33)$$

Knowing the low-frequency behavior of $S_0^0(\omega,x)$

$$S_0^{0(4)}(\omega,x) \underset{\omega\to 0}{\sim} 1 + \frac{i}{\omega} Q_0(x) \quad (34)$$

allows one to once more determine c_2 and c_1 by matching $\tilde{\Psi}$ and ϕ^+ at small k. So that, for $k\eta_0 < 2\pi$,

$$c_{2,1} = \frac{i}{2}\left[\gamma\left[\frac{k\eta_0}{2}\right]^{-(\nu+1/2)} \mp \gamma^{-1}\left[\frac{k\eta_0}{2}\right]^{\nu+1/2}\right] , \quad (35)$$

where $\gamma = \Gamma(\nu)/2\sqrt{\pi}$. For $k\eta_0 > 2\pi, c_1 \simeq 1, c_2 \simeq 0$.

Finally, if a radiation-dominated stage existed pri_r to inflation, the corresponding scalar field modes during inflation will be modified. In the absence of such a radiation stage, the state defined by the massless limit of the adiabatic vacuum state is unphysical, because the two-point function, and in some cases the energy-momentum tensor, have infrared divergences. An initial radiation stage (and its corresponding adiabatic vacuum) gives rise to an acceptable, physical, and finite quantum state in the subsequent de Sitter phase. Of course, different choices of quantum state (other than the $m\to 0$ limit of the adiabatic vacuum) for the de Sitter phase could be invoked to permit an infinite period of inflation with no initial radiation phase. It has been shown that any such quantum state would have to break de Sitter invariance.[19]

The state corresponding to the conformal vacuum during the radiation-dominated stage is given by

$$\phi^+ = \frac{1}{\sqrt{2k}}\frac{1}{a(\eta)}\exp(-ik\eta) \quad (36)$$

so that $\tilde{A} = \eta_0\sqrt{k/2}$, and $\tilde{B} = -i/\sqrt{2k}$, where $\tilde{A}$ and $\tilde{B}$ are defined by the low-frequency limit of (36) using (6). During inflation the resulting quantum state will once more be a linear superposition of positive- and negative-frequency solutions of (2), so that

$$\tilde{\phi}_k(\eta) = c_1\tilde{\phi}_k^{(+)} + c_2\tilde{\phi}_k^{(-)} ,$$

where, just as in (22),

$$\tilde{\phi}_k^{(+,-)}(\eta) = \left[\frac{\pi\eta_0}{4}\right]^{1/2}\left[\frac{\eta}{\eta_0}\right]^{\nu} H_\nu^{(2,1)}(k\eta), \quad \nu > 0 . \quad (37)$$

Matching ϕ^+ and $\tilde{\phi}$, for low frequencies, using the low-frequency limit of (37) given in (6) and (8a), we obtain

$$c_1 + c_2 = \frac{\tilde{B}}{B}, \quad c_1 - c_2 = \frac{\tilde{A}}{A} , \quad (38)$$

where A and B are defined in (8a). From (37) and (38) we finally get

$$\tilde{\phi}_k(\eta) = \left[\frac{\eta}{\eta_0}\right]^{\nu}\left[a\left[\frac{k\eta_0}{2}\right]^{-\nu-1/2} J_\nu(k\eta) - b\left[\frac{k\eta_0}{2}\right]^{\nu+1/2} Y_\nu(k\eta)\right] , \quad (39)$$

where

$$a = -i\frac{\sqrt{\eta_0}\,\Gamma(1+\nu)}{4\nu}, \quad b = \frac{\pi\sqrt{\eta_0}}{\Gamma(\nu)} . \quad (40)$$

From (39) we see that $\lim_{k\to 0}|\tilde{\phi}|^2k^2dk \propto k\,dk \to 0$, so that infrared divergences in $\langle \phi^2\rangle = 1/2\pi^2\int dk\, k^2|\tilde{\phi}|^2$ are indeed absent in this case. Infrared divergences are also absent in $\langle T_{\mu\nu}\rangle$.

IV. PRODUCTION OF MASSLESS SCALAR PARTICLES FROM INFLATION

It is also interesting to evaluate the energy density and pressure for massless scalar field quanta created during the transition from exponential inflation to a FRW universe containing matter with the equation of state $p=\tilde{\alpha}\epsilon$ with $-\frac{1}{3}<\tilde{\alpha}<1$. [Since $a=(t/t_0)^c \equiv (\eta/\eta_0)^{1/2-\mu}$, this range in $\tilde{\alpha}$ corresponds to $\frac{1}{3}<c<1$ and $\mu<0$. The parameters $\tilde{\alpha}$, c, and μ being related as follows: $\tilde{\alpha}=(1+\frac{2}{3}\mu)(1-2\mu)^{-1}=(2/3c)-1$].

As pointed out by Wise,[20] an axion field with a small mass may play the role of an effectively massless minimally coupled scalar field, especially if the Peccei-Quinn symmetry is spontaneously broken during the inflationary era, in which case the axion field is essentially massless during inflation. Other possibilities for the existence of extremely light, effectively massless, scalars and their influence on cosmology, have been recently discussed by Weiss.[20]

For a minimally coupled massless scalar field,[12]

$$\epsilon_\phi = \langle T^0_0\rangle = \frac{1}{4\pi^2a^2}\int_0^{\eta_0^{-1}} dk\, k^2(|\dot{\tilde{\phi}}_k|^2+k^2|\tilde{\phi}_k|^2) , \quad (41a)$$

$$p_\phi = -\langle T^l_l\rangle = \frac{1}{4\pi^2a^2}\int_0^{\eta_0^{-1}} dk\, k^2\left[|\dot{\tilde{\phi}}_k|^2-\frac{k^2}{3}|\tilde{\phi}_k|^2\right] . \quad (41b)$$

Where $\tilde{\phi}_k$ is defined in (10c) for arbitrary values of ν. For exponential inflation $\nu=\frac{3}{2}$, and

$$\tilde{\phi}_k(\eta)=\left[\frac{\pi\eta_0}{4}\right]^{1/2}\left[\frac{\eta}{\eta_0}\right]^{-|\mu|}F_{|\mu|}(k\eta) , \quad (42)$$

where

$$F_{|\mu|}(k\eta)=i\gamma\left[\frac{k\eta_0}{2}\right]^{-3/2-|\mu|}J_{|\mu|}(k\eta) + \gamma^{-1}\left[\frac{k\eta_0}{2}\right]^{3/2+|\mu|}Y_{|\mu|}(k\eta)$$

and $\gamma=\Gamma(1+|\mu|)/2\sqrt{\pi}$.

Then, noting that[13]

$$\dot{\tilde{\phi}}_k=-\left[\frac{\pi\eta_0}{4}\right]^{1/2}\left[\frac{\eta}{\eta_0}\right]^{-|\mu|}F_{|\mu|+1}(k\eta)$$

we obtain

$$\epsilon_\phi=\frac{\eta_0}{16a^2}\left[\frac{\eta}{\eta_0}\right]^{-2|\mu|}\int_0^{\eta_0^{-1}} dk\, k^4(|F_{|\mu|+1}|^2+|F_{|\mu|}|^2)$$

and

$$p_\phi=\frac{\eta_0}{16a^2}\left[\frac{\eta}{\eta_0}\right]^{-2|\mu|} \times\int_0^{\eta_0^{-1}} dk\, k^4(|F_{|\mu|+1}|^2-\tfrac{1}{3}|F_{|\mu|}|^2) . \quad (43)$$

As in the case of gravitons, the main contribution to (43) for equations of state $p<\epsilon/3$, ($\mu<-\frac{1}{2}$), comes from the vicinity $k=2\pi\eta^{-1}$, corresponding to those modes which are entering the horizon today. The integrals in (43) may be evaluated exactly so that we finally obtain (for $\mu<-\frac{1}{2}$) (Ref. 21)

$$\epsilon_\phi \simeq \frac{A}{8\pi^2a^2\eta^2\eta_0^2},\quad p_\phi\simeq\frac{B}{8\pi^2a^2\eta^2\eta_0^2} , \quad (44)$$

where

$$A=\mu\frac{1-2\mu}{1+2\mu},\quad B=\mu\frac{1+\frac{2}{3}\mu}{1+2\mu} .$$

and $H_{\rm inf}=|\eta_0|^{-1}$ is the Hubble constant during inflation.

Integrating (41a) for a radiation-dominated expansion $p=\epsilon/3$, ($\mu=-\frac{1}{2}$) we obtain

$$\epsilon_\phi=\frac{1}{4\pi^2(a\eta_0)^4}\left\{\ln\left[\frac{\eta}{\eta_0}\right]-{\rm ci}\left[2\left[\frac{\eta}{\eta_0}\right]\right]+C\right\} , \quad (45)$$

where ${\rm ci}(2\eta/\eta_0)$ is the integral cosine function, and C is a constant, $C=\frac{1}{16}+\ln 2-\gamma-\frac{1}{2}$ (γ being Euler's constant).

From (45) we see that ϵ_ϕ drops off more slowly in time than $H^2(\eta)=1/(a\eta_0)^4$. This is due to the fact that the spectral energy density is scale invariant in this case, with all wave numbers contributing equally to ϵ_ϕ. [This was also true for gravitons as shown in (18)]. The high-frequency cutoff in (41) is an absolute cutoff in this case since no massless particle production can occur in a metric which is conformally flat.[11] ($R=0$ for $p=\epsilon/3$, so that minimally and conformally coupled massless scalar field equations are equivalent in this case.)

As we have pointed out earlier, the dominant contribution in (43) for $\mu<-\frac{1}{2}$ arises from modes just entering the horizon for whom $k\eta\simeq 2\pi$. This accounts for the presence of the term $H_{\rm inf}a^{-2}\eta^{-2}=H_{\rm inf}[(1-c)/2]^2t^{-2}$ in (44), and also explains the close similarity between ϵ_ϕ and ϵ_g for exponential inflation.

From (44) we see that

$$\frac{p_\phi}{\epsilon_\phi}=\frac{1+\frac{2}{3}\mu}{1-2\mu}=\tilde{\alpha} , \quad (46)$$

where $p=\tilde{\alpha}\epsilon$ is the equation of state of the background matter. We thus see that the created particles possess exactly the same equation of state as the background matter driving the expansion of the Universe.

The fact that $\langle T_{\mu\nu}\rangle$ for created massless scalar particles mimics the behavior of ordinary matter driving the expansion of the Universe, leads to the interesting possibility that, perhaps, self-consistent solutions of the Einstein equations can be constructed in which the expansion of the Universe is sustained solely by the ongoing creation of massless scalar quanta (or equivalently of gravitons, since as shown in the previous section, gravitons created as a result of exponential inflation also have the property $\epsilon_g \propto \epsilon_m$), so that $G_{\mu\nu} = 8\pi G \langle T_{\mu\nu} \rangle$.

Solving for the trace of the Einstein equations

$$R = \frac{8\pi G}{c^4} \langle T \rangle , \tag{47}$$

where

$$R = \frac{6(\mu^2 - \frac{1}{4})}{a^2 \eta^2}$$

and

$$\langle T \rangle = \epsilon_\phi - 3p_\phi = -\frac{\mu}{4\pi^2 a^2 \eta^2 \eta_0^2} \quad (\mu < -\tfrac{1}{2})$$

we obtain the following algebraic equation for μ:

$$\mu^2 + b\mu - \tfrac{1}{4} = 0 , \tag{48}$$

where $b = s^2/3\pi$, and $s = m_p^{-1} H_{\rm inf}$ is the dimensionless parameter introduced earlier. Solving for μ we get

$$\mu = -\tfrac{1}{2}(b + \sqrt{1+b^2}) . \tag{49}$$

From Eqs. (47)–(49) we see that, given an *a priori* value of the Hubble parameter during inflation $H_{\rm inf}$, the corresponding self-consistent FRW expansion of the Universe, $a = (\eta/\eta_0)^\mu$ ($\mu < 0$), can always be determined from the relation (49) linking μ with $H_{\rm inf}$. To reexpress the expansion of the Universe in terms of the real-time coordinate $a \propto t^c$, we may use the relation

$$c = \frac{1 - 2\mu}{3 - 2\mu} , \tag{50}$$

linking the expansion index c with μ.

It is worthwhile to recall that quantum gravitational effects usually play a prominant role only very near the initial big-bang singularity, when the space-time curvature becomes comparable to its Planckian value.[11] The results of this and the previous section, however, indicate that it is possible for quantum gravitational effects to be significant even during the late stages of the Universe's expansion provided the Universe underwent an initial inflationary stage. The fact that most known solutions of $G_{\mu\nu} = 8\pi G \langle T_{\mu\nu} \rangle$ are almost exclusively sustained by the quantum vacuum polarization[22] makes the solutions to $G_{\mu\nu} = 8\pi G \langle T_{\mu\nu} \rangle$ obtained above rather unique, since in this case, the self-consistent expansion of the Universe is sustained solely by the ongoing creation of massless particles.

It is interesting to note that one cannot rule out large values of Ω_ϕ today, purely on the basis of primordial nucleosynthesis constraints which are usually invoked to limit the number density of other massless particles. This is so because massless scalar particles (and gravitons), created during an early inflationary epoch do not have the same equation of state as radiation, so that $\epsilon_g \propto a^{-4}$ is true only at early times when the Universe was radiation dominated.[23,24] The main constraint to Ω_ϕ comes, as in the case of Ω_g, from the observed upper limits on the isotropy of the cosmic-microwave-background radiation, which as in the case of gravitons, provides the constraint[2,3] $\Omega_\phi < 10^{-8}$, ruling out the possibility that massless scalar particles may play an important role in the dynamics of the present-day Universe.

So far in our treatment of particle production due to inflation we have restricted ourselves to soft equations of states for the matter driving the expansion of the Universe: $p \le \epsilon/3$. For stiffer equations of state, $\epsilon/3 < p \le \epsilon$, we find that, in addition to contributions to the energy density and pressure given by (44), there also exist terms in the energy-momentum tensor proportional to a^{-4}, which arise due to the production of high-frequency massless scalars. (The existence of such a contribution to the energy density of created massless particles was first noted by Grishchuk[7].) Since, for $p > \epsilon/3$, the background energy density ϵ_m ($\propto a^{-3(1+\bar{\alpha})}$) drops off faster than ϵ_ϕ ($\propto a^{-4}$), this will result in the energy density of created massless scalars rapidly overtaking the background energy density, thereby making the Universe effectively radiation dominated very soon[7,25] (so that $a \propto \sqrt{t}$ generically). The resulting energy density of created particles at late times will then be generically described by (45) for the case when the background equation of state for matter is stiffer than that of radiation. This argument is also applicable to the case of gravitons.[7,25]

V. CONCLUSIONS

We have evaluated both the spectral energy density and the total density for gravitational radiation produced during the transition from generalized inflation to a FRW-type expansion. We have shown that for power-law inflation, the spectral energy density for gravity waves has more power on larger scales than for purely exponential inflation. For the total density we find the surprising result that for both power-law inflation,[26] and for quasiexponential inflation,[27] the ratio of the energy density of gravity waves to the total matter density, $\Omega_g \simeq \epsilon_g/\epsilon_m$, grows slowly with time, as waves with larger amplitudes, originating earlier on during inflation, reenter the FRW horizon and contribute to the energy density an amount $\epsilon_g \sim \omega^2 h^2/16\pi G$, where h is the dimensionless amplitude of the gravity wave: $h \propto 16\pi G H_{\rm inf}^2(\eta)$, and $\omega = 2\pi/a\eta$ is the associated frequency at horizon crossing. $H_{\rm inf}(\eta)$ is the value of the inflationary Hubble parameter at a time when the wave crossed the inflationary horizon. In the case of primordial Planck-scale inflation, this behavior of Ω_g has led us to speculate that $\epsilon_g \approx \epsilon_m$ eventually, when gravity waves which left the inflationary horizon during the Planckian era, begin to reenter the FRW horizon with a dimensionless amplitude of order unity. Because of the steady growth in Ω_g, and the accompanying increasing back reaction of gravity waves, the expansion of the Universe will be modified at late times,[14] causing the Universe to enter into a period of "coasting expansion"[28] during which its scale factor grows linearly with time: $a \propto t$. The expansion law $a \propto t$, describes a stable asymptotic regime of expansion for which no new waves enter the horizon, signaling the absence of further graviton creation. This expansion law prevents all correlations which occurred during the pre-Planck era from reentering the FRW horizon, thereby restricting our knowledge of the epoch immediately following the initial big-bang singularity. This behavior resembles the operation of a principle of cosmic censorship. (All other quantum gravitational effects are absent as well, and $\langle T_{\mu\nu}\rangle=0$ if $a\propto t$, as was demonstrated in Ref. 29.)

In the idealized case of exactly exponential inflation, our treatment has been extended to include both minimally coupled massless scalar fields as well as gravitons. We have shown that both for gravitons and for massless scalars the effective equation of state for created particles mimics the background equation of state for matter driving the expansion of the Universe. This has led us to construct self-consistent solutions of the Einstein equations, which originate in an inflationary stage, and in which the subsequent postinflationary expansion of the Universe is sustained solely by the ongoing particle creation, so that $G_{\mu\nu}=8\pi G\langle T_{\mu\nu}\rangle$. (This effect remains a unique feature of massless scalar and spin-2 fields, since there is no production of massless spin-$\frac{1}{2}$ and spin-1 particles in a FRW cosmology.[11,30])

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[13]We have used the following useful relationship between the Bessel functions $Z_\nu(x)$ and $Z_{\nu\pm1}(x)$ to establish (12) and (43):

$$\frac{d}{dx}[x^{\pm|\mu|}Z_{|\mu|}(x)]=\pm x^{\pm|\mu|}Z_{|\mu|\mp1}(x)\ .$$

[See M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Methods* (Dover, New York, 1972).]

[14]A simple calculation shows the corresponding time scale to be $t_g=t_0(t_{0*}/t_{\rm Pl})^{3(c-1)}$, where t_0 is the present age of the Universe, $t_{\rm Pl}$ is the Planck time $\sim10^{-44}$ sec, and t_{0*} is the time when waves currently entering the FRW horizon left the inflationary horizon. Since $t_{0*}\simeq H_{\rm inf}^{-1}(\eta_{\rm HC})>10^4t_{\rm Pl}$ [$H_{\rm inf}(\eta)$ being the Hubble parameter during inflation], this implies $t_g\geq t_0 10^{12(c-1)}$ which corresponds to the Universe having expanded by the expansion factor $a(t_g)/a(t_0)\geq10^{8(c-1)}$. We see that even for nominal values of c, t_g is very large. For quasiexponential inflation t_g assumes still greater values since $H_{\rm inf}(\eta)$ decreases even more slowly with time in this case, so that $\epsilon_g/\epsilon_m\propto H_{\rm inf}^2(\eta)\approx$const over a period of several Hubble time scales. It should be mentioned, however, that the close similarity between gravity waves and a minimally coupled massless scalar field, upon which the present analysis is based, has been unambiguously demonstrated only for linearized waves with $h\ll1$ (Ref. 7). Therefore it is not clear to what extent the above arguments can be carried over to the case when $h\simeq1$.

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$$\int x^{1-2\lambda}J_\lambda^2(x)dx=\frac{1}{2(1-2\lambda)}x^{2-2\lambda}[J_\lambda^2(x)+J_{\lambda-1}^2(x)]\ ,$$

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