Self-energy of a thin charged shell in general relativity

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(Received 14 March 1990)

The self-energy of a thin charged shell is calculated within general relativity with the help of the Gauss-Codazzi and Lanczos equations. For the special case of a dust shell, our result agrees with that of Arnowitt, Deser, and Misner, but we find a different value for the lower bound on the radius of the shell and thus also for the total mass.

It has long been suggested that gravity in both its classical and quantized forms might play the role of a regulator for infinite self-energies of other interactions (see, e.g., Refs. 1 and 2). Within the context of general relativity, it has been demonstrated long ago by Arnowitt, Deser, and Misner[3] that a charged dust shell possesses a finite self-energy. Their result agreed with the naive computation within the Newtonian framework where the *ad hoc* implementation of the strong equivalence principle leads to a finite result, even in the limit of a point particle. In the relativistic calculation, this was achieved through the existence of a lower bound on the size of the shell. The purpose of this Brief Report is to generalize and simplify the derivation of these results. Firstly, not only the constraints are used, but also the dynamical equations of motion for the thin shell. Secondly, the equation of state is left unspecified as long as possible, obtaining in this way a more general result. Finally, the value of the minimally allowed radius of the shell is examined more carefully and is found to differ by a factor 2 from the result in Ref. 3. Technically, the formalism developed by Israel[4] using the Gauss-Codazzi and Lanczos equations will be applied to the case of a charged shell. This approach has also been proven fruitful in the discussion of domain walls,[5] and we will keep our notation close to Ref. 5.

Let us consider a three-dimensional timelike hypersurface Σ embedded in spacetime (see Fig. 1). The Gauss-Codazzi equations (constraint equations) for this hypersurface read

$$-2G_{ab}\xi^a\xi^b = {}^{(3)}R + K_{ab}K^{ab} - K^2 = -16\pi G T_{ab}\xi^a\xi^b \ , \quad (1)$$

$$G_{bc}h^b{}_a\xi^c = h_{ab}D_cK^{bc} - D_aK = 8\pi G T_{bc}h^b{}_a\xi^c \ . \quad (2)$$

Here ${}^{(3)}R$ denotes the Ricci scalar with respect to the three-metric h_{ab}, D the corresponding covariant derivative, K_{ab} the extrinsic curvature of Σ, K its trace, ξ a spacelike normal vector field, and G_{ab} the Einstein tensor. T_{ab} denotes the energy-momentum tensor which has a δ singularity on Σ. While the four-metric is continuous across Σ, K_{ab} exhibits a jump discontinuity. One thus introduces the integral of T_{ab} normally to Σ:

$$S_{ab} \equiv \int dl\, T_{ab} \quad (3)$$

and

$$\gamma_{ab} \equiv (K^+_{ab} - K^-_{ab}) \ , \quad (4)$$

$$\tilde{K}_{ab} \equiv \tfrac{1}{2}(K^+_{ab} + K^-_{ab}) \ , \quad (5)$$

where $\pm$ refers to the two sides of the shell. From the remaining Einstein equations one can derive the Lanczos equation for S_{ab} (see, e.g., Ref. 6):

$$S_{ab} = -\frac{1}{8\pi G}(\gamma_{ab} - h_{ab}\gamma) \ . \quad (6)$$

Taking (1) and (2) on opposite sides of Σ and using (6) one finds

$$h_{ab}D_cS^{bc} = 8\pi G[T_{bc}]h^b{}_a\xi^c \ , \quad (7)$$

$$\tilde{K}_{ab}S^{ab} = [T_{ab}]\xi^a\xi^b \ . \quad (8)$$

Here $[T]$ denotes the difference of the energy-momentum tensor on both sides of the hypersurface.

The surface stress-energy tensor is now taken to be of the general form

$$S_{ab} = \sigma u_a u_b - \tau(h_{ab} + u_a u_b) \ , \quad (9)$$

where σ and τ are the surface energy density and tension, respectively, as measured by an observer whose world line lies within Σ and who has four-velocity u. With this choice for S_{ab} one finds, from (7),

$$D_a(\sigma u^a) - \tau D_a u^a = -8\pi G[T_{ab}]u^a\xi^b \ . \quad (10)$$

Consider now the accelerations of observers on both sides of Σ. Their normal components $(\xi, \nabla_u u)_\pm$ can be found with the help of (8) to be (∇ denotes the four-dimensional

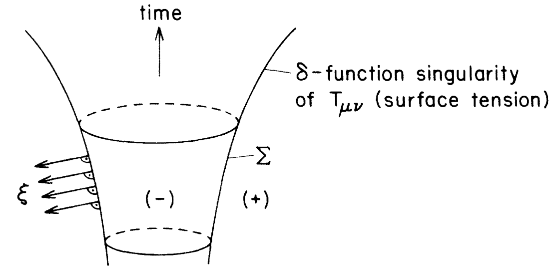

FIG. 1. Timelike hypersurface Σ embedded in spacetime. Depicted are also members of the normal vector field ξ.

covariant derivative)

$$(\xi,\nabla_u u)_+ + (\xi,\nabla_u u)_- = -\frac{2\tau}{\sigma}(h^{ab}+u^a u^b)\tilde{K}_{ab} - \frac{2}{\sigma}[T_{ab}]\xi^a\xi^b \quad (11)$$

and

$$(\xi,\nabla_u u)_+ - (\xi,\nabla_u u)_- = 4\pi G(\sigma-2\tau) . \quad (12)$$

At this stage we *restrict* ourselves to spherical charged shells, so that (according to Birkhoff's theorem) the metric outside the shell is given by the Reissner-Nordström solution

$$g_+ = -e^{2a(r)}dt^2 + e^{-2a(r)}dr^2 + r^2(d\vartheta^2+\sin^2\vartheta\, d\varphi^2) \quad (13)$$

with

$$e^{2a(r)} = 1 - \frac{2GM}{r} + \frac{GQ^2}{r^2} \quad (14)$$

(where Q is the electric charge) and inside the shell by the flat metric

$$g_- = -dt^2 + dr^2 + r^2(d\vartheta^2+\sin^2\vartheta\, d\varphi^2) . \quad (15)$$

The radius of the shell is given by $R(t)$. For the energy-momentum tensor $[T_{ab}]$ one has, with respect to the natural orthonormal frame $e^a dt$, etc., defined by (13) (remembering that there is vacuum inside the shell),

$$[T_{ab}] = \frac{Q^2}{8\pi r^4}\mathrm{diag}(1,-1,1,1). \quad (16)$$

The components of u and ξ in this basis are

$$u_+ = e^{-a}(\beta,\dot{R},0,0), \quad u_- = (\alpha,\dot{R},0,0) ,$$
$$\xi_+ = e^{-a}(\dot{R},\beta,0,0), \quad \xi_- = (\dot{R},\alpha,0,0) , \quad (17)$$

where the overdot denotes the derivative with respect to proper time and α and β are given, respectively, by

$$\alpha = \sqrt{1+\dot{R}^2} \quad (18)$$

$$\beta = \left[1 - \frac{2GM}{R} + \frac{GQ^2}{R^2} + \dot{R}^2\right]^{1/2} . \quad (19)$$

One can now evaluate (11) and (12) with these expressions and eventually finds the following equations of motion for the radius of the shell $R(t)$:

$$(\alpha-\beta)\ddot{R} = -\alpha G\left[\frac{M}{R_2} - \frac{Q^2}{R^3}\right] + 4\pi G\alpha\beta(\sigma-2\tau) , \quad (20)$$

$$(\alpha+\beta)\ddot{R} = -\alpha G\left[\frac{M}{R^2} - \frac{Q^2}{R^3}\right] - \frac{2\tau\alpha\beta(\alpha+\beta)}{\sigma R} + \frac{\alpha\beta Q^2}{4\pi\sigma R^4} . \quad (21)$$

Elimination of $\ddot{R}$ leads to

$$(\sigma-2\tau)\left[\frac{GM}{R^2} - 2\pi G\sigma(\alpha+\beta)\right] = (\sigma-\tau)\frac{GQ^2}{R^3} - (\alpha-\beta)\frac{Q^2}{8\pi R^4} \quad (22)$$

which through the use of the identity

$$\frac{GM}{R^2} = \frac{GQ^2}{2R^3} + \frac{\alpha^2-\beta^2}{2R} \quad (23)$$

can be transformed into a quadratic equation for β:

$$\beta^2 + \left[4\pi G\sigma R + \frac{Q^2}{4\pi R^3(\sigma-2\tau)}\right]\beta + \frac{GQ^2\sigma}{R^2(\sigma-2\tau)} - \alpha^2 + 4\pi G\sigma R\alpha - \frac{\alpha Q^2}{4\pi R^3(\sigma-2\tau)} = 0 . \quad (24)$$

One of the roots is always negative, while the other one reads

$$\beta = \alpha - 4\pi G\sigma R . \quad (25)$$

Using (23) one finds the result for the total energy:

$$M = \frac{Q^2}{2R} + 4\pi\sigma R^2(\alpha - 2\pi G\sigma R) . \quad (26)$$

It is interesting to note that this expression is *independent* of the value of the tension τ, on which it depends only implicitly through (10). If one specializes to *dust* ($\tau=0$) one finds from (10) that

$$4\pi\sigma R^2 = \mathrm{const} \equiv M_0 , \quad (27)$$

where M_0 is the rest mass of the shell. In Ref. 3, M_0 was artifically set equal to zero, which does not seem to be allowed.

An external value R_m for the radius is obtained by setting $\dot{R}=0$ in (26) (i.e., $\alpha=1$), thus leading to the final expression for the total energy:

$$M = M_0 - \frac{GM_0^2}{2R_m} + \frac{Q^2}{2R_m} , \quad (28)$$

which agrees with the naive Newtonian expression for the self-energy. This result is in accordance with Ref. 3. We disagree, however, with the value for the lower bound on R_m. The demand for β to be positive implies that

$$R_m \geq GM_0 , \quad (29)$$

as can be seen from (25). Using (28) this is equivalent to

$$R_m \geq GM + \sqrt{G^2M^2 - GQ^2} . \quad (30)$$

The right-hand side is the well-known expression for the horizon in the Reissner-Nordström solution. This is only consistent, if all quantities are real positive, so that $|Q| \leq \sqrt{G}M$, and a singularity is avoided. For $Q\to 0$, one obtains the well-known result that $R_m \geq 2GM$.[7] Thus (28) is bounded by

$$M \geq \frac{M_0}{2} + \frac{Q^2}{2GM_0} \equiv \bar{M} . \quad (31)$$

The lowest possible value for $\bar{M}$ is obtained for $M_0 = |Q|/\sqrt{G}$. In this case

$$\bar{M} = M_0 = \frac{|Q|}{\sqrt{G}} , \quad (32)$$

which also follows within the Newtonian framework, if in

(28) GM_0^2 is replaced by GM^2 (strong equivalence principle) and the limit $R \rightarrow 0$ is taken. Here the finiteness of the self-energy is achieved through the existence of a lower bound for R_m according to (29).

The binding energy of the shell, defined by the value of $M_0 - M$, satisfies

$$M_0 - M \leq \frac{M_0}{2} - \frac{Q^2}{2GM_0} \tag{33}$$

and thus can at most reach half of the rest mass M_0 (in the limit where Q^2 vanishes). This is in contrast with Ref. 3, where it was stated that the binding energy could even compensate the rest mass to yield vanishing total energy.

This work was supported in part by the Swiss National Science Foundation.

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